Bergman's Condition and varieties with near unanimity terms

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Abstract

Bergman showed that systems of projections of algebras in a variety will satisfy a certain consistency condition if the variety has a near-unanimity term. The converse of this theorem was left open. This paper investigates this open question, and whether the Bergman Condition is equivalent to having a near-unanimity term whose arity is a function on an integer k.

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1 Introduction

This paper was motivated by the result of George Bergman in his 1977 paper [3]. In 1975 Baker and Pixley [1] showed an equivalence between varieties which contain a $(k+1)$-ary near unanimity term and varieties in which any subalgebra of a direct product of algebras in V can be uniquely determined by its projections on each k-fold subproduct. In his paper, Bergman furthered this with an existence result. Namely, that if a variety satisfied the conditions of the theorem of Baker-Pixley, and for a collection of n algebras, a system of k-fold subproducts is given; this system of k-fold subproducts is the system of projections of a single n-fold product if and only if these subproducts are suitably "consistent" with each other. Bergman(k) is the property that if a system is suitably consistent then it is the projection of an n-fold product. The work in this paper is concerned with strengthening this result; both whether it can be strengthened, and by how much.

For a detailed introduction to Universal Algebra, see [4], but here we will include some necessary background and basic definitions.

Definition 1.1. A *type* of algebras is a set $\mathscr{F}$ of function symbols such that each $f \in \mathscr{F}$ is assigned a nonnegative integer n, called the *arity* of f.

Definition 1.2. An *algebra* $\mathbb{A}$ of type $\mathscr{F}$ is an ordered pair $\langle A, F\rangle$, where A is a nonempty set called the universe of $\mathbb{A}$, and F is a collection of finitary operations of A (called the *basic operations* of $\mathbb{A}$) indexed by $\mathscr{F}$ such that corresponding to each n-ary function symbol f in F there is an n-ary operation $f^{\mathbb{A}}$ on A. A *term operation*, or simply *term*, of $\mathbb{A}$ is any operation on A which is built from the composition of basic operations of $\mathbb{A}$.

For example, a ring is an algebra $\langle R, +, \cdot, -, 0\rangle$ where the set of symbols $\{+, \cdot, -, 0\}$ is the type. Notice that 0 is in the type, this is because constants are considered as nullary operations.

Definition 1.3. Let $\mathbb{A}$ and $\mathbb{B}$ be algebras of the same type. $\mathbb{B}$ is a *subalgebra* of $\mathbb{A}$, written $\mathbb{B} \leq \mathbb{A}$, if $B \subseteq A$ and every fundamental operation of $\mathbb{B}$ is the restriction to B of the corresponding operation of $\mathbb{A}$.

Definition 1.4. Given an algebra $\mathbb{A}$ define, for every $X \subseteq A$,

$$\mathrm{Sg}_{\mathbb{A}}(X) = \bigcap\{B \mid X \subseteq B \text{ and } B \text{ is a subuniverse of } A\}.$$

We read $\mathrm{Sg}_{\mathbb{A}}(X)$ as "the subuniverse of $\mathbb{A}$ generated by X".

Definition 1.5. A *homomorphism* of algebras of the same type, $\mathscr{F}$, is a map which preserves the fundamental operations of $\mathscr{F}$. Direct products of algebras of the same type are formed in the natural way. A *variety*, V, is a class of algebras that is closed under subalgebras, homomorphisms, and direct products. If $\mathbb{A}$ is an algebra, we call $V = \mathrm{HSP}(\mathbb{A})$ the variety generated by $\mathbb{A}$.

A common example of a variety is that of groups, $\mathrm{HSP}(\mathbb{G})$, where $\mathbb{G} = \langle G, \cdot, ^{-1}, 1\rangle$. It is interesting to note that the variety of abelian groups does not satisfy Bergman(k) for any k as illustrated in [3].

Definition 1.6. Let $\mathbb{A}$ be an algebra of type $\mathscr{F}$, and let θ be an equivalence relation on $\mathbb{A}$. Then θ is a *congruence* on $\mathbb{A}$ if θ satisfies the following compatibility property:

For each n-ary function symbol $f \in \mathscr{F}$ and elements $a_i, b_i \in A$, if $a_i \theta b_i$, holds for all $1 \leq i \leq n$ then

$$f^{\mathbb{A}}(a_1, \ldots, a_n)\theta f^{\mathbb{A}}(b_1, \ldots, b_n)$$

holds as well.

The notion of congruences allow us to unify and generalize the notions of Normal Subgroups in Group Theory and Ideals in Ring Theory via the following definition.

Definition 1.7. If θ is a congruence on $\mathbb{A}$, then we can form the *quotient algebra of* $\mathbb{A}$ *by* θ, written $\mathbb{A}/\theta$, as the algebra whose universe is A/θ and whose fundemental operations satisfy

$$f^{\mathbb{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = f^{\mathbb{A}}(a_1, \ldots, a_n)/\theta$$

where $a_1, \ldots, a_n \in A$ and f is an n-ary function symbol in $\mathscr{F}$.

Indeed, if $\mathbb{G}$ is a group, θ a congruence on $\mathbb{G}$, and $\mathbb{N}$ a normal subgroup of $\mathbb{G}$ then we have the following construction:

1. $1/\theta$ is the universe of a normal subgroup of $\mathbb{G}$, and for $a, b \in G$ we have $\langle a, b\rangle \in \theta$ iff $a \cdot b^{-1} \in 1/\theta$.
2. If $\mathbb{N}$ is a normal subgroup of $\mathbb{G}$, then the binary relation defined on G by

$$\langle a, b\rangle \in \theta \quad \text{iff} \quad a \cdot b^{-1} \in N$$

is a congruence on $\mathbb{G}$ with $1/\theta = N$.

A similar construction can easily be produced for congruences over rings, and ideals.

Definition 1.8. An *identity* of type $\mathscr{F}$ over X, for a set of variables X, is an expression of the form

$$p \approx q$$

where p and q are terms over X. An algebra $\mathbb{A}$ of type $\mathscr{F}$ satisfies an identity,

$$p(x_1, \dots, x_n) \approx q(x_1, \dots, x_n)$$

for $x_i \in X$ if, for all $a_1, \dots, a_n \in A$ we have

$$p(a_1, \dots, a_n) \approx q(a_1, \dots, a_n).$$

Identities are what allows Universal Algebra to describe any mathematical structure precisely. Observe that for all of the algebras with the type $\{\cdot, ^{-1}, 1\}$, only those which satisfy the identities:

G1: $x \cdot (y \cdot z) \approx (x \cdot y) \cdot z$

G2: $x \cdot 1 \approx 1 \cdot x \approx x$

G3: $x \cdot x^{-1} \approx x^{-1} \cdot x \approx 1$

are called *groups*. A class of algebras defined by the use of identities is called an *equational class*, and a theorem by Birkhoff shows that being an equational class is a necessary and sufficient condition for being a variety.

It will also be prudent to provide a basic introduction to lattices since the work in this paper relies heavily on varieties with lattice operations. See (Burris & Sankappanavar) for a more detailed introduction.

Definition 1.9. A non empty set L together with two binary operation $\wedge$ and $\vee$ (read *meet* and *join* respectively) on L is called a *lattice* if it satisfies the following identies:

L1: (a) $x \wedge y \approx y \wedge x$
(b) $x \vee y \approx y \vee x$ (commutative laws)

L2: (a) $x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$
(b) $x \vee (y \vee z) \approx (x \vee y) \vee z$ (associative laws)

L3: (a) $x \wedge x \approx x$
(b) $x \vee x \approx x$ (idempotent laws)

L4: (a) $x \wedge (x \vee y) \approx x$
(b) $x \vee (x \wedge y) \approx x$ (absorption laws)

EXAMPLE: Let L be the set of propositions, let $\vee$ as the logical connective OR, and $\wedge$ as the logical connective AND. Then L1 to L4 are well know properties of propositional logic modulo logical equivalence.

EXAMPLE: Let $L = \mathbb{N}$, let $\vee$ denote the least common multiple, let $\wedge$ denote the greatest common divisor. Then L1 to L4 are easily verifiable.

There is an alternative definition of a lattice which uses the notion of posets with $\leq$ as a partial order.

Definition 1.10. A poset L is a *lattice* iff for every a, b in L both $\sup\{a, b\}$ and $\inf\{a, b\}$ exist in L.

If L is a lattice by Definition 1.9, we can convert it to a lattice by Definition 1.10, and vice versa using the following two constructions.

1. If L is a lattice by the first definition, then define $\leq$ on L by $a \leq b$ iff $a = a \wedge b$;
2. If L is a lattice by the second definition, then define the operations $\vee$ and $\wedge$ by $a \vee b = \sup\{a, b\}$ and $a \wedge b = \inf\{a, b\}$.

Universal Algebra and Lattice theory share many connections. For example, the set of congruences of an algebra, $\mathrm{Con}(\mathbb{A})$, form a lattice under inclusion, called the *congruence lattice* of $\mathbb{A}$. Furthermore, any algebra with a meet and (or) join operation is a lattice (or semilattice), and can be visualized as such by using construction 2 above. In the case of a join-semilattice, the partial order can be induced by defining $a \leq b$ whenever $a \vee b = b$. We will utilize this in a later section.

2 Near Unanimity Terms

Definition 2.1. A *near unanimity term* is any term $t(x_1, \ldots, x_n)$ which satisfies the following identities,

$$t(y, x, x, \ldots, x) \approx x$$
$$t(x, y, x, \ldots, x) \approx x$$

$$\dots$$

$$t(x, x, \dots, x, y) \approx x$$

NU terms define a Mal'cev condition for a variety, meaning we can discern information about the structure of a variety simply by knowing it satisfies the identities of an NU term. For example, if a variety V has a ternary NU term then V is congruence-distributive [4, Theorem 12.3]. NU terms arise in many algebras. For example, in lattices

$$m(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$$

is an NU term. In general, for any variety with lattice operations, the $n+1$-ary term,

$$h(x_0 \dots, x_n) = \bigvee_{i=0}^{n} \bigwedge_{j \neq i} x_j$$

will always be a near unanimity term.

3 Baker-Pixley Theorem and Bergman(k)

For a variety of algebras, V, and $\mathbb{A}_i \in V, i \in \{i, \dots, n\}$, let $\mathbb{B} \leq \prod_1^n \mathbb{A}_i$ be a subalgebra. For $I \subseteq \{1, \dots, n\}$ let $\pi_I : \prod_1^n \mathbb{A}_i \to \prod_{i \in I} \mathbb{A}_i$ denote the natural projection map so that $\pi_I((a_1, \dots, a_n)) = (a_i \mid i \in I)$ and $\pi_I(\mathbb{B}) = \{\pi_I(\bar{b}) \mid \bar{b} \in \mathbb{B}\}$. For the sake of brevity, we will write $\pi_I(\mathbb{B}) = \mathbb{B}_I$ when the context is clear.

Let

$$\Gamma_{\mathbb{B}}(k) = \{\pi_I(\mathbb{B}) \mid I \subseteq \{1, \dots, n\},\ |I| = k\}.$$

We will say that Γ is a k-fold system of projections over $\mathbb{A}_i$, $1 \leq i \leq n$ if, for every subset $I \subseteq \{1, \dots, n\}$, $|I| = k$, Γ contains a unique set $\pi_I(\mathbb{S})$ for some $\mathbb{S} \leq \prod \mathbb{A}_i$.

Γ_I will denote the set $\pi_I(\mathbb{S})$

The natural question then is, under what conditions will Γ be of the form $\Gamma_{\mathbb{B}}(k)$ for some $\mathbb{B} \leq \prod_1^n \mathbb{A}_i$? The Baker-Pixley Theorem provides us with part of an answer. We will restate this theorem from [1] here:

Theorem 1 (Baker-Pixley Theorem)**.** *For a variety V and integer $k \geq 2$, the following conditions are equivalent:*

(i) *V has a (k+1)-variable near unanimity term $m(x_0, \ldots, x_k)$.*

(ii) *In V, if A is a subalgebra of a direct product $A_1 \times \ldots \times A_r$, $k \leq r < \infty$, then A is uniquely determined by its images under the projections of $A_1 \times \ldots \times A_r$ on all products $\prod_I A_i$ with $I \subseteq \{1, \ldots r\}$, $|I| = k$.*

(iii) *In any algebra $A \in V$, if r congruences $x \equiv a_i mod \theta_i$, $1 \leq i \leq r(k \leq r)$, are solvable k at a time, then they are solvable simultaneously.*

(iv) *For any algebra $A \in V$, integer $n \geq 1$, and finite partial function $f : A^n \to A$, if the restriction of f to each subset of its domain with k or fewer elements has an interpolating term operation, then so does f itself.*

(v) *f, as given in (iv), has an interpolating term operation if and only if all subalgebras of A^k are closed under f (where defined).*

Items (i) and (ii) of Theorem 1 provide us with a hint on how to approach our previous question. Namely, in V and in the presence of a $(k+1)$-ary near unanimity term, $\mathbb{B}$ is uniquely determined by $\Gamma_{\mathbb{B}}(k)$.

To see an application of the Baker-Pixley Theorem, we look to the algebra $\langle \mathbb{Z}, + \rangle$. Consider the subalgebra

$$\mathbb{S}_n = \left\{ (a_1, \ldots, a_n) \mid \sum a_i = 0 \right\} \lneq \mathbb{Z}^n.$$

Observe that for $I = \{1, \ldots, n-1\}$, $\pi_I(\mathbb{S}_n) = \mathbb{Z}^{n-1}$ since for any $\bar{a} = (a_1, \ldots, a_{n-1}) \in \mathbb{Z}^{n-1}$, let $a_n = -\sum_1^{n-1} a_i$, then

$$\bar{a}' = \left(a_1, \ldots, a_{n-1}, -\sum_1^{n-1} a_i \right) \in \mathbb{S}_n.$$

However, for any $J \subseteq \{1, \ldots, n\}$ with $|J| = n - 1$, we have $\pi_J(\mathbb{Z}^n) = \pi_J(\mathbb{S}_n) = \mathbb{Z}^{n-1}$. Thus we have found two distinct subalgebras of $\mathbb{Z}^n$ which have the same $(n-1)$-fold projections, then by the Theorem of Baker-Pixley, we can conclude that $\langle \mathbb{Z}, + \rangle$ has no near-unanimity term of any arity.

Definition 3.1 (Consistency)**.** Let Γ be a k-fold system of projections over $\mathbb{A}_i$, $1 \leq i \leq n$. For $J \subseteq \{1, \ldots, n\}$, Γ is *consistent on* J if for all $I \subseteq J$ $|I| = k$ and $\bar{a} \in \Gamma_I$, $\bar{a}$ can be extended to some $\bar{a}' \in \prod_{i \in J} \mathbb{A}_i$ such that for all $L \subseteq J$, $|L| = k$, $\pi_L(\bar{a}') \in \Gamma_L$.

For $r \geq k$, Γ is *r-consistent* if Γ is consistent on J for all $J \subseteq \{1, \ldots, n\}$, $|J| = r$.

If Γ is an r-consistent, k-fold system of projections as above, we will say that Γ is $C(k, r)$ or $\Gamma \models C(k, r)$.

This means that a k-fold system, Γ, over $\mathbb{A}_1, \ldots, \mathbb{A}_n$, is said to be $C(k, r)$ if every k-tuple in Γ can be extended to an r-tuple that is "consistent" with each k-fold projection.

Definition 3.2 (Bergman(k))**.** For $k > 2$, a variety V satisfies the Bergman(k) condition if whenever Γ, a k-fold system of projections over algebras $\mathbb{A}_1, \ldots, \mathbb{A}_n \in V$, is $C(k, k+1)$ then it is $C(k, r)$ for all $k < r \leq n$ as well. We will sometimes denote this as

$$C(k, k+1) \implies C(k, n).$$

We are now ready to return to our previous question; under what conditions will a k-fold system of projections, Γ, be of the form $\Gamma_{\mathbb{B}}(k)$ for some $\mathbb{B} \leq \prod_1^n \mathbb{A}_i$?

Claim. *A k-fold system of projections, Γ, is $C(k, n)$ if and only if $\Gamma = \Gamma_{\mathbb{B}}(k)$ for some $\mathbb{B} \leq \prod_1^n \mathbb{A}_i$.*

Proof. If Γ is $C(k, n)$ then let

$$\begin{aligned} \mathbb{B} &= \{\bar{a} = (a_1, \ldots, a_n) \mid \pi_I(\bar{a}) \in \Gamma_I \ \forall\ I \subseteq \{1, \ldots, n\},\ |I| = k\} \\ &= \bigcap_{|I|=k} \pi_I^{-1}(\Gamma_I). \end{aligned}$$

Clearly, $\pi_I(\mathbb{B}) \subseteq \Gamma_I$ for each $I \subseteq \{1, \ldots, n\}$, $|I| = k$ by definition.

Conversely, since Γ is $C(k, n)$, then for each $I \subseteq \{1, \ldots, n\}$, with $|I| = k$ and each $\bar{a}' \in \Gamma_I$, $\pi_I^{-1}(\bar{a}') \subseteq \mathbb{B}$. So $\pi_I(\mathbb{B}) \supseteq \Gamma_I$. Thus $\Gamma_{\mathbb{B}}(k) = \Gamma$. ∎

4 Known Results

In his paper, Bergman showed that if a variety, V, has a $(k+1)$-ary NU term, then any k-fold system of projections over n algebras from V is n-consistent so long as it is $(k+1)$-consistent. This means it satisfies Bergman(k). Notationally, we will write this as:

$$(k+1)\text{-ary NU} \implies \text{Bergman}(k). \tag{1}$$

Below is a rephrased statement of Bergman's Theorem from [3].

Theorem 2 (Bergman)**.** *For a variety, V, which has a $(k+1)$-ary NU term, $\mathbb{A}_1, \ldots, \mathbb{A}_n \in V$, and when Γ is a k-fold system of projections over the $\mathbb{A}_i$, and positive integer k; Γ is n-consistent if and only if it is $(k+1)$-consistent.*

Since Bergman first posed the converse to his theorem in [3], much work has been done in the area. In his paper he showed that in the case of $k = 2$

$$[C(2,3) \implies C(2,4)] \not\Longrightarrow \text{ternary NU}.$$

This leaves open the possibility that the stronger condition, Bergman(2), may be enough to satisfy the converse of (1).

We wish to better understand which varieties satisfy Bergman(k). To that end, this paper will explore whether there is a function, $f(k)$, such that

$$\text{Bergman}(k) \iff (f(k))\text{-ary NU}. \tag{2}$$

In [2], Barto, Kozik, Tan, and Valeriote have shown that

$$\text{Bergman}(k) \implies [C(k,k+1) \implies C(k,k+2)] \implies (2k)\text{-ary NU}, \tag{3}$$

A natural question is whether the converse

$$(2k)\text{-ary NU} \implies \text{Bergman}(k) \tag{4}$$

is true in V as well.

[2] also complemented this in the locally finite case by showing

$$(k+2)\text{-ary NU} \implies \text{Bergman}(k). \tag{5}$$

So for the case $k = 2$ we have that

$$(4)\text{-ary NU} \iff \text{Bergman}(2), \tag{6}$$

However, the result in (6) agrees with both $f(k) = k + 2$ and $f(k) = 2k$.

To help settle whether $f(k) = k + 2$ or $2k$ we will look at the case $k = 3$. We will try to show that (4) does not hold, thus eliminating $f(k) = 2k$ as a potential candidate. To accomplish this, we need a variety of algebras which has a 6-ary NU term, but no 5-ary NU term, and which does not satisfy Bergman(3); in particular, fails $C(3,4) \implies C(3,5)$. We had two strategies to accomplish this goal. The first was to computationally construct a system of projections with the necessary properties, ensuring they would be $C(3,4)$, then use software to test if the system were $C(3,5)$. The second was a direct, algebraic approach. For the sake of simplicity and without loss of generality, from this point forward we will assume that for any $\mathbb{P} = \mathbb{A}_1 \times \ldots \times \mathbb{A}_n$, $\mathbb{A}_1 = \ldots = \mathbb{A}_n = \mathbb{A}$ so that $\mathbb{P} = \mathbb{A}^n$, unless stated otherwise.

5 Computational Attempts

The main engine of our computational attempts was UACalc, a universal algebra calculator; see [6] for more information on that software. Beginning with a candidate algebra, $\mathbb{A}$, the four generated free algebra would be constructed, $\mathbb{F}_A(\{a,b,c,d\})$; where $\{a,b,c,d\}$ is a set of free generators. In order to build a system of 3-fold projections over $\mathbb{A}^5$ that is guaranteed to be $C(3,4)$, we built copies of the subpower of the cube of $\mathbb{F}_A(\{a,b,c,d\})$ generated by, X that consists of all triples of distinct members of $\{a,b,c,d\}$. See Lemma 1 for a proof that this will indeed create a system that is $C(3,4)$. The proof lifts very simply to the general case. With this system of projections in hand, we now use software developed by Valeriote to test if the system is $C(3,5)$. It is important to note that an algebra with many, or high arity, basic operations will result in a large free algebra.

We then spent time and effort to search for candidate algebras to apply the above construction to. Each step in the above construction drastically increases the size of the resulting algebra, so we needed to be judicious in our choice of candidates. To keep things as simple as possible, we looked for algebras with the smallest universe possible while having as few basic operations as necessary. This led us to 2-element algebras. Luckily, the class of 2-element algebras has been completely categorized. Emil Post compiled a complete list of all 2-element algebras and their finite bases in what is now called, Post's Lattice. Figure 2 below shows Post's Lattice while Table 1 contains the descriptions and bases of each 2-element algebra, up to equivalence. Looking through this list we see that the family of algebras $S_{10}^n = \langle \{0,1\},\ x \wedge (y \vee z),\ h_n \rangle$, S_{10}^5 in particular, suits our needs.

We can also construct algebras with NU terms of a specific arity by using the findings in [5]. They show that if $V_0 = \text{HSP}(\mathbb{A})$ and $V_1 = \text{HSP}(\mathbb{B})$ are idempotent varieties of the same type; where V_0 and V_1 have m and n-ary NU terms respectively, then $V = \text{HSP}(\mathbb{A} \times \mathbb{B})$ will have an $(m+n-1)$-ary NU term. Using this, we constructed the following candidate algebras:

1. $\mathbb{A} = \mathbb{A}_0 \times \mathbb{A}_1$ where;

$$\mathbb{A}_0 = \langle \{0,1\},\ x \wedge y \wedge z \wedge w,\ maj(x,y,z) \rangle$$

$$\mathbb{A}_1 = \langle \{0,1\},\ maj(x,y,z,w),\ x \wedge y \wedge z \rangle$$

2. $\mathbb{B} = \mathbb{B}_0 \times \mathbb{B}_1 \times \mathbb{B}_2$, where;

$$\mathbb{B}_0 = \langle \{0,1\},\ maj(x,y,z),\ x \vee y \vee z,\ x \wedge y \rangle$$

$$\mathbb{B}_1 = \langle \{0,1\},\ x \vee y \vee z,\ maj(x,y,z),\ x \vee y \rangle$$

$$\mathbb{B}_2 = \langle \{0,1\},\ x \vee y \vee z,\ x \vee y \vee z,\ x \wedge y \rangle$$

3. $\mathbb{C} = \mathbb{C}_0 \times \mathbb{C}_1 \times \mathbb{C}_2 \times \mathbb{C}_3$, where;

$$\mathbb{C}_0 = \langle \{0,1\},\ x \wedge y,\ x \vee y,\ x \vee y,\ x \vee y \rangle$$

$$\mathbb{C}_1 = \langle \{0,1\},\ x \vee y,\ x \wedge y,\ x \vee y,\ x \vee y \rangle$$

$$\mathbb{C}_2 = \langle \{0,1\},\ x \vee y,\ x \vee y,\ x \wedge y,\ x \vee y \rangle$$

$$\mathbb{C}_0 = \langle \{0,1\},\ x \vee y,\ x \vee y,\ x \vee y,\ x \wedge y \rangle$$

By the results in [5], each of these algebras has a 6-ary NU term and we verified that they all had no 5-ary NU term. Unfortunately each time the direct product of two algebras is taken, the universe of the resulting algebra increases multiplicatively in the sizes of the factors. For example, $|C| = 2^4 = 16$.

For all of the candidate algebras that were considered, the systems generated by them using the generating set, $X = \{\text{distinct triples of } \{a,b,c,d\}\}$, were far too large to be feasible. We then constructed a collection of $C(3,4)$ generating sets, one for each 3-fold projection, which were far smaller. Even so, the results were inconclusive. At this point, the computational route was abandoned.

6 Direct Approach

Since the computational approach did not yield fruitful results, we changed tactics and attempted to directly show that

$$\text{6-ary NU} \not\Longrightarrow \text{Bergman}(3) \tag{7}$$

by finding an algebra that has a 6-ary NU term but does not satisfy Bergman(3). However, before attempting to show (7), we have shown a simpler result that we hope will lift to the more complicated case. Once again, we turn to Post's Lattice and the S_{10}^n family of algebras, to find our counter-example. Note that if $n > m$, then $S_{10}^n < S_{10}^m$ with S_{10} at the bottom of the chain. This means that term operations of S_{10}^n are also term operations of S_{10}^m. So, if we can show that S_{10} is not $C(3,5)$, then we hope the proof will lift to S_{10}^6.

6.1 Constructing the System of Projections

Let $\mathbb{S}_{10} = (\{0,1\}, x \wedge (y \vee z))$ be an algebra, $V = HSP(\mathbb{S}_{10})$ the variety that it generates, and $\mathbb{F}_{10} = \mathbb{F}_V(\{a,b,c,d\})$ the four generated free algebra on V. See Figure 1 below for a sketch of $\mathbb{F}_{10}$, considered as a partially ordered set. Let $\mathbb{P} = \mathbb{A}_1 \times \ldots \times \mathbb{A}_5$, where each $\mathbb{A}_i = \mathbb{F}_{10}$ so that $\mathbb{P} = (\mathbb{F}_{10})^5$. Let $X = \{(x_0, x_1, x_2) \mid x_i \in \{a,b,c,d\}$ and $i \neq j \implies x_i \neq x_j\}$. For every $(i,j,k) \in \{1,\ldots,5\}^3$, i,j,k distinct, let $\mathbb{P}_{ijk} = \mathrm{Sg}_{{\mathbb{F}_{10}}^3}(X)$, and let $X_{ijk} = X$. Now let $G = \{X_{ijk}\}_{(i,j,k)}$. Finally, let $\Gamma = \{\mathbb{P}_{ijk}\}_{(i,j,k)}$, so that Γ is a system of 3-fold projections over $\mathbb{P}$.

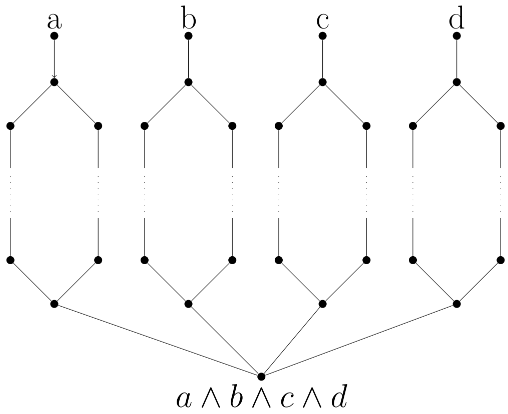

FIGURE 1: Sketch of $\mathbb{F}_{10}$

Lemma 1. Γ *is* $C(3,4)$

Proof. By construction, any 3-tuple in G is of the form (g_i, g_j, g_k) with $g_i \neq g_j \neq g_k$. It is then easy to see that a 4-tuple which extends this 3-tuple is (g_i, g_j, g_k, g_l) where g_l is the generator which did not appear in $\{g_i, g_j, g_k\}$. So clearly, on the level of sets, G is $C(3,4)$.

By symmetry, to show that Γ is $C(3,4)$, it is enough to consider only the coordinates 1,2,3,4.

Let $(\alpha, \beta, \gamma) \in \mathbb{P}_{123}$. We must show there is a $\delta \in \mathbb{A}_4$ with:

$$(\alpha, \beta, \delta) \in \mathbb{P}_{124} \tag{8}$$
$$(\alpha, \gamma, \delta) \in \mathbb{P}_{134} \tag{9}$$
$$(\beta, \gamma, \delta) \in \mathbb{P}_{234} \tag{10}$$

There exists a term, $t_{123}(\bar{x}_1, \ldots, \bar{x}_k)$ and generators $\bar{g}_1, \ldots, \bar{g}_k$, where $\bar{x}_i$ and $\bar{g}_i$ are 3-tuples, with

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = t_{123} \begin{pmatrix} g_{11}, & \cdots & , g_{k1} \\ g_{12}, & \cdots & , g_{k2} \\ g_{13}, & \cdots & , g_{k3} \end{pmatrix} \tag{11}$$

Since G is $C(3, 4)$, there is $u_i \in \mathbb{A}_4$, $i = 1, \ldots, k$, such that $(\bar{g}_i, u_i)$ is consistent with G. Let $d = t_{123}(u_1, \ldots, u_k)$, so that

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = t_{123} \begin{pmatrix} g_{11}, & \cdots & , g_{k1} \\ g_{12}, & \cdots & , g_{k2} \\ g_{13}, & \cdots & , g_{k3} \\ u_1 & \ldots & u_k \end{pmatrix} \tag{12}$$

,

$$(\alpha, \beta, \delta) \in \mathbb{P}_{124} \text{ by } t_{123} \begin{pmatrix} g_{11}, & \cdots, & g_{k1} \\ g_{12}, & \cdots & , g_{k2} \\ u_1, & \cdots & , u_k \end{pmatrix} \tag{13}$$

$$(\alpha, \gamma, \delta) \in \mathbb{P}_{124} \text{ by } t_{123} \begin{pmatrix} g_{11}, & \cdots & , g_{k1} \\ g_{13}, & \cdots & , g_{k3} \\ u_1, & \cdots & , u_k \end{pmatrix} \tag{14}$$

and

$$(\beta, \gamma, \delta) \in \mathbb{P}_{124} \text{ by } t_{123} \begin{pmatrix} g_{12}, & \cdots & , g_{k2} \\ g_{13}, & \cdots & , g_{k3} \\ u_1, & \cdots & , u_k \end{pmatrix} \tag{15}$$

and so (α, β, γ) has been consistently extended to a tuple over the coordinates $\{1, 2, 3, 4\}$. □

Theorem 3. $\mathbf{\Gamma}$ *is not* $C(3, 5)$

We will prove Theorem 3 by identifying a triple (α, β, γ) over $\mathbb{A}_1 \times \mathbb{A}_2 \times \mathbb{A}_3$ that can not be extended to a consistent 5-tuple. i.e. we will show that there are no $u \in \mathbb{A}_4$, $v \in \mathbb{A}_5$ such that $(\alpha, \beta, \gamma, u, v)$ is consistent. To do this we will need to prove two lemmas.

Lemma 2. *For a fixed* (i, j, k), *if* $(x_0, x_1, x_2) \in \mathbb{P}_{ijk}$ *then* $\exists$ *generators,* g_0, g_1, g_2 *distinct in* $\{a, b, c, d\}$ *with each* $x_i \leq g_i$.

Proof. Clearly, for any triple of distinct generators we have that $(g_i, g_j, g_k) \leq (g_i, g_j, g_k)$. Let $\mathbb{T} = \{(x_0, x_1, x_2) \in \ {\mathbb{F}_{10}}^3 \mid \exists\ g_i$, distinct with $x_i \leq g_i$, for $i = 0, 1, 2\}$, note that (g_i, g_j, g_k) belongs to $\mathbb{T}$.

Recall that for algebras $\mathbb{A}, \mathbb{B}$; if $\mathbb{B} \leq \mathbb{A}$ and $X \subseteq B$, then $\mathrm{Sg}(X) \leq \mathbb{B} \leq \mathbb{A}$. Note that $X_{ijk} \subseteq T$. So it is enough to show that $\mathbb{T}$ is a subalgebra of ${\mathbb{F}_{10}}^3$. Let

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} \in \ \mathbb{T} \tag{16}$$

with $x_i \leq g_i$ for distinct generators g_0, g_1, and g_2. Then

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \wedge \left[\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} \vee \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} \right] \leq \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \tag{17}$$

$$\leq \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix} \tag{18}$$

which shows that $\mathbb{T}$ is closed under the basic operations of $\mathbb{F}_{10}$ and so is a subalgebra of $\mathbb{F}_{10}^3$. □

Lemma 3. *Let (g_0, g_1, g_2, g_3) be a permutation of $\{a, b, c, d\}$ and $s, t \in \mathbb{F}_{10}$. If $(g_0, s, t) \in \mathbb{P}_{ijk}$, then $g_1 \wedge g_2 \wedge g_3 \leq s$ and $g_1 \wedge g_2 \wedge g_3 \leq t$.*

Proof. We will, prove this by induction on the length of the shortest term that produced (g_0, s, t) from generators. For the base case, let $\bar{g}$ be any triple of distinct generators. By symmetry, suppose $\bar{g} = (a, b, c)$. Clearly $b \wedge c \wedge d \leq b$ and $b \wedge c \wedge d \leq c$. Now suppose

$$\bar{h} = \begin{pmatrix} g_0 \\ s \\ t \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \wedge \left[\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} \vee \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} \right] \tag{19}$$

and assume that the result holds for triples that arise from shorter terms. Then $g_0 = x_0 \wedge (y_0 \vee z_0) \implies x_0 = g_0$ and either $y_0 = g_0$ or $z_0 = g_0$. By symmetry, suppose $y_0 = g_0$. Then,

$$\begin{pmatrix} g_0 \\ s \\ t \end{pmatrix} = \begin{pmatrix} g_0 \\ x_1 \\ x_2 \end{pmatrix} \wedge \left[\begin{pmatrix} g_0 \\ y_1 \\ y_2 \end{pmatrix} \vee \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} \right] \tag{20}$$

By induction, the hypothesis holds for (g_0, x_1, x_2) and (g_0, y_1, y_2). So

$$g_1 \wedge g_2 \wedge g_3 \leq x_1 \text{ and } g_1 \wedge g_2 \wedge g_3 \leq x_2 \tag{21}$$
$$\text{and } g_1 \wedge g_2 \wedge g_3 \leq y_1 \text{ and } g_1 \wedge g_2 \wedge g_3 \leq y_2 \tag{22}$$

$$\implies g_1 \wedge g_2 \wedge g_3 \leq x_1 \wedge (y_1 \vee z_1) = s \tag{23}$$
$$g_1 \wedge g_2 \wedge g_3 \leq x_2 \wedge (y_2 \vee z_2) = t \tag{24}$$

as required. □

6.2 Proving Theorem 3

We now have what we need to prove Theorem 3. Suppose the triple (a, b, c) over $\mathbb{P}_{123}$ can be extended consistently to (a, b, c, u, v). If we apply Lemma 2 to $(a, b, u) \in \mathbb{P}_{124}$, $(a, c, u) \in \mathbb{P}_{134}$, and $(b, c, u) \in \mathbb{P}_{234}$ we see that either $u \leq d$ or $u \leq a \wedge b \wedge c$ since,

$$(a, b, u) \in \mathbb{P}_{124} \implies u \leq c \text{ or } u \leq d \tag{25}$$
$$(a, c, u) \in \mathbb{P}_{134} \implies u \leq b \text{ or } u \leq d \tag{26}$$
$$(b, c, u) \in \mathbb{P}_{234} \implies u \leq a \text{ or } u \leq d \tag{27}$$

Suppose $u \not\leq d$, then $u \leq a, u \leq b$, and $u \leq c \implies u \leq a \wedge b \wedge c$. So

$$u \leq d \text{ or } u \leq a \wedge b \wedge c \tag{28}$$

Similarly,

$$v \leq d \text{ or } v \leq a \wedge b \wedge c \tag{29}$$

We now apply Lemma 3 to (a, u, v), (b, u, v), and (c, u, v) from $\mathbb{P}_{145}$, $\mathbb{P}_{245}$, and $\mathbb{P}_{345}$ respectively. From this we see that

$$u \geq b \wedge c \wedge d \text{ and } a \wedge c \wedge d \text{ and } a \wedge b \wedge d \tag{30}$$
$$v \geq b \wedge c \wedge d \text{ and } a \wedge c \wedge d \text{ and } a \wedge b \wedge d \tag{31}$$

Combining (28) and (30) along with (29) and (31) we see that,

$$u \leq d \text{ and } u \nleq a, b \textit{ or } c \tag{32}$$
$$v \leq d \text{ and } v \nleq a, b \textit{ or } c \tag{33}$$

since, if $u \geq b \wedge c \wedge d$ and $a \wedge c \wedge d$ and $a \wedge b \wedge d$ then $u \nleq a \wedge b \wedge c$. Similarly for v.

Which means, in the lattice of $\mathbb{F}_{10}$, both u and v lie below d, but not below a, b, or c. This means that both u and v can not be the meet or join of any generator other than d, i.e. $u = d$ and $v = d$. Then $(a, b, c, u, v) = (a, b, c, d, d)$ which is clearly not a 5-consistent tuple, since consistency would fail for any projection involving $\mathbb{A}_4$ and $\mathbb{A}_5$.

7 Conclusion

Bergman left the converse of his theorem open in [3], namely: if a system of projections over algebras in a variety V satisfies his consistency condition then does V have a $(k+1)$-ary NU term. This work does not settle that question, however it does shed light on the possibility that the Bergman(k) condition is equivalent to the existence of an NU term, the arity of which is based on k. The goal of this work was to narrow down the potential list of functions, $f(k)$ for which,

$$\text{Bergman}(k) \iff (f(k))\text{-ary NU}.$$

A likely candidate for f was $f(k) = 2k$, since it agrees with known results from [2]. Most of our efforts were focused on eliminating this function as a possibility, which required us to examine the case $k = 3$. This necessitated finding a variety which has a 6-ary, but no 5-ary, NU term and which does not satisfy Bergman(3). To that end, this work shows a simpler result, the proof of which we hope will lift to the more complicated case. We have found an algebra, $\mathbb{S}_{10}$, in which the variety it generates does not satisfy Bergman(3). Note that $\mathbb{S}_{10}$ does not have a 6-ary NU term, but the algebra $\mathbb{S}_{10}^5$ does, and the clone of $\mathbb{S}_{10}$ is contained in that of $\mathbb{S}_{10}^5$. To further our result, all that would be required is to expand the proofs of Lemmas 1, 2, and, 3 to ensure that they agree on the 6-ary NU basic operation.

8 Figures & Tables

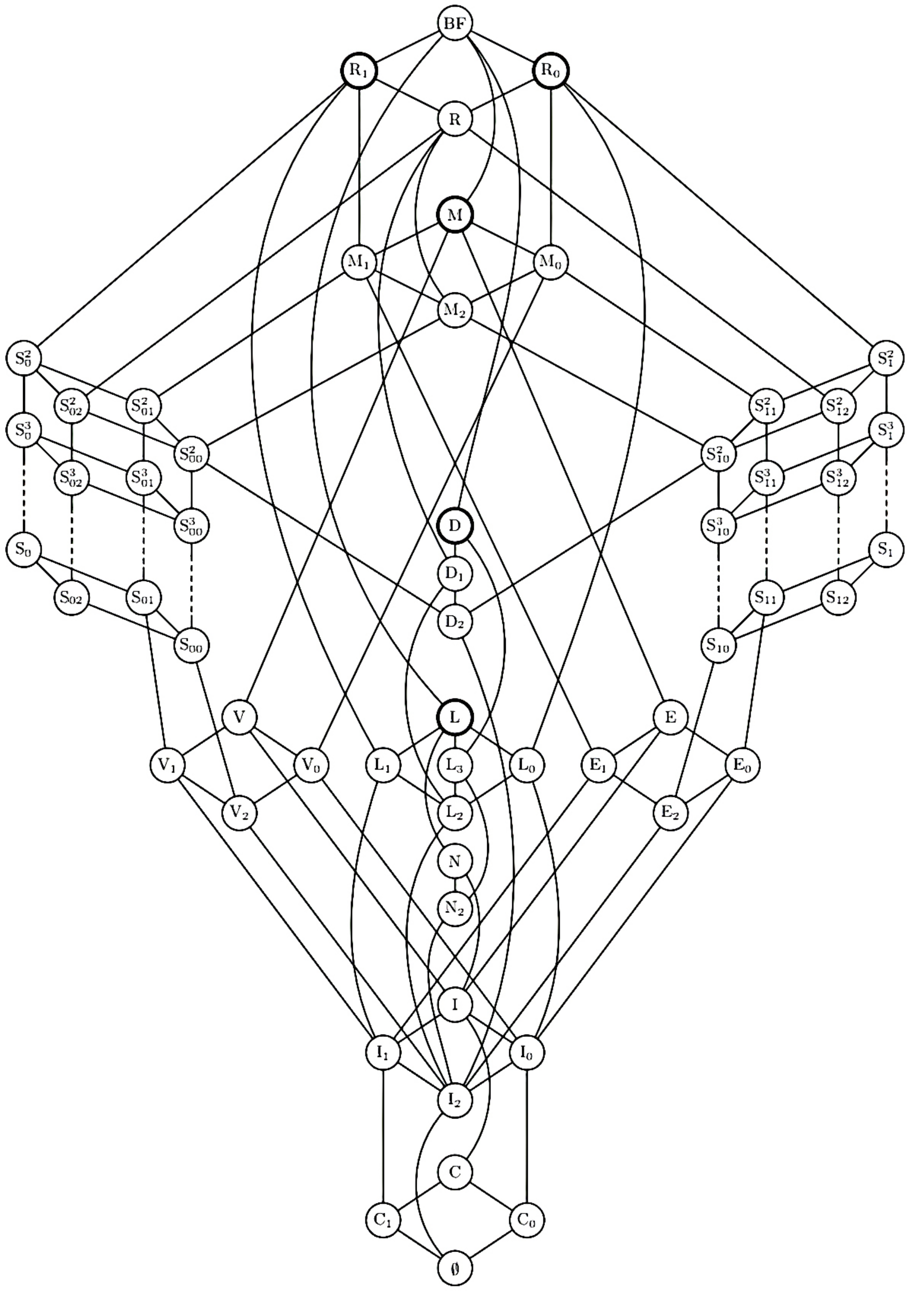

FIGURE 2: Graph of all closed classes of Boolean functions.

Class	Definition	Base(s)
BF	all Boolean functions	$\{\mathrm{AND}, \mathrm{NOT}\}$, $\{\mathrm{OR}, \mathrm{NOT}\}$, $\{\mathrm{NAND}\}$
R_0	$\{ f \in \mathrm{BF} \mid f$ is 0-reproducing $\}$	$\{\mathrm{AND}, \mathrm{XOR}\}$
R_1	$\{ f \in \mathrm{BF} \mid f$ is 1-reproducing $\}$	$\{\mathrm{OR}, x \oplus y \oplus 1\}$
R	$\mathrm{R}_1 \cap \mathrm{R}_0$	$\{\mathrm{OR}, x \wedge (y \oplus z \oplus 1)\}$
M	$\{ f \in \mathrm{BF} \mid f$ is monotonic $\}$	$\{\mathrm{AND}, \mathrm{OR}, \mathrm{c}_0, \mathrm{c}_1\}$
M_1	$\mathrm{M} \cap \mathrm{R}_1$	$\{\mathrm{AND}, \mathrm{OR}, \mathrm{c}_1\}$
M_0	$\mathrm{M} \cap \mathrm{R}_0$	$\{\mathrm{AND}, \mathrm{OR}, \mathrm{c}_0\}$
M_2	$\mathrm{M} \cap \mathrm{R}$	$\{\mathrm{AND}, \mathrm{OR}\}$
S_0^n	$\{ f \in \mathrm{BF} \mid f$ is 0-separating of degree $n \}$	$\{\mathrm{IMP}, \mathrm{dual}(h_n)\}$
S_0	$\{ f \in \mathrm{BF} \mid f$ is 0-separating $\}$	$\{\mathrm{IMP}\}$
S_1^n	$\{ f \in \mathrm{BF} \mid f$ is 1-separating of degree $n \}$	$\{x \wedge \overline{y}, h_n\}$
S_1	$\{ f \in \mathrm{BF} \mid f$ is 1-separating $\}$	$\{x \wedge \overline{y}\}$
S_{02}^n	$\mathrm{S}_0^n \cap \mathrm{R}$	$\{x \vee (y \wedge \overline{z}), \mathrm{dual}(h_n)\}$
S_{02}	$\mathrm{S}_0 \cap \mathrm{R}$	$\{x \vee (y \wedge \overline{z})\}$
S_{01}^n	$\mathrm{S}_0^n \cap \mathrm{M}$	$\{\mathrm{dual}(h_n), \mathrm{c}_1\}$
S_{01}	$\mathrm{S}_0 \cap \mathrm{M}$	$\{x \vee (y \wedge z), \mathrm{c}_1\}$
S_{00}^n	$\mathrm{S}_0^n \cap \mathrm{R} \cap \mathrm{M}$	$\{x \vee (y \wedge z), \mathrm{dual}(h_n)\}$
S_{00}	$\mathrm{S}_0 \cap \mathrm{R} \cap \mathrm{M}$	$\{x \vee (y \wedge z)\}$
S_{12}^n	$\mathrm{S}_1^n \cap \mathrm{R}$	$\{x \wedge (y, \overline{z}), h_n\}$
S_{12}	$\mathrm{S}_1 \cap \mathrm{R}$	$\{x \wedge (y \vee \overline{z}\}$
S_{11}^n	$\mathrm{S}_1^n \cap \mathrm{M}$	$\{h_n, \mathrm{c}_0\}$
S_{11}	$\mathrm{S}_1 \cap \mathrm{M}$	$\{x \wedge (y \vee z), \mathrm{c}_0\}$
S_{10}^n	$\mathrm{S}_1^n \cap \mathrm{R} \cap \mathrm{M}$	$\{x \wedge (y \vee z), h_n\}$
S_{10}	$\mathrm{S}_1 \cap \mathrm{R} \cap \mathrm{M}$	$\{x \wedge (y \vee z)\}$
D	$\{ f \mid f$ is self-dual $\}$	$\{x\overline{y} \vee x\overline{z} \vee \overline{y} \wedge \overline{z}\}$
D_1	$\mathrm{D} \cap \mathrm{R}$	$\{xy \vee x\overline{z} \vee y\overline{z}\}$
D_2	$\mathrm{D} \cap \mathrm{M}$	$\{xy \vee yz \vee xz\}$
L	$\{ f \mid$ there exists a formula of the form $c_0 \oplus c_1x_1 \oplus \ldots \oplus c_nx_n$ where c_i are constants for all $0 \leq i \leq n$, that describes $f^n \}$	$\{\mathrm{XOR}, \mathrm{c}_1\}$
L_0	$\mathrm{L} \cap \mathrm{R}_0$	$\{\mathrm{XOR}\}$
L_1	$\mathrm{L} \cap \mathrm{R}_1$	$\{\mathrm{EQ}\}$
L_2	$\mathrm{L} \cap \mathrm{R}$	$\{x \oplus y \oplus z\}$
L_3	$\mathrm{L} \cap \mathrm{D}$	$\{x \oplus y \oplus z \oplus \mathrm{c}_1\}$
V	$\{ f \mid$ there exists a formula of the form $c_0 \vee c_1x_1 \vee \ldots \vee c_nx_n$ where c_i are constants for all $0 \leq i \leq n$, that describes $f^n \}$	$\{\mathrm{OR}, \mathrm{c}_0, \mathrm{c}_1\}$
V_0	$[\{\mathrm{OR}\}] \cup [\{\mathrm{c}_0\}]$	$\{\mathrm{OR}, \mathrm{c}_0\}$
V_1	$[\{\mathrm{OR}\}] \cup [\{\mathrm{c}_1\}]$	$\{\mathrm{OR}, \mathrm{c}_1\}$
V_2	$[\{\mathrm{OR}\}]$	$\{\mathrm{OR}\}$
E	$\{ f \mid$ there exists a formula of the form $c_0 \wedge (c_1 \vee x_1) \wedge \ldots \wedge (c_n \vee x_n)$ where c_i are constants for all $0 \leq i \leq n$, that describes $f^n \}$	$\{\mathrm{AND}, \mathrm{c}_0, \mathrm{c}_1\}$
E_0	$[\{\mathrm{AND}\}] \cup [\{\mathrm{c}_0\}]$	$\{\mathrm{AND}, \mathrm{c}_0\}$
E_1	$[\{\mathrm{AND}\}] \cup [\{\mathrm{c}_1\}]$	$\{\mathrm{AND}, \mathrm{c}_1\}$
E_2	$[\{\mathrm{AND}\}]$	$\{\mathrm{AND}\}$
N	$[\{\mathrm{NOT}\}] \cup [\{\mathrm{c}_0\}] \cup [\{\mathrm{c}_1\}]$	$\{\mathrm{NOT}, \mathrm{c}_1\}$, $\{\mathrm{NOT}, \mathrm{c}_0\}$
N_2	$[\{\mathrm{NOT}\}]$	$\{\mathrm{NOT}\}$
I	$[\{\mathrm{ID}\}] \cup [\{\mathrm{c}_1\}] \cup [\{\mathrm{c}_0\}]$	$\{\mathrm{ID}, \mathrm{c}_0, \mathrm{c}_1\}$
I_0	$[\{\mathrm{ID}\}] \cup [\{\mathrm{c}_0\}]$	$\{\mathrm{ID}, \mathrm{c}_0\}$
I_1	$[\{\mathrm{ID}\}] \cup [\{\mathrm{c}_1\}]$	$\{\mathrm{ID}, \mathrm{c}_1\}$
I_2	$[\{\mathrm{ID}\}]$	$\{\mathrm{ID}\}$
C	$[\{\mathrm{c}_1\}] \cup [\{\mathrm{c}_0\}]$	$\{\mathrm{c}_1, \mathrm{c}_0\}$
C_0	$[\{\mathrm{c}_0\}]$	$\{\mathrm{c}_0\}$
C_1	$[\{\mathrm{c}_1\}]$	$\{\mathrm{c}_1\}$

TABLE 1: List of all closed classes of all Boolean functions along with bases. Here, $h_n := \bigvee_i x_1 \wedge \ldots \wedge x_{i-1} \wedge x_{i+1} \wedge \ldots \wedge x_{n+1}$.

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