A Review of "Commutator Theory for Congruence Modular Varieties"

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In this book, Freese and McKenzie present a detailed overview of the recently developed commutator theory for congruence modular varieties. This commutator is a generalization of the familiar commutator operation on normal subgroups of a group and shares many of the properties of this operation. The theory was developed quickly in the late 70's after the publication of the book **Mal'cev Varieties** by J. D. H. Smith. In this book Smith showed how a commutator operation could be defined on the congruence lattice of any algebra in a congruence permutable variety. The mathematicians J. Hagemann and C. Herrmann then proceeded to extend Smith's work to the general setting of congruence modular varieties. The authors of the book under review also played a role in the development of this theory, as well as proving several deep algebraic results using it.

The book opens with a brief introduction to the commutator for groups and rings and then moves on to the mandatory chapter on basic universal algebra. In this chapter two interesting "geometric" consequences of congruence modularity are proved. These are the so called Shifting Lemma and the Cube Lemma discovered by H. P. Gumm.

In chapter 3, three different commutator operations are defined for general algebraic structures, the first defining the commutator $C^{(\mathcal{V})}$ of a variety $\mathcal{V}$ as the largest binary operation defined on $\operatorname{Con}\mathbf{A}$, for every $\mathbf{A}$ in $\mathcal{V}$, such that $C^{(\mathcal{V})}$ is submeet and if π is the kernel of some surjective homomorphism $f : \mathbf{A} \longrightarrow \mathbf{B}$ in $\mathcal{V}$, then $C^{(\mathcal{V})}(\theta, \psi) \vee \pi = f^{-1}C^{(\mathcal{V})}(f(\theta \vee \pi), f(\psi \vee \pi))$ for any congruences θ and ψ of $\mathbf{A}$.

The other commutator operations are given in a more concrete way using the notion of centrality and it is from these definitions that the important properties of the modular commutator are proved in chapter 4. It is shown that the several definitions of the commutator give identical operations in the congruence modular setting and that besides being submeet and well behaved under homomorphisms, the modular commutator is monotone, symmetric and distributes over arbitrary joins. Finally in this chapter the commutator as defined is shown to coincide with the one given originally by Hagemann and Herrmann.

With a useful commutator at hand, Freese and McKenzie proceed with the development of the theory in chapters 5 through 10. The center of an algebra is defined as is the property of being Abelian. The related notions of solvability and nilpotency are also defined. In chapter 5 it is shown how an Abelian group structure can be naturally placed on any block of an Abelian congruence. This theme is picked up again in chapter 9 where rings are associated with any congruence modular variety in such a way that direct sums of these congruence blocks become modules over one of the rings. In the special case where the variety consists of only Abelian algebras, the variety actually is polynomially equivalent to the variety of all modules over one of the rings. This affine behaviour has been exploited by McKenzie and others to prove several interesting theorems.

In chapter 6 permutability is investigated and it is shown that every solvable congruence permutes with every other congruence in an algebra which generates a congruence modular variety. Gumm's characterization of congruence modularity is proved here as well as one due to S. Tschantz. Nilpotency is considered in chapter 7. Here it is shown that nilpotent algebras have regular and uniform congruences and that every such algebra supports a nilpotent loop structure.

The next chapter introduces several congruence identities. For example it is proved that a congruence modular variety is congruence distributive if and only if the commutator of any two congruences in any algebra in the variety is equal to the meet of these congruences, i.e., the variety satisfies the identity $[x, y] \approx x \wedge y$. The importance of another identity mentioned in this chapter is seen in chapter 10. There it is proved that if a congruence modular variety is residually small (i.e., there is a cardinal κ bounding the size of the subdirectly irreducible algebras in the variety) then the variety satisfies the identity $x \wedge [y, y] \approx [x \wedge y, y]$. If, in addition, the variety is finitely generated then satisfaction of the above identity is equivalent to being residually small.

Other properties of subdirectly irreducible algebras are looked at in this chapter, in particular a generalization of Jónsson's theorem is proved. This result is then specialized to the finitely generated case.

The closing chapters are devoted to several applications of the theory. In chapter 11 it is proved that the join of two residually small congruence modular varieties is residually small. Finite simple algebras in congruence modular varieties are looked at in chapter 12, as are minimal varieties. A sketch of a proof showing that there are infinitely many inequivalent lattice identities each having a Mal'cev condition is presented in chapter 13. In chapter 14 we find a generalization of Vaughan-Lee's finite basis result. It is proved that if $\mathbf{A}$ is a finite nilpotent algebra in a modular variety and is a direct product of algebras of prime power then $\mathbf{A}$ has a finite basis for its identities.

This book is packed with ideas and techniques that anyone interested in universal algebra should be aware of. As is evident from this review the modular commutator has become an important tool for researchers in this field. The authors have done a good job in organizing and presenting the material in this book and, thankfully, have provided very detailed proofs of most of the theorems under consideration. They have also included many exercises (and their solutions) which serve to introduce examples and other applications of the theory.