MAT2R3 TUTORIAL 7

- Thank you for the feedback! most people said the tutorials were okay but they wanted more review. I will add some more review to the notes and an example or two as well as a few questions to try at the end.
- These notes took me quite a while to type up. In the future I'm not sure how detailed I can be (I am not paid for the time I put into these), but if they are useful to you I will do my best to type faster!

Review:

(1) *l*inear transformations

A common theme in mathematics is to define a certain type of space (e.g. vector spaces), and then study them by looking at functions between spaces which "preserve" the structure of the space. In the case of linear algebra, we define vector spaces. The structure of a vector space (the operations on the vector space) are addition and scalar multiplication. We want to study functions between vector spaces which preserve these operations.

A *linear transformation* from a vector space V to a vector space W is a function $T\colon V \to W$ with the following properties;

- $T(v_1 + v_2) = T(v_1) + T(v_2)$ for every $v_1, v_2 \in V$,
- $T(cv) = cT(v)$ for every $v \in V$ and $c \in \mathbb{R}$ (or $\mathbb{C}$).

The first point says that the function preserves addition and the second that it preserves the scalar multiplication.

An important property about linear transformations is that they are completely determined by where they send a basis of V. If $\{v_1, \ldots, v_n\}$ is a basis for V, and $T(v_i) = w_i$, then for any $v \in V$ we can write $v = a_1v_1 + \ldots + a_nv_n$, and then $T(v) = T(a_1v_1 + \ldots + a_nv_n) = a_1T(v_1) + \ldots + a_nT(v_n) = a_1w_1 + \ldots + a_nw_n$. In other words, if you specify where T sends a basis of V, then you have completely determined T.

(2) *Kernel and Range*

The Kernel of a linear transformation T is analogous to the roots of a function from calculus. For instance 3 is a root of the polynomial $f(x) = (x-3)(x+3)$ because $f(x) = 0$. We define the Kernel of T to be the set of "roots" of T.

$$\text{Kernel}(T) = \{v \in V : T(v) = 0\}.$$

The Range of T is analagous to the range of a function from calculus. For instance the range of $f(x) = x^2$ is $[0, \infty)$ because for every value $y \in [0, \infty)$, we have $\sqrt{y}$ is in the domain of f, and $f(\sqrt{y}) = y$. We define the range of T in the same way. It is the set of all possible image points.

$$\text{Range}(T) = \{T(v) \in W : v \in V\}.$$

The Kernel is a subspace of V and the Range is a subspace of W.

example

Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the map $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix}$. T is a linear transformation. Geometrically, T is compressing every vector to the $y-$axis (it is an orthogonal projection onto the $y - axis$). Then $\text{Kernel}(T) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : T(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} 0 \\ y \end{pmatrix} = 0 \right\}$. We can easily see that the kernel is just the x axis, since $T\begin{pmatrix} x \\ y \end{pmatrix} = 0$ if and only if $y = 0$.

The range of T is the set of all image points. It is easy to see that this is the $y-$axis since if $\begin{pmatrix} 0 \\ y_0 \end{pmatrix}$ is a point on the $y - axis$ then $T\begin{pmatrix} x \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ y_0 \end{pmatrix}$ for any choice of x. In particular the point $\begin{pmatrix} 1 \\ y_0 \end{pmatrix}$ maps to $\begin{pmatrix} 0 \\ y_0 \end{pmatrix}$.

(3) *Isomoprhisms*

Consider the vector space $\mathbb{R}^4$ and the vector space of 2×2 matrices $M_2(\mathbb{R})$. They are essentially the same vector space. For every vector I have to specify 4 numbers. In the first I write them down as a column, and in the second I write them down in a grid. They aren't literally the same because they are different sets of objects, but they are "morally" the same. It shouldn't matter if we think of $\mathbb{R}^4$ in terms of columns or in terms of matrices. We would like some way to formally say that these are the "same" vector space, and we do that using linear transformations.

To make this example more specific, we have a correspondence

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and a correspondence of addition and scalar multiplication,

$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \\ d_1 + d_2 \end{pmatrix} \leftrightarrow \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix},$$

and

$$k \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} ka \\ kb \\ kc \\ kd \end{pmatrix} \leftrightarrow \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix} = k \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

All of this can be summed up by saying the map

$$T\colon \mathbb{R}^4 \to M_2(\mathbb{R}), \qquad T \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

is a bijective linear transformation (i.e. one-to-one and onto).

A linear transformation is called a (linear) isomorphism if it is bijective, and this captures what we mean by two vector spaces being the "same". In this case we say that the vector spaces are isomorphic.

Problems:

Problem 1

(1) Let $T\colon \mathbb{R}^4 \to P_2$ be $T \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = a - b + cx + dx^2$. Find the kernel of T and the range of T.

(2) For any linear transformation $T\colon V \to W$, show that T is an injective (one-to-one) map if and only if $\text{Kernel}(T) = \{0\}$.

(3) Is T from part (1) injective? surjective? an isomorphism?

Solution:

(1) If $T\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = a - b + cx + dx^2 = 0$ then we get the equations

$$a - b = 0, \quad c = 0, \quad d = 0,$$

so $a = b$, and we get

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} t \\ t \\ 0 \\ 0 \end{pmatrix}.$$

Hence the Kernel of T is span $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

(2) If T is injective, then whenever $T(v) = T(w)$ we have $v = w$. If $v \in \text{Kernel}(T)$, then $T(v) = 0 = T(0)$, so $v = 0$. This shows that $\text{Kernel}(T) = \{0\}$.

If $\text{Kernel}(T) = \{0\}$, then whenever $T(v) = T(w)$, we have $T(v) - T(w) = 0$, and since T is linear, $T(v - w) = 0$. Hence $v - w$ is in the kernel of T. Since the kernel of T consists of only the zero vector, we have $v - w = 0$, so $v = w$ and T is injective.

(3) T is not injective because the kernel is not 0. T is surjective because if $a_0 + a_1 x + a_2 x^2 \in P_2$, then $T\begin{pmatrix} a_0 \\ 0 \\ a_1 \\ a_2 \end{pmatrix} = a_0 + a_1 x + a_2 x^2$. T is not a bijection so it is not an isomorphism.

Problem 2

(1) Show that P_3, the vector space of polynomials of degree at most 3, is isomorphic to $\mathbb{R}^4$.

(2) Show that P_n is isomorphic to $\mathbb{R}^{n+1}$.

(3) (a little harder) Let $\mathbb{R}^\infty$ be the vector space of sequence $(a_1, a_2, \ldots)$ where only finitely many terms are non-zero (in other words, for each $(a_1, a_2, \ldots)$, there is some index N so that $a_i = 0$ for every $i > N$). Define addition and scalar multiplication component wise. Show that P, the vector space of polynomials (of any degree), is isomorphic to $\mathbb{R}^\infty$.

Solution:

(1) The isomorphism is given by

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \mapsto a + bx + cx^2 + dx^3.$$

(2) The isomorphism is given by the same idea,

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} \mapsto a_0 + a_1 x + \ldots + a_n x^n.$$

(3) The isomorphism here is the same basic idea. If $(a_0, a_1, \ldots, a_N, 0, 0, \ldots) \in \mathbb{R}^\infty$, then the map is

$$(a_0, a_1, \ldots, a_N, 0, 0, \ldots) \mapsto a_0 + a_1 x + \ldots + a_N x^N.$$

We gave the proof that (1) is a bijective linear transformation in tutorial. You should be able to prove the rest by looking at (1) as an example. The idea is the same in each part. As a reminder, the steps were as follows:

- Show that the function is a linear transformation. This ammounts to showing that $T(v_1 + v_2) = T(v_1) + T(v_2)$ for any $v_1, v_2 \in V$, and showing that $T(kv) = kT(v)$ for any $k \in \mathbb{R}$ and $v \in V$.

- Show that the function is one-to-one. As shown in question 1, this amounts to showing that the only vector in the kernel is the 0 vector. So starting with $v \in \text{Kernel}(T)$, show that v must be 0.

- Show that the function is onto. This amounts to showing that for every vector w in the target space, we can find some v in the domain, such that $T(v) = w$.