A structure theorem for strongly abelian varieties with few models

Bradd Hart
University of California
at Berkeley

Matthew Valeriote*
McMaster University

By a variety, we mean a class of structures in some language $\mathcal{L}$ containing only function symbols which is equationally defined or equivalently is closed under homomorphisms, submodels and products.

If K is a class of $\mathcal{L}$-structures then $I(K, \lambda)$ denotes the number of non-isomorphic models in K of cardinality λ. To say that K has few models, we mean that $I(K, \lambda) < 2^\lambda$ for some $\lambda > |\mathcal{L}|$. If $I(K, \lambda) = 2^\lambda$ for all $\lambda > |\mathcal{L}|$ then we say K has many models. In [9, 10], Shelah has shown that for an elementary class K, having few models is a strong structural condition.

Before we give the definition of strongly abelian, let us motivate how it arises in this context. A variety $\mathcal{V}$ is locally finite if every finitely generated algebra in $\mathcal{V}$ is finite. If $\mathcal{A}$ and $\mathcal{B}$ are subvarieties of $\mathcal{V}$ then $\mathcal{V} = \mathcal{A} \otimes \mathcal{B}$ means that $\mathcal{V}$ is the variety generated by $\mathcal{A}$ and $\mathcal{B}$ and moreover there is a term $\tau(x, y)$ so that $\tau(x, y) = x$ holds in $\mathcal{A}$ and $\tau(x, y) = y$ holds in $\mathcal{B}$. $\mathcal{V}$ is called the varietal product of $\mathcal{A}$ and $\mathcal{B}$. As a consequence, if $\mathcal{V} = \mathcal{A} \otimes \mathcal{B}$ then for every $M \in \mathcal{V}$ there is a unique up to isomorphism $A \in \mathcal{A}$ and $B \in \mathcal{B}$ so that $M \cong A \times B$.

In [4], McKenzie and Valeriote proved the following theorem

Theorem 0.1 *If $\mathcal{V}$ is a locally finite decidable variety then there are three subvarieties of $\mathcal{V}$, $\mathcal{A}$, $\mathcal{S}$ and $\mathcal{D}$ so that $\mathcal{V} = \mathcal{A} \otimes \mathcal{S} \otimes \mathcal{D}$ and $\mathcal{A}$ is an affine variety, $\mathcal{S}$ is a strongly abelian variety and $\mathcal{D}$ is a discriminator variety.*

For the exact definitions of the terms affine and discriminator one can see [4], however for us here it is important to know that an affine variety is

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polynomially equivalent to a variety of left R-modules over some ring R and that any nontrivial discriminator variety contains an algebra whose complete theory is unstable.

The proof in [4] of Theorem 0.1 shows as well if $\mathcal{V}$ is locally finite and is not the varietal product of a strongly abelian variety, an affine variety and a discriminator variety then $\mathcal{V}$ has many models. Moreover, from [9], we know that the class of models of an unstable theory has many models so we conclude

Theorem 0.2 *If $\mathcal{V}$ is a locally finite variety with few models then there are two subvarieties of $\mathcal{V}$, $\mathcal{A}$ and $\mathcal{S}$, which are respectively affine and strongly abelian so that $\mathcal{V} = \mathcal{A} \otimes \mathcal{S}$.*

It is time now to give the definition of strongly abelian.

Definition 0.3 *An algebra $\mathcal{A}$ is said to be strongly abelian if for every term τ of $\mathcal{L}$ and tuples a, b, c, d and e from A so that length(a) = length(c) and length(b) = length(d) = length(e) then*

$$\text{if } \tau(a, b) = \tau(c, d) \text{ then } \tau(a, e) = \tau(c, e).$$

A variety is said to be strongly abelian if all its algebras are.

Saying that an algebra $\mathcal{A}$ is strongly abelian places very strong restrictions on the polynomials of $\mathcal{A}$. If $\tau(x, y)$ is a term in $\mathcal{L}$ then let τ_a be the polynomial $\tau(a, y)$. To say that $\mathcal{A}$ is strongly abelian means that for all terms τ and all a, b, if the ranges of the polynomials τ_a and τ_b intersect then $\tau_a = \tau_b$.

In [11], Valeriote gives an algebraic characterization of the locally finite decidable strongly abelian varieties. This characterization is also a description of such varieties which have few models. In [1], Baldwin and McKenzie give a reasonable description of the affine varieties which have few models. In some sense then the problem of giving an algebraic description of the locally finite varieties with few models has been "solved".

We have conjectured that if $\mathcal{V}$ is a variety with few models then there is an affine subvariety $\mathcal{A}$ and a strongly abelian subvariety $\mathcal{S}$ so that $\mathcal{V} = \mathcal{A} \otimes \mathcal{S}$. In the locally finite case, Valeriote's description of the decidable strongly abelian varieties was necessary to achieve the decomposition in Theorem 0.1. We decided to begin by trying to achieve a similar description in the general case. The main theorem in this paper is

Theorem 0.4 *If $\mathcal{V}$ is a strongly abelian variety with few models then it is equivalent to a multi-sorted unary variety.*

In section 1, we show how to convert a strongly abelian variety with few models into a special type of multi-sorted variety. The main difference here between the locally finite case and the case with few models is that local finiteness is replaced by superstability.

In section 2, we finish the proof of Theorem 0.4 and give a complete description of the strongly abelian varieties with few models.

In section 3, we consider the spectrum function for both locally finite varieties and strongly abelian varieties. In [6, 7], Palyutin, Starchenko and Saffe calculate the spectrum functions for any Horn theory. The approach taken is much different than the one taken here. Instead of obtaining an algebraic characterization for the class in question, they apply the techniques of Shelah's classification theory to obtain a characterization of the algebras in terms of "independent trees of subalgebras". Although they obtain a complete list of spectrum functions, it is difficult to determine which of these is possible when one imposes some algebraic condition like local finiteness.

The notation we use is, for the most part, standard. One convention not used everywhere is that when no confusion can arise or the meaning is clear from context, we often treat tuples like elements. For example, if we write $a \in A$ and don't say otherwise, a is a tuple all of whose elements belong to A. Similarly, unless it matters we don't explicitly mention the arity of functions or the length of tuples. If the length does matter (usually when a is a singleton) we will explicitly say so.

At the end of section 1 we deal with multi-sorted algebras. We include here a few words about them and some notation.

A multi-sorted language $\mathcal{L}$ is a collection of sorts $\{U_i : 1 \leq i \leq n\}$ and function symbols where the variables of the function symbols are assigned a particular sort. We assume that the syntax of $\mathcal{L}$ is understood.

A multi-sorted algebra $\mathcal{A}$ for the language $\mathcal{L}$ is an assignment of non-empty sets A_i for each sort U_i together with an interpretation for each function symbol consistent with the designated sorts for its variables. Without loss, we may assume that the sorts are disjoint. Throughout the paper, if $\mathcal{A}$ is a multi-sorted algebra then we use the notation A_i to stand for the underlying set of the i^{th} sort. Also if $a \in A$ (here a singleton) then we write $\text{sort}(a) = i$ to mean that $a \in A_i$. If τ is a term of $\mathcal{L}$, we also use $\text{sort}(\tau) = i$

to mean that the range of τ is the i^{th} sort.

If $\mathcal{A}$ is a multi-sorted algebra then a subuniverse of $\mathcal{A}$ is a subset of A which is closed under the functions. A subalgebra is a subuniverse which has non-empty intersection with every sort. We use the notation $\langle X \rangle$ to mean the subuniverse generated by the set X. A product of multi-sorted algebras in a language $\mathcal{L}$ is done sort by sort with the usual interpretation for the functions. The definition of homomorphism is standard.

A congruence on a multi-sorted algebra $\mathcal{A}$ is a collection of equivalence relations θ_i on A_i which respect the functions on $\mathcal{A}$. If $C_i \subseteq A_i \times A_i$ then $\mathrm{Cg}\langle C_1, \ldots, C_n \rangle$ is the least congruence on $\mathcal{A}$ which contains each C_i.

A variety for a multi-sorted language $\mathcal{L}$ is a class of multi-sorted algebras which is closed under homomorphisms, subalgebras and products. The theory of such a class is equational; that is, it is axiomatized by a collection of universally quantified atomic formulas of the form $\eta = \nu$ where η and ν are terms in $\mathcal{L}$ of the same sort.

We want to include some illustrative examples for the reader to refer to throughout the paper.

Example 0.5 The simplest example of a strongly abelian algebra is any algebra in a language $\mathcal{L}$ which contains only constant symbols and unary function symbols.

Example 0.6 Suppose $\mathcal{L} = \{\tau\}$, a single binary function symbol. Let $\mathcal{A}_i$ have universe $\{0, 1\}$ for $i = 0, 1$. Let $\tau(x_0, x_1) = x_i$ be the interpretation of τ in $\mathcal{A}_i$. Both $\mathcal{A}_0$ and $\mathcal{A}_1$ are strongly abelian since they are essentially unary however strong abelianness is preserved under products (and submodels) so $\mathcal{A} = \mathcal{A}_0 \times \mathcal{A}_1$ is also strongly abelian. $\mathcal{A}$ is not essentially unary but is built up from unary algebras. It is easy to see that the variety generated by $\mathcal{A}$ has few models.

Example 0.7 Let $\mathcal{A}$ and $\mathcal{L}$ be as in the previous example. Define an algebra $\mathcal{B}$ with the same universe as $\mathcal{A}$ together with a single new element, a. Let τ be defined the same as on $\mathcal{A}$ except $\tau(x, a) = \tau(x, (0, 0))$ and $\tau(a, x) = \tau((0, 0), x)$ for all $x \in A$ and $\tau(a, a) = (0, 0)$. Although this example is only a minor variant of the previous example, the variety generated by $\mathcal{B}$ has many models. In fact, $\mathcal{B}^\omega$ is unstable. To see this, let's make a couple of definitions.

Definition 0.8 *1. φ is said to be a product-factor formula if whenever $\mathcal{A}_i$ is an $\mathcal{L}$-structure for each $i \in I$ then*

$$\prod_{i \in I} \mathcal{A}_i \models \varphi(a) \textit{ iff } \mathcal{A}_i \models \varphi(a(i)) \textit{ for each } i \in I.$$

2. A formula $\varphi(x, y)$ is normal in $\mathcal{A}$ if whenever $\varphi(A, a) \cap \varphi(A, b) \neq \emptyset$ for $a, b \in A$ then $\varphi(A, a) = \varphi(A, b)$.

Proposition 0.9 *If $\varphi(x, y)$ is a product-factor formula which is not normal for $\mathcal{A}$ then $\mathcal{A}^\omega$ is unstable.*

Proof: Suppose we have $c, d \in A$ so that $\varphi(A, c) \cap \varphi(A, d) \neq \emptyset$ and $\varphi(A, c) \neq \varphi(A, d)$. Without loss, we may assume that there is $a, b \in A$ with

$$\mathcal{A} \models \varphi(a, c) \wedge \neg\varphi(a, d) \wedge \varphi(b, c) \wedge \varphi(b, d).$$

Let $e_n, f_n \in A^\omega$ be defined by $e_n(i)$ is a if $i \leq n$ and b otherwise and $f_n(i)$ is c if $i \leq n$ and d otherwise. Easily,

$$\mathcal{A}^\omega \models \varphi(e_m, f_n) \text{ iff } m \leq n.$$

This says φ has the order property in $\mathcal{A}^\omega$ so $\mathcal{A}^\omega$ is unstable. $\square$

Now let us return to the $\mathcal{B}$ of the example. Define the formulas

$$\begin{aligned} x \sim y &:=: \forall z(\tau(x, z) = \tau(y, z) \wedge \tau(z, x) = \tau(z, y)) \\ \theta(x, y, z) &:=: z \sim \tau(x, y) \\ \psi(x, y, z) &:=: z = \tau(x, y) \end{aligned}$$

In general, if θ and ψ are product-factor formulas then

$$\chi(x, y) :=: \exists z\theta \wedge \forall z(\theta \rightarrow \psi)$$

is as well. Consider χ for our particular θ and ψ. Noticing that $\mathcal{B} \models (0, 0) \sim a$ it is easy to see that

$$\mathcal{B} \models \forall x \chi(x, (0, 1))$$

but

$$\mathcal{B} \models \chi((1, 0), (0, 0)) \wedge \neg\chi((0, 0), (0, 0))$$

This says that χ is not normal and hence $\mathcal{B}^\omega$ is unstable.

Example 0.10 Let $\mathcal{L} = \{d_n : n \geq 2\}$ where d_n is an n-ary function symbol. Let $\mathcal{M} = \langle 2^\omega, d_n \rangle_{n \geq 2}$ where we define

$$d_n(x_0, \ldots, x_{n-1})(i) = \begin{cases} x_i(i) & \text{if } i < n-1 \\ x_{n-1}(i) & \text{if } i \geq n-1 \end{cases}$$

The variety generated by $\mathcal{M}$ has many models. To see why, see the comments after the statement of Theorem 1.7.

The next three examples give the different unary possibilities.

Example 0.11 Let $\mathcal{L} = \{f, g\}$ where f and g are both unary. The axioms for the variety are $f^2(x) = f(x)$ and $g^2(x) = g(x)$. This variety has many models (see section 2.2).

Example 0.12 Let $\mathcal{L} = \{f\}$ where f is a unary function symbol. The variety of all algebras in this language has many models (see section 2.3).

Example 0.13 Let $\alpha > 0$ be an ordinal. Let $\mathcal{L} = \{f_\beta : \beta < \alpha\}$ be a collection of unary function symbols. The axioms for the variety $\mathcal{V}_\alpha$ are

$$f_\beta f_\gamma(x) = f_{\max\{\beta,\gamma\}}(x) \text{ for } \beta, \gamma < \alpha.$$

It is not hard to show that

$$I(\mathcal{V}_\alpha, \aleph_\beta) = \begin{cases} \min\{2^{\aleph_\beta}, \beth_\alpha(|\beta + \omega|)\} & \text{if } \alpha < \omega \\ \min\{2^{\aleph_\beta}, \beth_{\alpha+2}(|\beta + \omega|)\} & \text{if } \alpha \geq \omega \end{cases}$$

so these examples are examples with few models.

1 Some introductory lemmas

The notion of a theory being unsuperstable has many equivalent definitions. We use the following here (note this definition doesn't demand that T is complete.)

Definition 1.1 *T is unsuperstable if for every κ there is $\lambda > \kappa$ and $M \models T$, $A \subseteq M$, $|A| \leq \lambda$ and $\{p_i : i < \mu\}$ where the p_i's are pairwise contradictory partial types over A consistent with $Th(M)$ and $\mu > \lambda$.*

M is said to be unsuperstable if $Th(M)$ is unsuperstable. A class, K, of structures is unsuperstable if it contains an unsuperstable model. Any one of T, M or K is said to be superstable if it is not unsuperstable.

The main consequence of unsuperstability we use is the following theorem which can be found in [9].

Theorem 1.2 *If T is unsuperstable then $I(T,\lambda) = 2^\lambda$ for all $\lambda > |T|$.*

The following Theorem is analogous to the case of modules where if $\mathcal{M}$ is a module which is not ω-stable then $\mathcal{M}^\omega$ is not superstable. (See for example [8].)

Theorem 1.3 *Suppose $\mathcal{A}$ is an $\mathcal{L}$-structure and for every $i \in \omega$,*

1. *$E_i(x,y)$ is a product formula and $length(x) = length(y)$,*
2. *E_i is an equivalence relation on A and*
3. *E_{i+1} properly refines E_i on A*

then $\mathcal{A}^\omega$ is unsuperstable.

Remark: Note that the fact that E_{i+1} refines E_i implies that there is an $n > 0$ so that E_j is an equivalence relation on A^n for all j.

Proof: Suppose $a_0^i, a_1^i \in A$ so that

$$\mathcal{A} \models E_i(a_0^i, a_1^i) \wedge \neg E_{i+1}(a_0^i, a_1^i).$$

We'll prove that $\mathcal{A}^{\omega\times\omega}$ is unsuperstable.

For $f : \omega \times \omega \to 2$ define $b_f \in A^{\omega\times\omega}$ by

$$b_f(m,n) = a^n_{f(m,n)}.$$

It is not hard to show

$$\mathcal{A}^{\omega\times\omega} \models E_i(b_f, b_g) \text{ iff } f\restriction_{\omega\times i} = g\restriction_{\omega\times i}.$$

Let λ be a cardinal so that $\lambda^\omega > \lambda$. Choose constants c_η for each $\eta \in \lambda^{<\omega}$ and consider the set

$$Th(\mathcal{A}^{\omega\times\omega}) \cup \{E_i(c_\eta, c_\mu) : \eta\restriction_i = \mu\restriction_i, i < \omega\} \cup \{\neg E_i(c_\eta, c_\mu) : \eta\restriction_i \neq \mu\restriction_i, i < \omega\}.$$

Using the b_f's, we see that this set is consistent. If $\mathcal{B}$ is a model of this set then consider the types p_η for $\eta \in \lambda^\omega$ over the set $\{c_\nu : \nu \in \lambda^{<\omega}\} \subseteq B$ where

$$p_\eta = \{E_i(x, c_{\eta_i}) : i < \omega\}.$$

These types are pairwise contradictory and this suffices to show that $Th(\mathcal{A}^{\omega\times\omega})$ is unsuperstable. $\square$

From now on assume that $\mathcal{V}$ is multi-sorted strongly abelian and T is the theory of $\mathcal{V}$.

Definition 1.4 *1. A term $\tau(x, \bar{z})$ depends on x in the variety $\mathcal{V}$ if the equation*

$$\tau(x, \bar{z}) = \tau(y, \bar{z})$$

does not hold in $\mathcal{V}$.

2. A term $d(x_1, \ldots, x_n, \bar{y})$ where $x_1, \ldots, x_n$ are all singletons of the same sort is said to be diagonal if it depends exactly on the variables $x_1, \ldots, x_n$ and

$$T \models d(x, x, \ldots, x, \bar{y}) = x.$$

We usually suppress the mention of the $\bar{y}$.

Comment: Note that if $\tau(x_1, \ldots, x_m)$ is a term so that

$$T \models \tau(x, \ldots, x) = x$$

then τ is a diagonal term in the variables that it depends on.

Lemma 1.5 *1. If d is diagonal then*

$$T \models d(d(x_1^1, \ldots, x_n^1), \ldots, d(x_1^n, \ldots, x_n^n)) = d(x_1^1, \ldots, x_n^n).$$

2. If $d(x, y, \bar{z})$ is diagonal then $d(x, x, \bar{z})$ is diagonal.

Proof:

1. Since d is diagonal, we have

$$T \models d(d(y_1, \ldots, y_n), \ldots, d(y_1, \ldots, y_n)) = d(y_1, \ldots, y_n).$$

Since $\mathcal{V}$ is strongly abelian, we have

$$T \models d(\bar{u},\ \underset{\stackrel{\uparrow}{i}}{d(y_1, \ldots, y_n)}\ , \bar{v}) = d(\bar{u}, y_i, \bar{v})$$

for any $\bar{u}$ and $\bar{v}$. Hence the following equations hold in any $\mathcal{A} \in \mathcal{V}$:

$$
\begin{aligned}
d(d(x_1^1, \ldots, x_n^1), \ldots, d(x_1^n, \ldots, x_n^n)) &= d(x_1^1, \ldots, d(x_1^n, \ldots, x_n^n)) \\
&= d(x_1^1, x_2^2, \ldots, d(x_1^n, \ldots, x_n^n)) \\
&= \ldots \\
&= d(x_1^1, \ldots, x_n^n).
\end{aligned}
$$

2. Easy using the strongly abelian property. □

Now suppose $\mathcal{V}$ is a multi-sorted strongly abelian variety and $d(x_1, \ldots, x_k)$ is a diagonal term in $\mathcal{V}$. We will define another multi-sorted strongly abelian variety $\mathcal{V}[d]$ as follows:

First, if $\mathcal{L}$ is the language of $\mathcal{V}$ then we define $\mathcal{L}[d]$, the language of $\mathcal{V}[d]$. Suppose $U_1, \ldots, U_n$ are the sorts of $\mathcal{L}$ and U_1 is the sort of the x_i's. $\mathcal{L}[d]$ will have $n + k - 1$ sorts $V_1, \ldots, V_k, U_2, \ldots, U_n$.

Now if $f \in \mathcal{L}$ is m-ary then without loss, assume that the variables of f, $y_1, \ldots, y_m$ are arranged so that the sort of y_i is U_1 iff $i \leq j$ for some fixed $j \geq 0$. There are now two cases.

Case 1: If the sort of f is not U_1 then let $\bar{f} \in \mathcal{L}[d]$ where $\bar{f}$ has $jk+m-j$ arguments written

$$\bar{f}(y_1^1, \ldots, y_k^1, \ldots, y_1^j, \ldots, y_k^j, y_{j+1}, \ldots, y_m).$$

The sort of $\bar{f}$ is the same as f, the sort of y_l^i is V_l and the sort of y_l for $l > j$ is the same.

Case 2: If the sort of f is U_1 then there are k new symbols $f_1, \ldots, f_k \in \mathcal{L}[d]$. The sort of f_i is V_i and f_i has $jk + m - j$ arguments exactly as above.

Now if $\mathcal{A} \in \mathcal{V}$ then we define $\mathcal{A}[d]$, an algebra in the language $\mathcal{L}[d]$ as follows:

Let $\sim_i$ be the equivalence relation defined on $U_1(A)$ by $a \sim_i b$ for $a, b \in A$ iff for all $\bar{x}, \bar{y} \in A$

$$d(\bar{x}, \underset{\substack{\uparrow \\ i}}{a}, \bar{y}) = d(\bar{x}, \underset{\substack{\uparrow \\ i}}{b}, \bar{y}).$$

Let $V_i(\mathcal{A}[d]) = U_1(A)/{\sim_i}$ and $U_i(\mathcal{A}[d]) = U_i(A)$ for $i > 1$.

Now if $g \in \mathcal{L}[d]$ then there are two possibilities. First, $g = \bar{f}$ for some $f \in \mathcal{L}$ where the sort of f is not U_1. In this case, interpret g as

$$g(a^1_1/{\sim_1},\ldots,a^1_k/{\sim_k},\ldots,a^j_1/{\sim_1},\ldots,a^j_k/{\sim_k},a_{j+1},\ldots,a_m) = \\ f(d(a^1_1,\ldots,a^1_k),\ldots,d(a^j_1,\ldots,a^j_k),a_{j+1},\ldots,a_m).$$

Second, $g = f_i$ for some $f \in \mathcal{L}$ where the sort of f is U_1. In this case

$$g(a^1_1/{\sim_1},\ldots,a^1_k/{\sim_k},\ldots,a^j_1/{\sim_1},\ldots,a^j_k/{\sim_k},a_{j+1},\ldots,a_m) = \\ f(d(a^1_1,\ldots,a^1_k),\ldots,d(a^j_1,\ldots,a^j_k),a_{j+1},\ldots,a_m)/{\sim_i}\ .$$

If we let $\mathcal{V}[d]$ be the closure of the class containing $\mathcal{A}[d]$ for all $\mathcal{A} \in \mathcal{V}$ under isomorphism then $\mathcal{V}[d]$ is a multi-sorted strongly abelian variety. Moreover, $\mathcal{V}[d]$ is bi-interpretable with $\mathcal{V}$ via d and the $\sim_i$'s.

To be more precise, if $\mathcal{A} \in \mathcal{V}$ then if $a \in U_i(A)$ and $i > 1$ then a is interpreted as itself in $\mathcal{A}[d]$. If $a \in U_1(A)$ then a is interpreted as the tuple $\langle a/{\sim_1},\ldots,a/{\sim_k}\rangle$.

Consider example 0.6. There τ is a binary diagonal term. If one constructs $\mathcal{A}[\tau]$ from $\mathcal{A}$ it is easy to see that each sort is essentially one of the algebras $\mathcal{A}_i$ from which $\mathcal{A}$ was built.

Definition 1.6 *A term $s(x_1,\ldots,x_s)$ is called essentially unary if for some $i \le s$ and some variables $y_1,\ldots,y_s$ distinct and different from the x's, the equation*

$$s(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_s) = s(y_1,\ldots,y_{i-1},x_i,y_{i+1},\ldots,y_s)$$

holds in $\mathcal{V}$.

Theorem 1.7 *If $\mathcal{V}$ is a strongly abelian variety with few models then it is bi-interpretable with a multi-sorted strongly abelian variety $\mathcal{V}'$ so that the only diagonal terms in $\mathcal{V}'$ are essentially unary.*

Comment: To see how this could fail, consider example 0.10. Here each d_n is a diagonal term but none is "maximal". That is, if you use d_n to form $\mathcal{V}[d_n]$ then d_{n+1} will be converted into a diagonal term which is not unary in $\mathcal{V}[d_n]$. By considering the equivalence relations E_n formed by the kernels of the terms d_n in the first $n-1$ variables, it is easy to see that they satisfy Theorem 1.3 and so $\mathcal{V}$ is not superstable. The proof essentially follows this example.

Proof: We define a sequence of multi-sorted strongly abelian varieties $\mathcal{V}_i$ and binary diagonal terms d_i so that $\mathcal{V}_{i+1} = \mathcal{V}_i[d_i]$. The sorts will be indexed by finite sequences of 0's and 1's. We think of $\mathcal{V}$ as one-sorted with sort $U_{\langle\rangle}$.

Let $\mathcal{V}_0 = \mathcal{V}$. Suppose we have defined $\mathcal{V}_i$ for all $i \leq n$ and d_i for all $i < n$. There are now two cases.

Case 1: There is no diagonal term. In this case, let $\mathcal{V}' = \mathcal{V}_n$ and we are done.

Case 2: There is a diagonal term d for $\mathcal{V}_n$ which is not essentially unary. By lemma 1.5, we may assume d is binary. Suppose that the sort of d is U_η. Let $\mathcal{V}_{n+1} = \mathcal{V}_n[d]$ and let the sorts of $\mathcal{V}_{n+1}$ be the same as those of $\mathcal{V}_n$ except that U_η is replaced by $U_{\eta 0}$ and $U_{\eta 1}$. We label d by both d_n and d_η.

If this process never stops then by König's lemma there is $\eta \in 2^\omega$ so that $U_{\eta n}$ is defined for all $n \in \omega$.

Without loss of generality, we may assume η is the identically 1 sequence. By omitting steps in the construction, we can assume that $d_n = d_{\eta n}$ for all n.

Now suppose $\mathcal{F}$ is the free algebra on countably many generators in $\mathcal{V}$. If $x \in F$ then by tracing the interpretation of $\mathcal{V}$ to $\mathcal{V}_n$, we see x is interpreted as a tuple $\langle x_0^n, \ldots, x_n^n \rangle$. Moreover, by our assumptions about d_n, $x_i^l = x_i^m$ if $i < l, m$.

Define $E_n(x, y)$ iff $x_i^n = y_i^n$ for all $i < n$. By our observation, E_{n+1} refines E_n and this refinement is proper since d_n depends on both of its variables. One can show that E_n is definable in the language of $\mathcal{V}$ by a product formula. As a consequence then $\mathcal{V}$ has many models by Theorem 1.3 which contradicts the assumption that $\mathcal{V}$ has few models. $\square$

Corollary 1.8 *If $\mathcal{V}$ is a strongly abelian variety in a countable language and $I(\mathcal{V}, \aleph_0) < 2^{\aleph_0}$ then $\mathcal{V}$ is bi-interpretable with a multi-sorted strongly abelian variety with only essentially unary diagonal terms.*

Proof: This is more a corollary to the proof of Theorem 1.7. Follow the steps in the proof until we have constructed $\mathcal{V}_n$ for all $n \in \omega$. Suppose that $\mathcal{F}$ is the free algebra on the two generators a and b.

For $x \in F$, let $\langle x_0^n, \ldots, x_n^n \rangle$ be the interpretation of x traced to $\mathcal{V}_n$. Define the equivalence relations $S_n(x, y)$ iff $x_n^{n+1} = y_n^{n+1}$. These are definable in the language of $\mathcal{V}$.

Using the fact that a and b are free generators and the d_n's are diagonal terms, it is not hard to show that for every $U \subseteq \omega$,

$$p_U(x) = \{S_n(x, a) : n \in U\} \cup \{S_n(x, b) : n \in \omega \setminus U\}$$

is a consistent set of partial types over the parameters $\{a, b\}$. Hence there are 2^ω consistent types over a finite subset of $\mathcal{F}$ so $I(\mathcal{V}, \aleph_0) = 2^{\aleph_0}$ which is a contradiction. $\square$

2 The main construction

In this section we assume that $\mathcal{V}$ is a multi-sorted strongly abelian variety with k sorts which satisfies the condition:

* if for some term $\tau(x_1, x_2, \bar{y})$ the equation $\tau(x, x, \bar{y}) = x$ holds then τ does not depend on one of x_1 or x_2.

The main results of this section are patterned after similar results in [4] (and [11]) where the same conclusions are reached assuming $\mathcal{V}$ is a strongly abelian locally finite decidable variety. The subsection titles for this section indicate what is proved about $\mathcal{V}$ in that section.

2.1 $\mathcal{V}$ is unary

Definition 2.1 *1. If $t(x_1, \dots x_n)$ is a term we will say that t is left invertible at the variable x_j if there is a term $s(y_1, \dots, y_m)$ such that the variables $y_1, \dots, y_m$ are distinct and different from the x's, and*

$$s(t(\bar{x}), y_2, \dots, y_m) = x_j$$

holds in $\mathcal{V}$.

2. A term $s(x_1, \dots, x_n)$ will be called right invertible if there are terms $s_j(y, \bar{z})$ such that

$$s(s_1(y, \bar{z}), \dots, s_n(y, \bar{z})) = y$$

holds in $\mathcal{V}$.

Lemma 2.2 *If $\mathcal{A}$ is a strongly abelian algebra and a polynomial $\tau(x, a)$ is onto for some $a \in A$ then $\tau(x, y)$ depends only on x in $\mathcal{A}$.*

Proof: Since $\tau(x, a)$ is onto, the range of $\tau(x, a)$ and $\tau(x, b)$ intersect for any $b \in A$. By the comment after the definition of strongly abelian, this says that $\tau(x, a) = \tau(x, b)$ for all x and hence $\tau(x, y)$ does not depend on y. $\square$

Lemma 2.3 *If s is a right invertible term then it is essentially unary.*

Proof: Suppose there are terms $s_j(y, \bar{z})$ so that

$$s(s_1(y, \bar{z}), \ldots, s_n(y, \bar{z})) = y.$$

Then the term

$$s(s_1(y_1, \bar{z}), \ldots, s_n(y_n, \bar{z}))$$

must be essentially unary by condition $*$. Suppose that it depends on y_1. Then by lemma 2.2, since $s(x, s_2(y_2, \bar{z}), \ldots, s_n(y_n, \bar{z}))$ is onto, it follows that s is essentially unary. □

Now suppose that $\mathcal{V}$ is not essentially unary.

Lemma 2.4 *There is a term $q(x_1, \ldots, x_n)$ so that q depends on x_1 and x_2 and q is not left invertible at x_1 or x_2.*

Proof: We begin by choosing an arbitrary term $t(x_1, x_2, \ldots, x_n)$ that depends on x_1 and x_2. Suppose that t is left invertible at x_1. We can choose a term $s(y, \bar{z})$ such that

$$s(t(\bar{x}), \bar{z}) = x_1.$$

Since s is right invertible, s depends only upon y. Let

$$t'(y_1, x_2, \ldots, x_n, \bar{z}) = t(s(y_1, \bar{z}), x_2, \ldots, x_n).$$

Then since t depends on x_1 and x_2 it is not hard to conclude that t' depends on y_1 and x_2.

We now show that the term t' can't be left invertible at y_1. Consider the term

$$p(x, y, z, \bar{w}) = t'(t'(x, y, \bar{w}), z, \bar{w}).$$

It is not hard to show that if t' is left invertible at y_1 then the term p must depend on the variable y. But by unraveling the definition of p and t' it is possible to show that p is in fact independent of y.

We can use the same argument on the second variable of t' to finally arrive at a term that depends on x_1 and x_2 and is left invertible at neither. □

Definition 2.5 *For each i, define T_i to be the set of terms $t(y_1, \ldots, y_n)$ so that sort$(y_1) = i$, and t is not left invertible at y_1. If $\mathcal{A} \in \mathcal{V}$ and $a, b \in A_i$ then*

$$a \sim_i b \text{ iff for every } t \in T_i \text{ and } \bar{u} \in A,\ t(a, \bar{u}) = t(b, \bar{u}).$$

If $a \sim_i b$ then we say that a is an analogue of b.

Lemma 2.6 *Let $\mathcal{B}$, $\mathcal{D}_j \in \mathcal{V}$ for $j \in I$.*

1. *For each i, $\sim_i$ defines an equivalence relation on B_i.*
2. *If $s(x)$ is right invertible, sort$(x) = i$, sort$(s) = j$ and $a, b \in B_i$ then if $a \sim_i b$ then $s(a) \sim_j s(b)$.*
3. *$\mu \sim_i \nu$ in $\prod_{j \in I} \mathcal{D}_j$ if and only if $\mu(j) \sim_i \nu(j)$ in $\mathcal{D}_j$, for each $j \in I$.*

Proof: 1 and 3 are immediate from the definition.

To prove 2, suppose that $a, b \in B_i$ and $a \sim_i b$. If $t(y_1, \ldots, y_n) \in T_j$ then it follows that $t(s(a), \bar{u}) = t(s(b), \bar{u})$ for all $\bar{u}$ of the appropriate sort, since the term $t(s(x_1), \ldots, y_n)$ is in T_i. This shows that $s(a) \sim_j s(b)$. □

When it is clear from context which sort we mean we will often leave the subscript off of $\sim$.

We introduce a construction with free algebras in $\mathcal{V}$ that will be used in the upcoming many models proof. Let $\mathcal{F}$ be the free algebra in $\mathcal{V}$ with generators $\overline{X} = \langle X_1, \ldots, X_k \rangle$ where X_i is a set of generators of sort i. Choose some new generator z and let $\overline{X'} = \langle {X_1}', X_2, \ldots, X_k \rangle$ where ${X_1}' = X_1 \cup \{z\}$, and let $\mathcal{F}'$ be the free algebra in $\mathcal{V}$ with generators $\overline{X'}$. Choose an element 0 in the first sort of $\mathcal{F}$. The following claim is the technical heart of Theorem 2.8.

Claim 2.7 *There is a congruence θ on $\mathcal{F}'$ so that*

1. *$h : \mathcal{F} \to \mathcal{F}'/\theta$ defined by $h(a) = a/\theta$ is an embedding,*
2. *$z/\theta = \{z\}$ and*
3. *z/θ is an analogue of $0/\theta$.*

Proof: For $1 \leq i \leq k$, we define

$$\begin{aligned} C_i &= \{\langle t(0,\bar{u}), t(z,\bar{u})\rangle : t(x,\bar{y}) \text{ is a term, sort}(t) = i, \\ &\quad \bar{u} \in F \text{ and } t(x,\bar{y}) \text{ is not left invertible at } x\}. \end{aligned}$$

Then we define a congruence on $\mathcal{F}'$:

$$\theta = \mathrm{Cg}(\langle C_1, \ldots, C_k\rangle).$$

Clearly this is the minimal congruence necessary for z/θ to be an analogue of $0/\theta$.

To prove 1, it suffices to show that for $a \in F$, if $\mathrm{sort}(a) = i$ then $a/\theta = \{b \in F' : \langle a, b\rangle \in C_i\}$. To see this, note that if $b \in F$ is in a/θ then $b = \tau(z, \bar{u})$ for some τ where $a = \tau(0, \bar{u})$. But $\mathcal{F}'$ is free with z as a free generator so we obtain $b = \tau(0, \bar{u})$ from the first equation which gives $a = b$.

So suppose we have $a \in F$ with $\mathrm{sort}(a) = i$. Let

$$S = \{b \in F' : \langle a, b\rangle \in C_i\}.$$

Clearly, we have $S \subseteq a/\theta$, and using the term τ defined by $\tau(x, y) = y$, we get $a \in S$. To prove that $a/\theta = S$, it will suffice to prove that if $\langle u, v\rangle \in C_j$ for some $1 \leq j \leq k$ and p is a unary polynomial of $\mathcal{F}'$ with $p(u)$ or $p(v)$ in S, then $\{p(u), p(v)\} \subseteq S$.

There are two cases to consider, $p(u) \in S$ or $p(v) \in S$. Suppose that the former holds. Choose terms $t(x, \bar{y})$ and $g(x', y, \bar{y}')$, and elements $\bar{b}, \bar{c}$ in F such that

$$p(r) = g(r, z, \bar{b}) \text{ for all } r \text{ and}$$

$$\langle u, v\rangle = \langle t(0, \bar{c}), t(z, \bar{c})\rangle$$

where $t(x, \bar{y})$ is not left invertible at x. Since $p(u) \in S$, we can find a term $t'(x, \bar{y}'')$, not left invertible at x, and elements $\bar{d}$ in F, such that

$$\langle t'(0, \bar{d}), t'(z, \bar{d})\rangle = \langle a, p(u)\rangle.$$

Thus we have

$$g(t(0, \bar{c}), z, \bar{b}) = t'(z, \bar{d})$$

in $\mathcal{F}'$. Now since $\mathcal{F}'$ is free and $0, \bar{b}, \bar{c}, \bar{d}$ are in F, then we have

$$g(t(0, \bar{c}), q, \bar{b}) = t'(q, \bar{d})$$

in $\mathcal{F}'$, for any q. Thus $g(t(0,\bar{c}),0,\bar{b}) = t'(0,\bar{d}) = a$.

Now it follows from the above observations, and the strongly abelian property, that since t' is not left invertible at x, then the term

$$s(x_1, x_2, \bar{y}, \bar{y}') = g(t(x_1, \bar{y}), x_2, \bar{y}')$$

is not left invertible at x_2. Also, since $t(x,\bar{y})$ is not left invertible at x, s is not left invertible at x_1. Using condition $*$, it follows that the term $s'(x,\bar{y},\bar{y}') = s(x,x,\bar{y},\bar{y}')$ is not left invertible at x.

Therefore $\langle s'(0,\bar{c},\bar{b}), s'(z,\bar{c},\bar{b})\rangle \in C_i$. But we have

$$s'(0,\bar{c},\bar{b}) = t'(0,\bar{d}) = a \text{ and}$$

$$s'(z,\bar{c},\bar{b}) = g(t(z,\bar{c}),z,\bar{b}) = p(v).$$

The proof for the remaining case, when $p(v) \in S$, is similar. This shows that $a/\theta = S$.

To prove that $z/\theta = \{z\}$, we must show that if $\langle u,v\rangle \in C_j$ for some $1 \leq j \leq k$, and if p is a unary polynomial of $\mathcal{F}'$, then $p(u) = z$ if and only if $p(v) = z$. As above, we choose terms $t(x,\bar{y})$, $g(x',y,\bar{y})$ and elements $\bar{b},\bar{c}$ in F such that $p(r) = g(r,z,\bar{b})$ for all r, and $\langle u,v\rangle = \langle t(0,\bar{c}), t(z,\bar{c})\rangle$.

Now if $p(u) = z$, then $g(t(0,\bar{c}),z,\bar{b}) = z$. From this, using the strongly abelian property, we see that $g(t(x_1,\bar{y}),x_2,\bar{y}') = x_2$ in $\mathcal{F}'$ for all values of the variables (of the appropriate sorts). Thus

$$z = g(t(0,\bar{c}),z,\bar{b}) = g(t(z,\bar{c}),z,\bar{b}) = p(v).$$

On the other hand, if $p(v) = z$, then $g(t(z,\bar{c}),z,\bar{b}) = z$, and so we can conclude that $g(t(x,\bar{y}),x,\bar{y}') = x$ in $\mathcal{F}'$. By condition $*$ we have that the term $g(t(x_1,\bar{y}),x_2,\bar{y}')$ depends only on one of the two variables x_1 and x_2. Since t is not left invertible, it follows that the equation $g(t(x_1,\bar{y}),x_2,\bar{y}') = x_2$ is valid in $\mathcal{V}$. Thus

$$z = g(t(z,\bar{c}),z,\bar{b}) = g(t(0,\bar{c}),z,\bar{b}) = p(u),$$

which completes the proof. □

In example 0.7, a is an analogue of $(0,0)$. The only diagonal term is the identity function. We saw that the variety generated by this example is unstable. Unfortunately, we don't know whether this holds in the general situation. Precisely, we don't know that if $\mathcal{V}$ is a superstable, strongly abelian variety then it is bi-interpretable with a multi-sorted unary variety. The proof given below directly constructs many models. We are now ready to prove the main theorem of this section.

Theorem 2.8 *If $\mathcal{V}$ is not essentially unary then $I(\mathcal{V}, \lambda) = 2^\lambda$ for all $\lambda \geq |\mathcal{L}|$.*

Proof: A bipartite graph $G = \langle V, E \rangle$ is a set V with a symmetric binary relation E which can be partitioned into two sets V_1 and V_2 so that if $u, v \in V$ and $E(u, v)$ then exactly one of u and v is in V_1 and one is in V_2. We will construct, for every bipartite graph G, an algebra $\mathcal{B}^G$ with certain properties. At the end of the proof we will say why this is enough to prove the theorem.

To achieve this, we use the term $q(x_1, x_2, \ldots, x_n)$ from lemma 2.4. Let $\mathcal{F}$ be the free algebra on a, a', b, b' and a set X where a and a' have the same sort as x_1, b and b' have the same sort as x_2 and X contains exactly one other generator for each sort. Let $x \bullet y$ denote the $\mathcal{F}$-polynomial $q(x, y, \bar{u})$, where each element of $\bar{u}$ comes from X and is of the appropriate sort. Let $0 = a \bullet b$, $1 = a \bullet b'$ and $2 = a' \bullet b$. We may assume $\mathrm{sort}(q) = 1$. Choose a new element z of sort 1 and let $\mathcal{F}'$ be the free algebra on $X \cup \{z\}$. By the previous claim, there is a congruence θ on $\mathcal{F}'$ with the properties listed there. Let $\mathcal{C} = \mathcal{F}'/\theta$ and identify $\mathcal{F}$ with its image in $\mathcal{C}$ and z with z/θ.

The following is a critical observation.

Claim 2.9 $0 \sim z$, $0 \not\sim 1$ *and* $0 \not\sim 2$ *in* $\mathcal{C}$.

Proof: $0 \sim z$ follows immediately from Claim 2.7. We handle the case $0 \not\sim 1$. The case of $0 \not\sim 2$ is similar.

Suppose $0 \sim 1$ in $\mathcal{C}$. We must have $0 \sim 1$ in $\mathcal{F}$ as well. Let $\alpha = \mathrm{Cg}(\langle 0, 1 \rangle)$. Let $3 = a' \bullet b'$. We claim that $3/\alpha = \{3\}$. To see this we must show that if $p(x) = g(x, \bar{u})$ is a unary polynomial with $\bar{u} \in F$ then $p(0) = 3$ iff $p(1) = 3$.

If $g(a \bullet b, \bar{u}) = a' \bullet b'$ then since $\mathcal{F}$ is free, a, b, a', b' are generators and $\mathcal{F}$ is strongly abelian, we would immediately conclude that $g(a' \bullet b, \bar{u}) = a' \bullet b'$; i. e. p(1) = 3.

If $g(a \bullet b', \bar{u}) = a' \bullet b'$ then there are two cases. If g is not left invertible at x then by assumption, $p(0) = p(1)$ so there is nothing to prove. Otherwise, $p(x)$ is one-to-one. But then, since $\mathcal{F}$ is free, a, a', b are free generators and $\mathcal{F}$ is strongly abelian, $g(a' \bullet b', \bar{u}) = a' \bullet b'$; i. e. p(3) = p(1) which gives

$a \bullet b' = a' \bullet b'$. This says q does not depend on its first variable which is a contradiction.

So we have shown that $3/\alpha = \{3\}$. But then, in $\mathcal{F}/\alpha$, $a/\alpha \bullet b/\alpha = a/\alpha \bullet b'/\alpha$ so since $\mathcal{F}/\alpha$ is strongly abelian, $a'/\alpha \bullet b/\alpha = a'/\alpha \bullet b'/\alpha$; i. e. $a' \bullet b = a' \bullet b'$ which says that q does not depend on its second variable. Contradiction. We conclude then that $0 \not\sim 1$. $\square$

Suppose that $G = \langle V, E \rangle$ is a bipartite graph and V_1, V_2 is a partition of V so that $E \subseteq V_1 \times V_2 \cup V_2 \times V_1$. Choose p_1 and p_2 distinct and not in V and let $Y = V \cup \{p_1, p_2\}$. Let $\widehat{E} = \{\{u, v\} : (u, v) \in E\}$.

For $v \in V_1$, let $f_v : Y \to C$ be defined by

$$f_v(x) = \begin{cases} a' & \text{if } x = v \\ a & \text{otherwise} \end{cases}$$

For $v \in V_2$, let $f_v : Y \to C$ be defined by

$$f_v(x) = \begin{cases} b' & \text{if } x = v \\ b & \text{otherwise} \end{cases}$$

For $i \in \{1, 2\}$, $v \in V_1$, $w \in V_2$ and $e = \{v, w\} \in \widehat{E}$ let $f_e^i : Y \to C$ be defined by

$$f_e^i(x) = \begin{cases} 2 & \text{if } x = v \\ 1 & \text{if } x = w \\ z & \text{if } x = p_i \\ 0 & \text{otherwise} \end{cases}$$

Let

$$V_1^* = \{f_v : v \in V_1\}, V_2^* = \{f_v : v \in V_2\} \text{ and}$$
$$E^* = \{f_e^i : e \in \widehat{E} \text{ and } i \in \{1, 2\}\}.$$

We define $\mathcal{B}^G \subseteq \mathcal{C}^Y$ as follows: $\mathcal{B}^G$ is the subalgebra of $\mathcal{C}^Y$ generated by the set

$$V_1^* \cup V_2^* \cup E^* \cup \{\hat{x} : x \in X\}$$

where for $d \in C$, $\hat{d}$ denotes the constant valued function with value d. We call elements of this generating set the generators.

We now want to recover G up to isomorphism from $\mathcal{B}^G$. Let

$$R = \{b \in B^G : b \text{ is only in the range of right invertible terms}\}.$$

Claim 2.10 *R contains the generators.*

Proof: Suppose τ is some term and $\tau(a_1, \ldots, a_n) = f_u$ for some $u \in V_1$ where $a_i \in B^G$ for all i. Hence there are terms μ_i and tuples of generators $\bar{f}_i$ so that $\mu_i(\bar{f}_i) = a_i$. We want to show that τ is right invertible. Evaluating at $u \in Y$, we get

$$\tau(\mu_1(\bar{f}_1(u)), \ldots, \mu_n(\bar{f}_n(u))) = a'.$$

Since $\mathcal{F}$ is free and a' is one of the generators, we conclude that τ is right invertible. A similar proof works for f_v with $v \in V_2$ and $\hat{x}$ for $x \in X$.

Suppose $\tau(a_1, \ldots, a_n) = f_e^i$ where $e \in \widehat{E}$, $i \in \{1, 2\}$ and the other notation is as above. Consider the value of these terms at the co-ordinate p_i. We have $\tau(a_1(p_i), \ldots, a_n(p_i)) = z$. Since the θ-class of z contains only z, we have this equality in a free algebra as well where z is a generator. This is enough to conclude that τ is right invertible. $\square$

Now define a quasi-order, $\leq$, on R as follows: $c \leq d$ for $c, d \in R$ if there is an essentially unary term τ so that $\tau(d) = c$. Let $c \equiv d$ if $c \leq d$ and $d \leq c$.

Claim 2.11 *The maximal $\equiv$-classes in $R/\equiv$ with respect to $\leq$ are exactly those containing a generator and each generator is in a distinct $\equiv$-class.*

Proof: To see that every maximal $\equiv$-class contains a generator, suppose that $a \in R$ is in a maximal $\equiv$-class. Then a is in the range of only right invertible terms. But by lemma 2.3, any right invertible term is essentially unary. So for some essentially unary term τ, there is a generator, f, so that $\tau(f) = a$ where we suppress all but the variable on which τ depends. Therefore, $f \geq a$.

Now suppose that $a \in R$. As above, we can write $a = \tau(f)$ for some generator f and essentially unary term τ where only the variable that τ depends on is displayed. Suppose μ is essentially unary and $\mu(\tau(f)) = g$ where g is some generator. If we show that f must equal g then we will have shown that the generators are in distinct maximal $\equiv$-classes. We can assume then that $\tau(f) = g$ and f and g are generators and we want to show that $f = g$.

The cases when $f = f_u$ for some $u \in V$ or $f = \hat{x}$ for some $x \in X$ are not hard. We do the case when $f = f_e^i$ for some i and e. The case when $g = \hat{x}$ is easily seen to be impossible. Let's see that $g = f_u$ for some $u \in V$ is also impossible. Suppose $u \in V_1$. Then $\tau(f(u)) = a'$ which means that $\tau(j) = a'$

where $j = 0, 1$ or 2 which is either directly a contradiction or contradicts that q is not left invertible in either of its first two variables. We have a similar situation if $u \in V_2$.

So $\tau(f_e^i) = f_{e'}^j$ for some j and e'. Using the p_i's it is not hard to conclude that $i = j$. Suppose $e \neq e'$ and $x \in e \setminus e'$. Further suppose that $x \in V_1$. Then by considering $\tau(f(x)) = g(x)$ we see that $\tau(a' \bullet b) = a \bullet b$ which leads to the conclusion that q is independent of its first variable. We reach a similar contradiction if $x \in V_2$. Hence, $e = e'$ and so $f = g$. $\square$

We want to distinguish between the vertices and the edges of G. Let $\mathcal{P}$ be the set of maximal $\equiv$-classes which do not contain $\hat{x}$ for any $x \in X$. We say $C, D \in \mathcal{P}$ are $\sim$-related if there are $c \in C$ and $d \in D$ so that $c \sim d$.

Claim 2.12 *If $C \in \mathcal{P}$ then C is $\sim$-related only to itself in $\mathcal{P}$ iff $f_u \in C$ for some $u \in V$.*

Proof: If $e \in \widehat{E}$ then $f_e^1 \sim f_e^2$ so the direction from left to right is clear.

Now suppose $\tau(f_u) \sim \mu(f)$ for some $u \in V$ and some generator f so that $\mu(f) \equiv f$ and $\tau(f_u) \equiv f_u$. There is ν so that $\nu\mu(f) = f$ and by considering the components of f, we get the valid equation $\nu\mu(x) = x$. Hence ν is right invertible so by lemma 2.6, $\nu\tau(f_u) \sim f$.

Suppose $u \in V_1$. If $f = f_v$ for some $v \in V_1$ with $v \neq u$ then by lemma 2.6, $\nu\tau(a') \sim a$ which would contradict, among other things, that q depends on its first variable. The case when $f = f_v$ for $v \in V_2$ is similar.

Now suppose $f = f_e^i$ for some $e \in \widehat{E}$. If we choose $v \in V$ so that $v \neq u$ and $v \notin e$ then by lemma 2.6, we get $\nu\tau(a) \sim 0$. i. e. $\nu\tau(a) \sim a \bullet b$ in $\mathcal{F}$. From this we conclude that $0 \sim 1$ which contradicts claim 2.9.

The case when $u \in V_2$ is similar and so we conclude that if $\tau(f_u) \sim \mu(f)$ then $f = f_u$ which finishes the claim. $\square$

Call a $C \in \mathcal{P}$ which is $\sim$-related only to itself in $\mathcal{P}$, a vertex. We say there is an edge between two vertices C and D if there is $c \in C$, $d \in D$ and some $e \in E \in \mathcal{P}$ so that E is not a vertex and $c \bullet d \sim e$.

Claim 2.13 *There is an edge between $f_u/\equiv$ and $f_v/\equiv$ iff $\{u, v\} \in \widehat{E}$.*

Proof: The direction from right to left is by construction.

Now suppose $f_u \in C$, $f_v \in D$, there is an edge between C and D but $\{u, v\} \notin \widehat{E}$. We have then that

$$\nu(\tau(f_u) \bullet \mu(f_v)) \sim f_e^1$$

for some $e \in \widehat{E}$ and essentially unary terms ν, τ and μ.

Choose $x \in e$ so that $x \notin \{u, v\}$. Let $a = \nu(\tau(f_u) \bullet \mu(f_v))$. But then $a(p_1) = a(x)$ and by lemma 2.6,

$$f_e^1(p_1) \sim a(p_1) = a(x) \sim f_e^1(x).$$

But then $0 \sim z \sim j$ where $j = 1$ or 2 which contradicts claim 2.9. $\square$

In this way, we are able to recover the isomorphism type of the graph G from $\mathcal{B}^G$ as long as we fix the elements $\hat{x}$ for $x \in X$. Let $\bar{x} = \langle \hat{x} : x \in X \rangle$. Then if G and H are two bipartite graphs we have $\langle \mathcal{B}^G, \bar{x} \rangle \cong \langle \mathcal{B}^H, \bar{x} \rangle$ implies $G \cong H$. Moreover, $|G| \leq |\mathcal{B}^G| \leq |G| + |\mathcal{L}|$. Since there are the maximal number of non-isomorphic bipartite graphs in every infinite cardinality and we have fixed only finitely many elements $\bar{x}$, we can conclude that $I(\mathcal{V}, \lambda) = 2^\lambda$ for all $\lambda \geq |\mathcal{L}|$. $\square$

2.2 $\mathcal{V}$ is linear

Let $\mathcal{V}$ be a multi-sorted unary variety. We call a term τ constant valued if the equation $\tau(x) = \tau(y)$ holds in $\mathcal{V}$. If $\mathcal{A} \in \mathcal{V}$ then let $C(A)$ be the subuniverse generated by the constants and the values of the constant valued terms in A. (This may be empty.) Define a quasi-order on $A \setminus C(A)$ as follows: For $a, b \in A \setminus C(A)$,

$$a \leq b \text{ iff there is a term } \tau \text{ so that } \tau(b) = a.$$

Say $a \sim b$ if $a \leq b$ and $b \leq a$. $\langle A \setminus C(A)/{\sim}, \leq \rangle$ is a partial order.

Definition 2.14 *We say $\mathcal{V}$ is linear if for all $\mathcal{A} \in \mathcal{V}$ and $a \in A \setminus C(A)$,*

$$\{b/{\sim} : b \leq a, b \in A \setminus C(A)\}$$

is linearly ordered by $\leq$.

Comment: $\mathcal{V}$ is linear means that for all $\mathcal{A} \in \mathcal{V}$ and $a \in A \setminus C(A)$, the subuniverses of the form $\langle b \rangle \cup C(A)$ for $b \in \langle a \rangle$ are linearly ordered by inclusion.

Theorem 2.15 *If $\mathcal{V}$ is not linear then $I(\mathcal{V}, \lambda) = 2^\lambda$ for all $\lambda \geq |\mathcal{L}|$.*

Proof: We need some preparation. If there is a failure of linearity then it will occur in a free algebra. Let $\mathcal{F}$ be free on $x_1, y_1, \ldots, x_k, y_k$ where x_i, y_i are of sort i. We may assume that $\{b/\sim: b \leq x_1\}$ is not linearly ordered. So there are $a', b' \in F$ with $a' \leq x_1, b' \leq x_1$ but $a' \not\leq b'$ and $b' \not\leq a'$.

Let $D = \{z \in F : z \not\geq a' \text{ and } z \not\geq b'\}$ and $D_i = F_i \cap D$. Let

$$C_i = \{\langle u, y_i \rangle : u \in D_i\} \text{ and } \theta = Cg(\langle C_1, \ldots, C_k \rangle).$$

Claim 2.16 *If $z \in F \setminus D$ then $z/\theta = \{z\}$ and if $z \in D_i$ then $z/\theta = D_i$.*

Proof: The second follows from the first and the definition of C_i. To prove the first, suppose $z \geq a'$.

Suppose $\langle u, y_i \rangle \in C_i$ and there is a term τ so that $\tau(u) = z$. Then $u \geq z$ and $z \geq a'$ so $u \geq a'$ which is a contradiction to $u \in D_i$. Suppose there is a term τ so that $\tau(y_i) = z$. Since $a' \leq x_1$, there is a term μ so that $a' = \mu(x_1)$ and there is a term γ so that $\gamma(z) = a'$ so we have $\gamma(\tau(y_i)) = \mu(x_1)$. y_i and x_1 are free generators so $\gamma(\tau(u)) = a'$ which contradicts $u \in D_i$.

We have shown then that if $\langle u, y_i \rangle \in C_i$, it can never happen that there is a term τ so that $\tau(u) = z$ or $\tau(y_i) = z$ so $z/\theta = \{z\}$. □

Let $\mathcal{B} = \mathcal{F}/\theta$, $\mathcal{A} = \langle y_1/\theta, \ldots, y_k/\theta \rangle$, $a = a'/\theta$, $b = b'/\theta$, $c = x_1/\theta$, $\mathcal{B}_1 = \langle \mathcal{A}a \rangle$ and $\mathcal{B}_2 = \langle \mathcal{A}b \rangle$.

Claim 2.17 *$\mathcal{B}_1$ and $\mathcal{B}_2$ are the only minimal proper extensions of $\mathcal{A}$ in $\mathcal{B}$.*

Proof: If $d \in B \setminus A$ then $d = d'/\theta$ for some $d \in F \setminus D$. So $d' \geq a'$ or $d' \geq b'$. That is, $\mathcal{B}_1 \subseteq \langle \mathcal{A}d \rangle$ or $\mathcal{B}_2 \subseteq \langle \mathcal{A}d \rangle$ which demonstrates what we want. □

Now suppose G is a bipartite graph $\langle N, E \rangle$ where N is partitioned into N_1 and N_2 and there is a $n^* \in N_1$ so that for every $u \in N_2$, $\langle n^*, u \rangle \in E$. Note that there is the maximal number of non-isomorphic graphs of this type for any infinite cardinal. Let

$$\hat{E} = \{\{u, v\} : \langle u, v \rangle \in E\}.$$

Since we are dealing with a unary variety, we can define $\mathcal{B}^G \in \mathcal{V}$ so that:

$$\mathcal{B}^G = \mathcal{A} \cup \bigcup_{u \in N} \mathcal{B}_u \cup \bigcup_{e \in \hat{E}} \mathcal{B}_e$$

where

1. $\mathcal{A} \subseteq \mathcal{B}_u$ for all $u \in N$,
2. if $e = \{u, v\}$ then $\mathcal{B}_u, \mathcal{B}_v \subseteq \mathcal{B}_e$,
3. if $u \in N_1$, $\mathcal{B}_u \cong \mathcal{B}_1$ over $\mathcal{A}$,
4. if $u \in N_2$, $\mathcal{B}_u \cong \mathcal{B}_2$ over $\mathcal{A}$,
5. if $u \in N_1$, $v \in N_2$ and $e = \{u, v\} \in \hat{E}$ then there is an isomorphism $\mu : \mathcal{B} \to \mathcal{B}_e$ which fixes $\mathcal{A}$, $\mu(\mathcal{B}_1) = \mathcal{B}_u$ and $\mu(\mathcal{B}_2) = \mathcal{B}_v$,
6. if $u, v \in N$ and $u \neq v$ then $\mathcal{B}_u \cap \mathcal{B}_v = \mathcal{A}$ and
7. if $e, e' \in \hat{E}$, $e \neq e'$ then $\mathcal{B}_e \cap \mathcal{B}_{e'} = \mathcal{B}_{e \cap e'}$ where $\mathcal{B}_\emptyset = \mathcal{A}$.

Fix this notation for the rest of the proof.

Let C be the set of minimal subalgebras extending $\mathcal{A}$ in $\mathcal{B}^G$. By claim 2.17, $C = \{B_u : u \in N\}$.

To recover N_2, recall that $n^* \in N_1$ so if $u \in N \setminus \{n^*\}$ then $u \in N_2$ iff there is a $c \in B^G$ and terms μ and τ so that $\mu(c) \in B_u \setminus A$ and $\tau(c) \in B_{n^*} \setminus A$.

For the direction from left to right, the generator of $\mathcal{B}_e$ corresponding to x_1/θ in $\mathcal{B}$ where $e = \{n^*, u\}$ will suffice. For the direction from right to left, if $u \in N_1$ then if the c mentioned on the right hand side is in $\mathcal{B}_e$, then there would be at least three minimal proper extensions of $\mathcal{A}$ in $\mathcal{B}_e$ which contradicts claim 2.17.

Say that $\mathcal{D} \in C$ is a 2-vertex if there is $c \in B^G$ and μ, τ so that $\mu(c) \in D \setminus A$ and $\tau(c) \in B_{n^*} \setminus A$.

Hence, by naming the generator of $\mathcal{B}_{n^*}$ and using C, we can recover N_2 via the 2-vertices. Say that $\mathcal{D} \in C$ is a 1-vertex if it is not a 2-vertex.

We say there is an edge between a 1-vertex $\mathcal{D}$ and a 2-vertex $\mathcal{E}$ if there is a $c \in B^G$ and μ, τ so that $\mu(c) \in D \setminus A$ and $\tau(c) \in E \setminus A$.

Claim 2.18 *There is an edge between a 1-vertex $\mathcal{B}_u$ and a 2-vertex $\mathcal{B}_v$ iff $\{u, v\} \in \hat{E}$.*

Proof: From right to left is by construction of $\mathcal{B}^G$. To go from left to right, suppose $\{u, v\} \notin \hat{E}$. Suppose $c \in B_e$ witnesses that there is an edge between $\mathcal{B}_u$ and $\mathcal{B}_v$. Then $\mathcal{B}_u$ and $\mathcal{B}_v$ are minimal proper extensions of $\mathcal{A}$ in $\mathcal{B}_e$. If $e \neq \{u, v\}$ then there is a third minimal proper extension of $\mathcal{A}$ in $\mathcal{B}_e$ which contradicts claim 2.17. $\square$

To conclude then, we have shown that we can recover the isomorphism type of G from $\mathcal{B}^G$ as long as we fix the generators for $\mathcal{A}$ and the generator for $\mathcal{B}_{n^*}$. Call these elements, $\bar{w}_G$. Hence if $\langle \mathcal{B}^G, \bar{w}_G \rangle \cong \langle \mathcal{B}^H, \bar{w}_H \rangle$ for two appropriate bipartite graphs G and H then $G \cong H$. Moreover, $|G| \leq |\mathcal{B}^G| \leq |G| + |\mathcal{L}|$. This is enough to conclude $I(\mathcal{V}, \lambda) = 2^\lambda$ for all $\lambda \geq |\mathcal{L}|$.

2.3 $\mathcal{V}$ has the ascending chain condition

We now assume that $\mathcal{V}$ is a multi-sorted unary variety which is linear.

Definition 2.19 *$\mathcal{V}$ has the ascending chain condition if there is no $\mathcal{A} \in \mathcal{V}$ and $a_i \in A$ for $i \in \omega$ so that $a_i < a_{i+1}$ for all i.*

Comment: To say that $\mathcal{V}$ has the ascending chain condition means that there is no infinite increasing chain of one-generated subuniverses in any algebra in $\mathcal{V}$. Example 0.12 is an example of a variety which does not have the ascending chain condition.

Theorem 2.20 *If $\mathcal{V}$ does not have the ascending chain condition then $I(\mathcal{V}, \lambda) = 2^\lambda$ for all $\lambda \geq |\mathcal{L}|$.*

Proof: Suppose $\mathcal{A} \in \mathcal{V}$ and $a_i \in A$ for $i \in \omega$ so that $a_i < a_{i+1}$ for all i. $\langle a_i : i \in \omega \rangle$ may not form a subalgebra of $\mathcal{A}$ so we must make the following small adjustment. Let $\mathcal{B}$ be an isomorphic copy of $\mathcal{A}$ so that $\mathcal{A} \cap \mathcal{B} = \langle a_0 \rangle$. $\mathcal{A} \cup \mathcal{B} \in \mathcal{V}$. Choose $b_j \in B_j$ for $1 \leq j \leq k$ and let $\mathcal{A}_i = \langle b_1, \ldots, b_k, a_i \rangle$ for $i \in \omega$. We use the subalgebras $\mathcal{A}_i$ for $i \in \omega$ to construct many models (with $\mathcal{A}_0$ fixed).

For us, a tree will be a partial order $\langle P, < \rangle$ so that for every $a \in P$, the set $\{b : b < a\}$ is well ordered. We write $L(a)$ for the order type of $\{b : b < a\}$. If b is an immediate successor of a we write $a \lhd b$. We say a tree is well-founded if it has no infinite increasing sequences. For a well-founded tree, $\langle P, < \rangle$ we define the following ordinal valued depth function, *dep*, for $a \in P$:

$$dep(a) = \bigcup \{dep(b) + 1 : a \lhd b\}.$$

Call a well-founded tree $\mathcal{P}$ everbranching if every $\eta \in \mathcal{P}$ has at least two immediate successors if it has any. It is easy to construct an everbranching well-founded tree $\mathcal{P}$ of tree depth α so that $|\mathcal{P}| \leq |\alpha + \omega|$.

Now fix $\lambda \geq |\mathcal{L}|$ and $X \subseteq \lambda$ with $|X| = \lambda$. Let $\mathcal{P}$ be a well-founded tree of cardinality λ so that the set of depths of nodes on the first level is exactly X and $\mathcal{P}$ is everbranching.

Since $\mathcal{V}$ is a unary variety, it is easy to construct an algebra $\mathcal{A}_X$ where

$$\mathcal{A}_X = \bigcup_{\eta \in \mathcal{P}} \mathcal{A}_\eta$$

so that

1. $\mathcal{A}_{\langle\rangle} = \mathcal{A}_0$,
2. $\mathcal{A}_\eta = \langle a_\eta \rangle$ for all $\eta \in \mathcal{P}$, $\eta \neq \langle\rangle$,
3. for all $\eta, \nu \in \mathcal{P}$, $\mathcal{A}_\eta \cap \mathcal{A}_\nu = \mathcal{A}_{\eta \wedge \nu}$ and
4. for all $\eta \in \mathcal{P}$, $\eta \neq \langle\rangle$, the map sending $a_{l(\nu)}$ to a_ν for all $\nu \leq \eta$ generates an isomorphism from $\mathcal{A}_{l(\eta)}$ to $\mathcal{A}_\eta$.

We try to recover X from the $\sim$-classes of $A_X \setminus A_0$. We call a $\sim$-class, $a/{\sim}$, good if $a \in A_X \setminus A_0$ and either it is maximal or whenever $b/{\sim} > a/{\sim}$ there is $c/{\sim} > a/{\sim}$ so that $b/{\sim}$ and $c/{\sim}$ are incomparable.

Claim 2.21 *The good $\sim$-classes are exactly the $\sim$-classes of a_η for $\eta \in \mathcal{P} \setminus \{\langle\rangle\}$.*

Proof: That each a_η is in a good $\sim$-class follows from the fact that $\mathcal{P}$ is everbranching. For if $b > a_\eta$ then η has at least two successors. Hence there is $c > a_\eta$ so that b and c are incomparable.

Now suppose $a \in A_X \setminus A_0$. Let η be the least so that $a \in A_\eta$. Hence $a_\eta \geq a$. If $a/{\sim}$ is good and $a_\eta > a$ then there should be $b \in A_\mu$, $\mu \neq \eta$ so that $b > a$ and a_η and b are incomparable. But then $a \in A_\mu \cap A_\eta$ which contradicts the minimality of η. So either $a \sim a_\eta$ or $a/{\sim}$ is not good. $\square$

From the good $\sim$-classes, one can recover X by considering the depth of the minimal good $\sim$-classes.

Hence, if we fix $a_0, b_1, \ldots, b_k$ we have

$$\langle \mathcal{A}_X, a_0, b_1, \ldots, b_k \rangle \cong \langle \mathcal{A}_Y, a_0, b_1, \ldots, b_k \rangle$$

implies $X = Y$. Since $|\mathcal{P}| \leq |\mathcal{A}_X| \leq |\mathcal{P}| + |\mathcal{L}|$, this is enough to imply that $I(\mathcal{V}, \lambda) = 2^\lambda$ for all $\lambda \geq |\mathcal{L}|$. $\square$

We shall conclude this section with a sketch of a characterization of strongly abelian varieties with few models. It follows in form the characterization of complete countable first order theories given by Shelah which is explained in either [2] or [3]. Any undefined terms can be found there.

Now if $\mathcal{V}$ is a strongly abelian variety with few models then the conclusion one can draw from Theorems 1.7 and Theorem 2.8 is that $\mathcal{V}$ is bi-interpretable with a multi-sorted unary variety. Theorems 2.15 and 2.20 allow us to conclude that this variety is also linear and satisfies the ascending chain condition.

Now suppose $\mathcal{A} \in \mathcal{V}$ and $\mathcal{V}$ is a multi-sorted unary variety which is linear and satisfies the ascending chain condition. $\mathcal{A}$ can be represented as the union of a well-founded tree of 1-generated subuniverses. To be precise, let's adopt some terminology.

Say two subuniverse of $\mathcal{A}$, C and D, are independent over a subuniverse B if $C \cap D \subseteq B$. Given three subuniverses of $\mathcal{A}$, B, C and D, say that D is dominated by C over B if $C \cap D \nsubseteq B$.

Note that to say that D is dominated by C over B is just to say that C and D are not independent over B but this terminology will be more analogous to the usage of classification theory in the tree constructed below.

Define a labelled tree of subuniverses as follows

1. The root of the tree is $C(A)$, the constant subuniverse of $\mathcal{A}$. This could be empty.

2. On the first level, choose a maximal collection of pairwise independent over $C(A)$ 1-generated subuniverses of $\mathcal{A}$ which are not contained in $C(A)$.

3. Suppose we have determined that B is on the n^{th} level and C is a successor of B on the $(n+1)^{\text{st}}$ level. For the successors of C, choose a maximal collection of pairwise independent over C 1-generated subuniverses of $\mathcal{A}$ which are dominated by C over B and are not contained in C.

Note that in the above construction, we demanded that the successors of any node in the tree be only pairwise independent. They are in fact totally "independent" since the variety is unary and this notion of independence is trivial.

To show that this tree exhausts all of $\mathcal{A}$, one uses linearity, which is analogous to NDOP from classification theory. To conclude that the depth of the tree is bounded by some ordinal which depends only on $\mathcal{V}$, one uses the ascending chain condition, which is analogous to the property of being shallow from classification theory.

It is not hard to show that if $\mathcal{A}$ and $\mathcal{B}$ are in $\mathcal{V}$ and give rise, via the above construction, to isomorphic labelled trees then $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.

Using standard arguments, (see [2] or [3]), one can show that if $\mathcal{V}$ is a multi-sorted unary variety which is linear and has the asscending chain condition then there is an ordinal δ so that

$$I(\mathcal{V}, \aleph_\alpha) \leq {}_\delta(|\alpha + \omega|).$$

Hence we have the following characterization

Theorem 2.22 *If $\mathcal{V}$ is a strongly abelian variety then $\mathcal{V}$ has few models iff $\mathcal{V}$ is bi-interpretable with a multi-sorted unary variety which is linear and has the ascending chain condition.*

3 The spectrum function

In this section we fulfill a promise made in [4] to enumerate all the spectrum functions for locally finite varieties. In fact, we also list the possible spectrum functions for all strongly abelian varieties and note that we have verified Vaught's conjecture in both cases.

Palyutin and Starchenko in [6] have calculated the spectrum functions for all Horn classes in a countable language and Palyutin, [5], has listed those which are spectrums of varieties. For completeness, we list the possible spectrum functions for a variety and give strongly abelian examples of each type.

Theorem 3.1 (Palyutin) *If $\mathcal{V}$ is a variety in a countable language then $I(\mathcal{V}, \aleph_\alpha)$ as a function of α for $\alpha > 0$ is the minimum of $2^{\aleph_\alpha}$ and exactly one of the following functions*

1. *some fixed finite number, $\aleph_0$, $2^{\aleph_0}$,*
2. *${}_\delta(|\alpha + \omega|)$ for some $\delta < \omega_1$, δ not a limit ordinal,*

3. ${}_n(|\alpha+\omega|^{\aleph_0})$ *for some* $n < \omega$,

4. ${}_n(|\alpha+\omega|^{2^{\aleph_0}})$ *for some* $n < \omega$,

5. ${}_n(|\alpha+2^{\aleph_0}|)$ *for some* $n < \omega$ *or*

6. $2^{\aleph_\alpha}$.

Example 3.2 An example of the first type listed above is the variety of all structures in a language with countably many constants.

Example 3.3 Example 0.13 provides an example of the second type.

Example 3.4 By combining example 0.13 with G-sets for particular groups G we can obtain examples for types 3 and 4. More precisely, fix a group G and $n < \omega$. Let

$$\mathcal{L} = \{h_g : g \in G\} \cup \{f_i : i < n\}$$

where each symbol is a unary function.

The axioms for the variety will be

$$f_i f_j(x) = f_{\max i,j}(x) \text{ for all } i, j < n$$

$$h_g f_i(x) = f_i h_g(x) = f_i(x) \text{ for all } i < n \text{ and } g \in G$$

$$h_e(x) = x \text{ where } e \text{ is the identity of } G$$

$$h_g h_k(x) = h_{gk}(x) \text{ for all } g, k \in G$$

If $\mathcal{A}$ is an algebra in this variety then consider $x \in A$ so that x is in the range of f_0. Let $S_x = \{y \in A : f_0(y) = x \text{ and } x \neq y\}$. The action of the functions $\{h_g : g \in G\}$ on S_x makes S_x into a G-set. Using this fact, it is easy to compute the spectrum for such varieties.

If G is countable and has countably many subgroups (for example, the integers) then this is an example of type 3 listed above. If G has $2^{\aleph_0}$ many subgroups then this is an example of type 4.

Example 3.5 An example of the fifth type is obtained by looking at a variety whose associated multi-sorted variety has two sorts and in one sort you have a copy of example 3.3 with $\delta = n$ and in the other sort you have a copy of type 4 with the subscript $n - 1$.

Example 3.6 Example 0.11 provides an example of the sixth type.

In the locally finite case, only the first possibility of case 1, only 2 with $\delta < \omega$ and 6 can occur. Baldwin and McKenzie gave a complete analysis of the spectrum for an affine variety in [1]. Looking at their proof, in the locally finite case we get

Theorem 3.7 (Baldwin-McKenzie) *If $\mathcal{V}$ is a locally finite affine variety in a countable language then $I(\mathcal{V}, \aleph_\alpha)$ as a function of α for $\alpha > 0$ is exactly one of the following functions*

1. *1*
2. $|\alpha + \omega|$
3. $2^{\aleph_\alpha}$

Actually, in their paper, there is the possibility that $I(\mathcal{V}, \aleph_\alpha) = |\alpha+\omega|^{\aleph_0}$. This could happen if for a ring R, every R-module was ω-stable but there were countably many indecomposables. Mike Prest has pointed out to us that by consulting [8] one sees that any Artin algebra all of whose right modules are ω-stable has only finitely many indecomposables. Since a finite ring is an Artin algebra, the extra possibility from [1] is ruled out.

Hence by Theorem 0.2,

Theorem 3.8 *If $\mathcal{V}$ is a locally finite variety in a countable language then $I(\mathcal{V}, \aleph_\alpha)$ as a function of α for $\alpha > 0$ is the minimum of $2^{\aleph_\alpha}$ and one of*

1. *some finite number*
2. $\beth_n(|\alpha + \omega|)$ *for some $n < \omega$ or*
3. $2^{\aleph_\alpha}$.

Now let's consider Vaught's conjecture for locally finite varieties. From the proofs in [4], if a locally finite variety $\mathcal{V}$ is not the varietal product of a strongly abelian, an affine and a discriminator variety then $I(\mathcal{V}, \aleph_0) = 2^{\aleph_0}$. Moreover, if $\mathcal{D}$ is a non-trivial discriminator variety then from [1] we know that $I(\mathcal{D}, \aleph_0) = 2^{\aleph_0}$. Hence by Theorem 0.1 and what we have said, if $\mathcal{V}$ is locally finite and $I(\mathcal{V}, \aleph_0) < 2^{\aleph_0}$ then there are affine and strongly abelian subvarieties $\mathcal{A}$ and $\mathcal{S}$ respectively so that $\mathcal{V} = \mathcal{A} \otimes \mathcal{S}$. Hence if one proves Vaught's conjecture for affine and strongly abelian varieties separately then one will have it for locally finite varieties. The affine case has been handled in [1] so let us say a few words about the strongly abelian case.

Suppose $\mathcal{V}$ is strongly abelian and $I(\mathcal{V}, \aleph_0) < 2^{\aleph_0}$. Then from Corollary 1.8, we may assume condition $*$ from section 2. Since Theorem 2.8 was proved for all $\lambda \geq |\mathcal{L}|$, we conclude that $\mathcal{V}$ is bi-interpretable with a multi-sorted unary variety. Theorem 2.15 was also proved for $\lambda \geq |\mathcal{L}|$ and so we may assume that $\mathcal{V}$ is linear.

Lemma 3.9 *If $\mathcal{V}$ is a linear multi-sorted unary variety with $I(\mathcal{V}, \aleph_0) < 2^{\aleph_0}$ then all non-constant 1-generated subuniverses are minimal.*

Proof: Suppose $\mathcal{A} \in \mathcal{V}$ and $C = \langle c \rangle \subseteq \mathcal{A}$ is not minimal. We may assume, by naming finitely many constants, that $C(A)$, the constant subuniverse, together with C is a subalgebra of $\mathcal{A}$ and that there is $B = \langle b \rangle \quad C$ and $B \not\subseteq C(A)$.

Fix $n \in \omega$ and define the algebra $\mathcal{C}_n \in \mathcal{V}$ by

$$\mathcal{C}_n = C(A) \cup B \cup \bigcup_{i<n} C_n^i$$

where $C_n^i \cong C$ over B and if $i \neq j$ then $C_n^i \cap C_n^j = B$.

Claim 3.10 *If $\mathcal{C}_m \cong \mathcal{C}_n$ then $m = n$.*

Proof: Consider the image under an isomorphism of the generators of $\mathcal{C}_m^i$ for $i < m$. By linearity, no two of them can be in the same $\mathcal{C}_n^j$. Hence, $m \leq n$. By symmetry, $m = n$. $\square$

For an algebra $\mathcal{D} \in \mathcal{V}$ with $C(A) \subseteq D$, say that $a, b \in D \setminus C(A)$ are in the same connected component if there is $c \in D \setminus C(A)$ so that $c \leq a$ and $c \leq b$. (See the definition before Claim 2.11.) Note that by linearity of $\mathcal{V}$, being in the same connected component is an equivalence relation on $D \setminus C(A)$.

Let $X \subseteq \omega$, $|X| = \aleph_0$ and

$$\mathcal{C}_X = C(A) \cup \bigcup_{n \in X} C_n^X$$

where $C_n^X \cong \mathcal{C}_n$ over $C(A)$ and for $m, n \in X$, $m \neq n$, $C_m^X \cap C_n^X = C(A)$. Note that the connected components for $\mathcal{C}_X$ are $C_n^X \setminus C(A)$ for $n \in X$.

Claim 3.11 *If $\mathcal{C}_X \cong \mathcal{C}_Y$ then $X = Y$.*

Proof: Any isomorphism must preserve connected components. The above observation and claim 3.10 shows that $X = Y$. $\Box$

Hence the existence of a non-minimal 1-generated subuniverse leads to $I(\mathcal{V}, \aleph_0) = 2^{\aleph_0}$ and we are done. $\Box$

Finally, the number of constant subuniverses is either $\leq \aleph_0$ or $2^{\aleph_0}$. One way to see this is that the property of being a constant subuniverse of an algebra in $\mathcal{V}$ is expressible by a very low rank Scott sentence.

For any particular constant subuniverse C one can consider the number of minimal 1-generated subuniverses in algebras containing C. If there are infinitely many such then by varying the number of copies of each, one easily produces $2^{\aleph_0}$ many non-isomorphic countable models containing C. If instead there are only finitely many then there are at most countably many non-isomorphic countable models containing C.

If $I(\mathcal{V}, \aleph_0) < 2^{\aleph_0}$ then there must be $\leq \aleph_0$ many non-isomorphic constant subuniverses and each is contained in at most countaby many non-isomorphic countable algebras. It easily follows that $I(\mathcal{V}, \aleph_0) \leq \aleph_0$.

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