MAT2R3 TUTORIAL 9

Review:

(1) *Similarity*

Consider a linear operator $T\colon V \to V$ where V is a finite dimensional vector space. Let B and B' be two bases for V.

Last week we saw that how to represent T as a matrix. If we take the same basis for the domain and the target space, then we have the matrix $[T]_B^B$, and we write this as $[T]_B$.

What if we chose to represent T with B' instead of B? We would expect the matrices to be related somehow since they represent the same transformation, and indeed they are!

If I is the identity transformation, i.e. $I(v) = v$, then we can consider the matrix $[I]_B^{B'}$. This is called the transition matrix, or change of basis matrix. If we call this matrix P, then we have $P^{-1} = [I]_{B'}^B$ and

$$[T]_B = P^{-1}[T]_{B'}P.$$

If we write a vector v in terms of the basis B, then P transforms v in B coordinates to v in B' coordinates, then we apply the transformation T in B' coordinates, and then P^{-1} takes us back from B' to B coordinates.

So any matrix representing T is related by an invertible matrix in this way. This is the definition of similarity.

We say that two matrices A and B are similar if there exists an invertible matrix P such that

$$A = P^{-1}BP.$$

(2) *Orthogonal and Unitary matrices*

We call a square matrix A orthogonal if $A^{-1} = A^T$. There are a bunch of useful facts about orthogonal matrices

- A matrix is orthogonal if and only if its rows / columns are orthonormal vectors in $\mathbb{R}^n$.

- The inverse of an orthogonal matrix is orthogonal, the product of orthogonal matrices are orthogonal, and the determinant of an orthogonal matrix is ± 1.

- We have $\|Ax\| = \|x\|$ (so A doesn't stretch the vectors at all), and $Ax \cdot Ay = x \cdot y$.

- The change of basis matrix from one orthonormal basis to another orthonormal basis is an orthogonal matrix.

We say A is orthogonally diagonalizable if it is orthogonally similar to a diagonal matrix. This is equivalent to

- A is orthogonally diagonalizable,

- A has a set of orthonormal eigenvectors,

- A is symmetric.

The complex analog to orthogonal is unitary. The conjugate transpose of a complex matrix A is $A^* = \overline{A^T}$. A matrix is unitary if $A^* = A^{-1}$, and hermitian if $A^* = A$. All of the same facts above about orthogonal matrices apply to unitary matrices.

- A is unitary if and only if the columns / rows of A form an orthonormal basis of $\mathbb{C}^n$.

- The inverse of a unitary matrix is unitary, the product of unitary matrices are unitary, and the determinant of a unitary matrix has length 1.

- $\|Ax\| = \|x\|$ and $Ax \cdot Ay = x \cdot y$.

We call a matrix normal if $A^*A = AA^*$. Then we have the following equivalences

- A is unitarily diagonalizable,

- A has an orthonormal set of eigenvectors,

- A is normal.

(3) Example

Let $[T]_{B'} = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$ and $P = \begin{pmatrix} 4 & 5 \\ 1 & -1 \end{pmatrix}$ be the transition matrix from B to B'. Then we can calculate $[T]_B$ as

$$[T]_B = P^{-1}[T]_{B'}P.$$

We have $P^{-1} = \frac{-1}{9}\begin{pmatrix} -1 & -5 \\ -1 & 4 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 1 & 5 \\ 1 & -4 \end{pmatrix}$,

so

$$[T]_B = \frac{1}{9}\begin{pmatrix}1 & 5\\ 1 & -4\end{pmatrix}\begin{pmatrix}3 & 2\\ -1 & 1\end{pmatrix}\begin{pmatrix}4 & 5\\ 1 & -1\end{pmatrix} = \frac{1}{9}\begin{pmatrix}1 & 5\\ 1 & -4\end{pmatrix}\begin{pmatrix}14 & 13\\ -3 & -6\end{pmatrix} = \frac{1}{9}\begin{pmatrix}-1 & -17\\ 26 & 37\end{pmatrix}.$$

Problems:

Problem 1

Let $T\colon P_4 \to P_4$ be the linear operator $T(P(x)) = P(2x+1)$. Use convenient bases to find a matrix representation of T and use this to determine if T is one-to-one and onto.

Solution:

Let $B = \{1, x, x^2, x^3, x^4\}$ and $B' = \{1, 2x+1(2x+1)^2, (2x+1)^3, (2x+1)^4\}$. Then we have

$$[T]_B^{B'} = \left([T(1)]_{B'} \quad | \ldots [T(x^4)]_{B'}\right) = \begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{pmatrix},$$

which is both one-to-one and onto.

Problem 2

Let $A = \begin{pmatrix}\frac{4}{5} & 0 & \frac{-3}{5}\\ \frac{-9}{25} & \frac{4}{5} & \frac{-12}{25}\\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25}\end{pmatrix}$. A is an orthogonal matrix. Let $x = \begin{pmatrix}-2\\ 3\\ 5\end{pmatrix}$. Verify that $\|Ax\| = \|x\|$.

Solution:

We have $Ax = \begin{pmatrix}\frac{-23}{5}\\ \frac{18}{25}\\ \frac{101}{25}\end{pmatrix}$, so $\|Ax\|^2 = 38$, and $\|x\|^2 = 38$, hence $\|Ax\| = \|x\|$.

Problem 3

Under what conditions will a diagonal matrix be orthogonal?

Solution:

A diagonal matrix has columns $a_i e_i$ where $a_i \in \mathbb{R}$ and e_i is the i'th standard basis vector. The matrix is orthogonal if and only if its columns are orthonormal. The columns are already orthogonal, so we must have that the columns are normalized, i.e., $\|a_i e_i\| = |a_i| = 1$. So we have $a_i = \pm 1$.