A Characterization of Decidable Locally Finite Varieties

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Abstract

We describe the structure of those locally finite varieties whose first order theory is decidable. A variety is a class of universal algebras defined by a set of equations. Such a class is said to be locally finite if every finitely generated member of the class is finite. It turns out that in order for such a variety to have a decidable theory it must decompose into the varietal product of three special kinds of varieties; a strongly Abelian variety; an affine variety; and a discriminator variety.

1 Introduction

Since the 1930’s, when precise notions of algorithm and decidability were introduced, the decidability or undecidability of many familiar varieties has been determined. For example, Tarski [21] proved that the theory of Boolean algebras is decidable and in a series of papers [20, 7, 25] it was shown that a variety of groups is decidable if and only if it contains no non-Abelian group.

In 1986 the authors, building on the work of Burris and McKenzie [3], Zamyatin [23, 24, 25] and others were able to give an algebraic description of the decidable locally finite varieties. In this paper we describe this structure and give a rough outline of the proof. Complete details can be found in [17]. One of the principle tools used in this work is the recently discovered theory

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called tame congruence theory. This theory was developed in the early 1980's by McKenzie and his student David Hobby. We will present a small fragment of this theory in section 3. The reader can consult [9] for more details.

By an algebra we mean simply any structure \( (A, f_i (i \in I) ) \) consisting of a nonvoid set \( A \) and a system of finitary operations \( f_i \) over \( A \). A variety, or equational class, is a class of similar algebras defined by some set of equations. A variety is called locally finite if every one of its finitely generated algebras is finite.

A variety \( \mathcal{V} \) is called decidable if and only if its first order theory is a recursive set of sentences. This means that there is an algorithm which will determine, for any given sentence \( \varphi \) in the language of \( \mathcal{V} \), whether \( \varphi \) is true for all members of \( \mathcal{V} \).

Usually, to establish the undecidability of a class of structures, another class known to be undecidable is semantically embedded into the first class. The class of finite graphs, \( \mathcal{G}_{fin} \), was shown to be undecidable by Lavrov in [13]. It turns out that if a locally finite variety fails to have the structure described in Theorem 2.14 then it allows a semantic embedding of this class \( \mathcal{G}_{fin} \) and hence is undecidable.

## 2 Special Kinds of Varieties

We introduce the families of Abelian varieties, strongly Abelian varieties, affine varieties, and discriminator varieties. These are the varieties that are needed in our description of the locally finite decidable varieties.

Given a set \( A \) and natural numbers \( n > 0 \) and \( i < n \), we can define the projection function \( p^n_i : A^n \to A \) by

\[
p^n_i(x_0, \ldots, x_{n-1}) = x_i \text{ for all } x_0, \ldots, x_{n-1} \in A.
\]

A clone on a set \( A \) is a set of finitary operations on \( A \) that contains the projections \( p^n_i \) for all \( 0 \leq i < n < \omega \), and is closed under composition of operations. By a polynomial clone on \( A \), we mean any clone on \( A \) that also contains all of the constant 0-ary operations on \( A \). (A general clone is not required by our definition to contain any 0-ary operations.)

Associated with any algebra \( A = (A, f^A (f \in \Phi)) \) are two important clones. The clone of term operations of \( A \), denoted \( \text{Clo} A \), is the clone on \( A \) generated by the set \( \{ f^A : f \in \Phi \} \) of basic operations of \( A \). An
n-ary operation $f$ on $A$ belongs to $\text{Clo}_A$ iff there exists a term $t$ in the language $L$ such that $f = t^A$—i.e., $f$ is the operation induced by the term $t$. The clone of polynomial operations of $A$, denoted $\text{Pol}_A$, is the clone on $A$ generated by the basic operations of $A$ along with all of the constant 0-ary operations on $A$. We write $\text{Clo}_nA$ for the set of all $n$-ary members of $\text{Clo}_A$ ($n$-ary term operations of $A$); and, similarly, we write $\text{Pol}_nA$ for the set of all $n$-ary members of $\text{Pol}_A$ ($n$-ary polynomial operations of $A$). It follows that if $p \in \text{Pol}_nA$, then for some $m \in \omega$ and $a_0, \ldots, a_{m-1} \in A$ and for some $f \in \text{Clo}_{m+n}A$, we have $p(\bar{x}) = f(\bar{x}, \bar{a})$ for all $\bar{x} \in A^n$ (where $\bar{a} = (a_0, \ldots, a_{m-1})$).

**Definition 2.1** Algebras $A$ and $B$ are said to be **polynomially equivalent** if they have the same universe and precisely the same polynomial operations, i.e., if $\text{Pol}_A = \text{Pol}_B$.

The concept of an Abelian algebra was introduced in the 1970’s and has played an important role in the study of both congruence modular varieties and locally finite varieties.

**Definition 2.2** Let $\alpha, \beta, \gamma \in \text{Con}_A$. We write $C(\alpha, \beta; \gamma)$, and say that $\alpha$ **centralizes** $\beta$ **modulo** $\gamma$, provided that the following condition holds:

For every $n \geq 1$, for every $f \in \text{Clo}_{n+1}A$, and for all $\langle a, b \rangle \in \alpha$ and $\langle c_1, d_1 \rangle, \ldots, \langle c_n, d_n \rangle \in \beta$ we have

$$f(a, c) \equiv_\gamma f(a, d) \iff f(b, c) \equiv_\gamma f(b, d).$$

**Definition 2.3** Let $A$ be any algebra. The **center** of $A$ is the binary relation $Z(A)$ defined by $\langle x, y \rangle \in Z(A)$ if and only if $\langle x, d \rangle \equiv f(x, y) \iff f(y, c) \equiv f(y, d)$. It is a congruence on $A$. The algebra $A$ is called **Abelian** iff $Z(A) = 1_A$, and called **centerless** iff $Z(A) = 0_A$.

**Definition 2.4** Let $A$ be any algebra, $\alpha, \beta, \gamma, \delta$ be congruences on $A$. 

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(1) If \( \alpha \leq \beta \), we say that \( \beta \) is **Abelian** over \( \alpha \) iff \( \text{C}(\beta, \beta; \alpha) \).

(2) If \( \delta \leq \gamma \), we say that \( \gamma \) is **solvable** over \( \delta \) iff there exists a finite chain of congruences \( \alpha_0 = \delta \leq \alpha_1 \leq \cdots \leq \alpha_n = \gamma \) with \( \alpha_{i+1} \) Abelian over \( \alpha_i \) for all \( i < n \).

(3) If \( \delta \leq \gamma \), we say that \( \gamma \) is **locally solvable** over \( \delta \) iff for every finitely generated subalgebra \( B \leq A \) and for the restricted congruences \( \delta|_B \) and \( \gamma|_B \) of \( B \), we have that \( \gamma|_B \) is solvable over \( \delta|_B \).

(4) We say that \( A \) is **solvable** iff \( 1_A \) is solvable over \( 0_A \); **locally solvable** iff \( 1_A \) is locally solvable over \( 0_A \).

We now strengthen the Abelian property in two mutually incompatible ways: to strongly Abelian algebras and to affine algebras.

**Definition 2.5** Let \( \alpha \leq \beta \) be congruences of an algebra \( A \). We say that \( \beta \) is **strongly Abelian** over \( \alpha \) iff for all \( n \geq 1 \), for all \( f \in \text{Clo}_{n+1}A \), and for all \( a \equiv b \equiv c \pmod{\beta} \) and \( (u_1, v_1), \ldots, (u_n, v_n) \in \beta \) we have

\[
    f(a, \bar{u}) \equiv_{\alpha} f(b, \bar{v}) \rightarrow f(c, \bar{u}) \equiv_{\alpha} f(c, \bar{v}).
\]

We say that \( A \) is **strongly Abelian** iff \( 1_A \) is strongly Abelian over \( 0_A \).

**Definition 2.6** Let \( \delta \leq \gamma \) be congruences of an algebra \( A \).

(1) We say that \( \gamma \) is **strongly solvable** over \( \delta \) iff there exists a finite chain of congruences \( \alpha_0 = \delta \leq \alpha_1 \leq \cdots \leq \alpha_n = \gamma \) such that \( \alpha_{i+1} \) is strongly Abelian over \( \alpha_i \) for all \( i < n \).

(2) We say that \( \gamma \) is **locally strongly solvable** over \( \delta \) iff for every finitely generated subalgebra \( B \leq A \) and for the restricted congruences \( \delta|_B \) and \( \gamma|_B \) of \( B \), we have that \( \gamma|_B \) is strongly solvable over \( \delta|_B \).

(3) The algebra \( A \) is said to be **strongly solvable** iff \( 1_A \) is strongly solvable over \( 0_A \); and is said to be **locally strongly solvable** iff \( 1_A \) is locally strongly solvable over \( 0_A \).

The proof of the following theorem can be found in [9].
THEOREM 2.7 For every locally finite variety $\mathcal{V}$, the class of all locally solvable algebras in $\mathcal{V}$, and the class of all locally strongly solvable algebras in $\mathcal{V}$, are varieties.

Two equivalence relations $\alpha$ and $\beta$ on a set $A$ are said to permute iff whenever $a \alpha b \beta c$ there exists some element $d$ such that $a \beta d \alpha c$. A. I. Mal'tsev [14] proved that a variety $\mathcal{V}$ has the property that every two congruences on any algebra in $\mathcal{V}$ permute iff there exists a term $t(x, y, z)$ in the language of $\mathcal{V}$ for which the equations $t(x, y, y) \approx x$ and $t(x, x, y) \approx y$ are valid in $\mathcal{V}$. When such a term exists, we say that $\mathcal{V}$ is Mal’tsev. An operation on a set $A$ that obeys these two equations on $A$ is called a Mal’tsev operation; also, an algebra having a term operation that obeys these equations is called a Mal’tsev algebra. A useful corollary of Mal’tsev’s result is that a variety $\mathcal{V}$ is Mal’tsev if and only if the free algebra $F_{\mathcal{V}}(3)$ has permuting congruences.

Definition 2.8 An algebra $A$ is called affine iff $A$ is polynomially equivalent with an algebra $M$ that is a module over a ring.

Definition 2.9

(1) If (P) is any one of the properties “Abelian”, “strongly Abelian”, “affine”, “locally solvable”, “locally strongly solvable” defined above, we say that a variety is (P) iff every algebra in the variety is (P).

(2) A variety $\mathcal{V}$ is said to be congruence-modular (or -distributive) iff the congruence lattice of each algebra in $\mathcal{V}$ is a modular (or distributive) lattice.

The basic results concerning affine algebras and affine varieties are fully detailed in Freese, McKenzie [8]. We describe these results here without giving proofs. Note that if $\mathcal{V}$ is affine, then, since the free algebra on three generators in $\mathcal{V}$ has a Mal’tsev term operation, it follows that $\mathcal{V}$ is Mal’tsev.

THEOREM 2.10 An algebra $A$ is affine if and only if it satisfies one of these conditions (which are equivalent).

(i) $A$ is Abelian and possesses a polynomial operation $p(x, y, z)$ obeying the equations $p(x, y, y) \approx x$ and $p(x, x, y) \approx y$. 

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A possesses a term operation \( p(x, y, z) \) obeying the above equations and such that for every basic operation \( f \) of \( A \), the equation

\[
p(f(\bar{x}), f(\bar{y}), f(\bar{z})) \approx f(p(x_0, y_0, z_0), \ldots, p(x_{n-1}, y_{n-1}, z_{n-1}))
\]

is valid in \( A \) (if \( f \) is \( n \)-ary).

**Theorem 2.11** A variety \( \mathcal{V} \) is affine if and only if it is congruence-modular and Abelian. If \( \mathcal{V} \) is affine then it is in fact Maltsev; and there exists a term \( t(x, y, z) \) in the language of \( \mathcal{V} \) and a ring \( R \) with unit such that every algebra in \( \mathcal{V} \) is polynomially equivalent with a unitary left \( R \)-module in which \( x - y + z = t(x, y, z) \), and every unitary left \( R \)-module is polynomially equivalent with an algebra in \( \mathcal{V} \).

The concept of a discriminator variety is in many respects the polar opposite of that of an Abelian variety. On any set \( U \) we can define an operation \( t_U(x, y, z) \) by stipulating that \( t_U(x, y, z) \) is \( z \) if \( x = y \), and is \( x \) if \( x \neq y \). This operation \( t_U \) is called the **ternary discriminator** on \( U \).

**Definition 2.12** A variety \( \mathcal{V} \) is called a **discriminator variety** iff there exists a term \( t(x, y, z) \) in the language of \( \mathcal{V} \) such that \( \mathcal{V} = \mathcal{V}(S) \) where \( S \) is the class of all \( A \in \mathcal{V} \) such that \( t^A = t_A \) (i.e., the term \( t \) defines the discriminator on the universe of \( A \)). Such a term \( t \) is called a **discriminator term** for \( \mathcal{V} \).

There is a very nice structure theory for discriminator varieties, the details of which can be found in Burris, Sankappanavar [5]. Several important facts about these varieties are given below without proof. An algebra \( A \) is called **hereditarily simple** iff \( |A| > 1 \) and every subalgebra \( B \leq A \) with more than one element is simple. A variety \( \mathcal{V} \) is called **arithmetical** iff \( \mathcal{V} \) is congruence-distributive and Maltsev.

**Theorem 2.13** Let \( \mathcal{V} \) be a discriminator variety with discriminator term \( t \). Then \( \mathcal{V} \) is an arithmetical variety. The equations

\[
t(x, y, y) \approx t(x, y, x) \approx t(y, y, x) \approx x
\]

are valid in \( \mathcal{V} \). Every algebra in \( \mathcal{V} \) is centerless; and every finite algebra in \( \mathcal{V} \) is isomorphic to a direct product of simple algebras. The following are equivalent for every \( A \in \mathcal{V} \): \( A \) is subdirectly irreducible; \( A \) is hereditarily simple; \( t^A = t_A \).
One last concept is needed before we can state our result. Varieties $V_1, \ldots, V_n$ in the same language $L$ are called independent iff there exists an $L$-term $t(x_1, \ldots, x_n)$ such that $V_i \models t \approx x_i$ for $i = 1, \ldots, n$. If $V_1, \ldots, V_n$ are independent, then every algebra $A$ in $V = V_1 \lor \cdots \lor V_n$ is isomorphic to a product $A_1 \times \cdots \times A_n$ with $A_1 \in V_1, \ldots, A_n \in V_n$ and the algebras $A_i$ are determined up to isomorphism. In this case, we write $V_1 \otimes \cdots \otimes V_n$ for the join variety $V = V_1 \lor \cdots \lor V_n$, and say that $V$ is the product of its subvarieties $V_1, \ldots, V_n$.

**Theorem 2.14** Let $V$ be any decidable locally finite variety. There exists a strongly Abelian variety $V_1$, an affine variety $V_2$, and a discriminator variety $V_3$ such that $V = V_1 \otimes V_2 \otimes V_3$. These three subvarieties of $V$ are uniquely determined and are all decidable.

### 3 Tame Congruence Theory

One of the key steps in the development of tame congruence theory was the realization that locally the behaviour of finite algebras is quite limited. This is made precise in the following definitions and theorems. The reader may wish to refer to [9] for further details and proofs. (For the basic theory of universal algebra, consult [5] or [16].)

**Definition 3.1** Let $A$ be a finite algebra and let $\alpha$ and $\beta$ be congruences of $A$.

1. We say that a function $f : A \to A$ collapses $\beta$ into $\alpha$ and write $f(\beta) \subset \alpha$ if $\langle f(a), f(b) \rangle \in \alpha$ for all $\langle a, b \rangle \in \beta$.

2. By a congruence quotient of $A$ we mean a pair $\langle \alpha, \beta \rangle$ of congruences of $A$ such that $\alpha < \beta$. A congruence quotient $\langle \alpha, \beta \rangle$ of $A$ is called a prime quotient iff $\beta$ covers $\alpha$ in $\text{Con} A$ (the congruence lattice of $A$). The relation of covering between two elements of $\text{Con} A$ is written $\alpha \prec \beta$.

3. Let $\langle \alpha, \beta \rangle$ be a congruence quotient of $A$ and let

$$U_A(\alpha, \beta) = \{ f(A) : f \in \text{Pol}_1 A \text{ and } f(\beta) \not\subset \alpha \}$$
and \( M_A(\alpha, \beta) \) be the set of all minimal members of \( U_A(\alpha, \beta) \) relative to the ordering of inclusion. A member of \( M_A(\alpha, \beta) \) is called an \( \langle \alpha, \beta \rangle \)-minimal set of \( A \).

**Definition 3.2** Let \( A \) be a finite algebra and suppose that \( \alpha \prec \beta \in \text{Con} A \).

By an \( \langle \alpha, \beta \rangle \)-trace in \( A \) we mean any set \( N \subset A \) such that for some \( U \in M_A(\alpha, \beta) \), \( N \subset U \) and \( N \) is of the form \( (x/\beta) \cap U \) for some \( x \in U \) such that \( (x/\alpha) \cap U \neq (x/\beta) \cap U \). The **body** and the **tail** of an \( \langle \alpha, \beta \rangle \)-minimal set \( U \) with respect to \( \langle \alpha, \beta \rangle \) are defined by

\[
\text{body} = \bigcup \{ \langle \alpha, \beta \rangle \text{-traces contained in } U \},
\]

\[
\text{tail} = U - \text{body}.
\]

For \( U \) a nonvoid subset of an algebra \( A \), we let \( (\text{Pol} A)|_U \) denote the set of all \( f|_U \) where \( f \in \text{Pol} A \) and \( U \) is closed under \( f \). The (non-indexed) algebra \( A|_U \) having universe \( U \) and fundamental operations \( (\text{Pol} A)|_U \) is called the **algebra induced by** \( A \) **on** \( U \).

In tame congruence theory we focus on the algebras induced on the minimal sets, bodies and traces of finite algebras. As the next theorem demonstrates, there are few possibilities for the algebraic structure induced on a trace by a finite algebra.

**Theorem 3.3** Let \( A \) be a finite algebra and let \( \langle \alpha, \beta \rangle \) be a prime quotient of \( A \). If \( N_1 \) and \( N_2 \) are \( \langle \alpha, \beta \rangle \)-traces then \( \alpha|_{N_i} \) is a congruence of \( A|_{N_i} \) for \( i = 1, 2 \) and the algebras \( (A|_{N_1})/(\alpha|_{N_1}) \) and \( (A|_{N_2})/(\alpha|_{N_2}) \) are isomorphic. Furthermore these quotient algebras are polynomially equivalent to exactly one algebra (up to isomorphism) from the following list:

1. a faithful \( G \)-set, for some finite group \( G \),
2. a vector space,
3. a two-element Boolean algebra,
4. a two-element lattice,
5. a two-element semilattice.
We say that the type of the prime quotient \( \langle \alpha, \beta \rangle \) is equal to \( i \) if the algebra \( (A|_{N_1})/(\alpha|_{N_1}) \) is polynomially equivalent to an algebra in the \( i \)th entry of this list. We denote this type by \( \text{typ}(\alpha, \beta) \).

**Definition 3.4**

(1) Let \( \langle \delta, \gamma \rangle \) be any congruence quotient of a finite algebra \( A \). We define \( \text{typ}\{\delta, \gamma\} \) to be the set

\[
\{\text{typ}(\alpha, \beta) : \delta \leq \alpha < \beta \leq \gamma\}.
\]

(2) For a finite algebra \( A \) we define \( \text{typ}\{A\} \) to be \( \text{typ}\{0_A, 1_A\} \).

(3) For a class \( K \) of algebras we define \( \text{typ}\{K\} \) to be the set

\[
\bigcup \{\text{typ}\{A\} : A \in K \text{ and } A \text{ is finite}\}.
\]

We say that a finite algebra **omits type** \( i \) if \( i \notin \text{typ}\{A\} \). A class \( K \) omits type \( i \) if every finite member of \( K \) does so.

**COROLLARY 3.5** Let \( A \) be a finite algebra and let \( \alpha \) and \( \beta \) be congruences of \( A \). \( \text{typ}\{\alpha, \beta\} \subseteq \{1, 2\} \) if and only if \( \beta \) is solvable over \( \alpha \).

One of the interesting aspects of the work of Hobby and McKenzie can be found in chapters 8 and 9 of [9]. There they show that certain Maltsev conditions (for locally finite varieties) are easily expressible in the language of tame congruence theory. For example, a locally finite variety \( V \) is \( n \)-permutable for some \( n \) if and only if \( V \) omits types 1, 4 and 5.

In chapter 11 of [9] the following theorem is proved.

**THEOREM 3.6** Let \( V \) be a locally finite decidable variety. Then \( V \) omits types 4 and 5.

### 4 A Sketch of the Proof

Throughout this section we fix a decidable locally finite variety \( V \). From Theorem 3.6 we know that \( \text{typ}\{V\} \subseteq \{1, 2, 3\} \). We first define three subvarieties, \( V_1, V_2 \) and \( V_3 \), of \( V \); and we prove that every member of \( V \) is a subdirect product of three algebras belonging to these subvarieties.
Definition 4.1

1. A **subdirect product** of the algebras $\langle B_i : i \in I \rangle$ is an algebra $A \leq \prod_{i \in I} B_i$ such that $A$ maps onto each of the algebras $B_i$ via the coordinate projection. An embedding $f : A \rightarrow \prod_{i \in I} B_i$ is called **subdirect** if $f(A)$ is a subdirect product of the $B_i$'s.

2. An **irredundant subdirect product** of $\langle B_i : i \in I \rangle$ is a subdirect product $A$ of $\langle B_i : i \in I \rangle$ such that for each $i \in I$, $A$ fails to be embedded in $\prod_{j \neq i} B_j$ via the natural projection.

3. A subdirect product $A$ of $\langle B_i : i \in I \rangle$ is called **direct** iff $A = \prod_{i \in I} B_i$.

4. An algebra $A$ is called **subdirectly irreducible** if and only if $|A| > 1$ and for every subdirect embedding $f : A \rightarrow \prod_{i \in I} B_i$, there is some $i \in I$ such that the composition of $f$ with the projection onto $B_i$ is an isomorphism between $A$ and $B_i$.

Equivalent to an algebra $A$ being subdirectly irreducible is the existence of a smallest nonzero congruence in $\text{Con} A$. We call this congruence (when it exists) the **monolith** of $A$.

Definition 4.2

For $1 \leq i \leq 3$, let $S_i$ be the class of all finite, subdirectly irreducible algebras $A \in V$ such that the type of $\langle 0_A, \beta \rangle$ is $i$, where $\beta$ is the monolith of $A$. We define $V_i$ to be the variety generated by $S_i$.

Here are three easy consequences of the definition.

**Theorem 4.3**

(i) Every finite subdirectly irreducible algebra in $V$ belongs to $V_1 \cup V_2 \cup V_3$.

(ii) Every locally solvable algebra in $V$ belongs to $V_1 \lor V_2$.

(iii) $V = V_1 \lor V_2 \lor V_3$; in fact, every algebra in $V$ is a subdirect product of three algebras $C_1, C_2$ and $C_3$ belonging, respectively, to $V_1, V_2$ and $V_3$. 
**Proof.** Since \( \mathcal{V} \) is decidable, then it follows from Theorem 3.6 that every finite, subdirectly irreducible algebra in \( \mathcal{V} \) belongs to \( S_1 \cup S_2 \cup S_3 \).

Let \( A \in \mathcal{V} \) be locally solvable. Since \( \mathcal{V} \) is locally finite, every finitely generated subalgebra of \( A \) is finite; and so \( A \) belongs to the variety generated by its finite subalgebras. Hence it suffices to prove that all finite subalgebras of \( A \) belong to \( \mathcal{V}_1 \lor \mathcal{V}_2 \). Let \( B \) be a finite subalgebra of \( A \). \( B \) is a subdirect product of subdirectly irreducible homomorphic images of \( B \), which are solvable, and hence have monoliths of types 1 or 2. Thus \( B \in \text{SP}_{\text{fin}}(S_1 \cup S_2) \), implying that \( B \in \mathcal{V}_1 \lor \mathcal{V}_2 \).

It follows easily from part (i) that every finite algebra in \( \mathcal{V} \) can be embedded into a product \( C_1 \times C_2 \times C_3 \) with \( C_i \in \mathcal{V}_i \). The class of algebras that can be so embedded is closed under the formation of ultraproducts and of subalgebras. The statement thus follows from the fact that every locally finite algebra can be embedded into an ultraproduct of its finite subalgebras.

With the three subvarieties \( \mathcal{V}_1 \), \( \mathcal{V}_2 \) and \( \mathcal{V}_3 \) defined, we now set out to describe the structure of each of them. The next lemmas are used to show that \( \mathcal{V}_3 \) is a discriminator variety.

**Lemma 4.4** Let \( F \) be a finite, subdirectly irreducible, centerless algebra in \( \mathcal{V} \). Every subalgebra of \( F \) having at least two elements is simple and non-Abelian.

**Lemma 4.5** Every irredundant subdirect product of finitely many algebras in \( S_3 \) is direct.

**Theorem 4.6** \( \mathcal{V}_3 \) is a discriminator variety.

**Proof.** By Lemma 4.4, \( \mathcal{V}_3 \) is generated by the class of finite, simple, non-Abelian algebras in \( \mathcal{V}_3 \). Thus if \( \mathcal{V}_3 \) can be shown to be Maltsev, then it will follow from Theorem 9.1 of [3] that \( \mathcal{V}_3 \) is a discriminator variety.

Let \( F \) be the free algebra on three generators in \( \mathcal{V}_3 \). Thus \( \mathcal{V}_3 \) is Maltsev iff \( F \) has permuting congruences. Since \( F \) is finite, and \( \mathcal{V}_3 = \text{HSP}(S_3) \) is locally finite, there exists a finite set \( K \subset S_3 \) such that \( F \in \text{HSP}(K) \). Thus our task is reduced to proving that \( \mathcal{V}' = \text{HSP}(K) \) is Maltsev. By Lemma 4.4, we can assume that every nontrivial subalgebra of an algebra in \( K \) belongs to \( K \). Then \( \text{SP}_{\text{fin}}(K) \) consists of the trivial one-element algebras and the algebras
isomorphic to irredundant subdirect products of finitely many members of \( K \). By Lemma 4.5, we have \( SP_{\text{fin}}(K) \subset P(K) \). Thus by Theorem 3.4 of [15], \( V' \) is Maltsev.

The following lemma and corollary are needed in our investigation of the subvarieties \( V_1 \) and \( V_2 \).

**Lemma 4.7** If \( A \) is any finite algebra in \( V \), then \( A/Z(A) \) is a centerless algebra.

**Corollary 4.8** If \( A \in V \) then \( Z(A) \) is the largest locally solvable congruence of \( A \). Every locally solvable algebra in \( V \) is Abelian.

We now introduce a concept that will allow us to characterize the subvarieties \( V_1 \) and \( V_2 \).

**Definition 4.9** Let \( A \) be a finite algebra. If \( i,j \in \{1, \ldots, 5\} \) are distinct types, we say that \( A \) possesses the \((i,j)\) transfer principle iff for all \( \chi_0, \chi_1, \chi_2 \in \text{Con} A \), if \( \chi_0 \prec \chi_1 \prec \chi_2 \) and \( \text{typ}(\chi_0, \chi_1) = i \) and \( \text{typ}(\chi_1, \chi_2) = j \), then there exists \( \beta \leq \chi_2 \) such that \( \chi_0 \prec \beta \) and \( \text{typ}(\chi_0, \beta) = j \).

A locally finite variety possesses the \((i,j)\) transfer principle if every finite member does.

**Theorem 4.10** Let \( i, j \) be distinct members of \( \{1,2,3\} \). Every finite algebra in \( V \) possesses the \((i,j)\) transfer principle.

**Theorem 4.11** \( V_1 \lor V_2 \) is the class of all Abelian algebras in \( V \).

**Proof.** By Theorem 4.3, every Abelian algebra in \( V \) belongs to \( V_1 \lor V_2 \). To prove the converse, it will suffice, by Corollary 4.8, to prove that every algebra in \( V_1 \lor V_2 \) is locally solvable. Since \( V_1 \lor V_2 \) is generated by the class \( S_1 \cup S_2 \), it will suffice, by Theorem 2.7, to prove that every algebra in \( S_1 \cup S_2 \) is solvable. Now \( S_1 \cup S_2 \) is just the class of finite subdirectly irreducible algebras in \( V \) with Abelian monolith. So let \( A \) be a finite subdirectly irreducible algebra belonging to \( V \) whose monolith \( \beta \) is Abelian. We shall show that \( A \) is solvable.

By Theorem 3.6, we have \( \text{typ}(A) \subset \{1,2,3\} \). By Corollary 3.5, \( A \) is solvable iff \( \text{typ}(A) \subset \{1,2\} \). Now in order to reach a contradiction, assume
that $3 \in \text{typ}\{A\}$. Among all the prime quotients $\langle \chi_0, \chi_1 \rangle$ in $\text{Con} \, A$ with $\text{typ}(\chi_0, \chi_1) = 3$, choose one $\langle \delta, \gamma \rangle$ such that the cardinality of the interval $I[0_A, \gamma]$ is as small as possible. Since $\text{typ}(0_A, \beta) \in \{1, 2\}$ where $\beta$ is the monolith, we have $\delta \neq 0_A$. Now choose any $\xi \in \text{Con} \, A$ such that $\xi \prec \delta$. By our choice of $\langle \delta, \gamma \rangle$, the type of $\langle \xi, \delta \rangle$ is not $3$—so it must be $1$ or $2$. Now by the $(1, 3)$ or the $(2, 3)$ transfer principle, there exists a congruence $\lambda$ such that $\xi \prec \lambda < \gamma$ and $\text{typ}(\xi, \lambda) = 3$. This contradicts our choice of $\langle \delta, \gamma \rangle$ and ends the proof.

We now focus on the Abelian subvariety $A = V_1 \lor V_2$. Using the transfer principals and some tame congruence theory it is possible to show that $\text{typ}\{V_1\} = 1$ and $\text{typ}\{V_2\} = 2$. From this we conclude the following.

**THEOREM 4.12** The subvariety $V_1$ is strongly Abelian and the subvariety $V_2$ is affine.

To establish the independence of the three subvarieties $V_1$, $V_2$ and $V_3$, and hence finish the proof of Theorem 2.14, we found it necessary to first characterize the decidable locally finite strongly Abelian varieties. There is a close correspondence between these varieties and multi-sorted unary varieties. A $k$-sorted unary algebra $A$ is a structure of the form

$$\langle A_1, A_2, \ldots, A_k; \{f_i : i \in I\} \rangle$$

where the $A_i$ are nonempty disjoint sets and for each $i \in I$, $f_i$ is a map from $A_{\sigma(i)}$ to $A_{\tau(i)}$ for some $\sigma(i)$ and $\tau(i)$ less than or equal to $k$.

By a derived operation of $A$ we mean an operation obtained by composing some of the $f_i$ or one of the identity functions $\text{id}_i : A_i \longrightarrow A_i$. Given a sequence $g_1, \ldots, g_k$ of derived operations having their ranges contained in $A_1, \ldots, A_k$ respectively and a sequence of numbers less than or equal to $k$, $\eta = \langle n_1, \ldots, n_k \rangle$, we can define a $k$-ary operation $[g_1, \ldots, g_k, \eta]$ on the set $A_1 \times \cdots \times A_k$ as follows:

$$[g_1, \ldots, g_k, \eta](\langle a^1_1, \ldots, a^1_k \rangle, \ldots, \langle a^k_1, \ldots, a^k_k \rangle) = \langle g_1(a^1_{n_1}), \ldots, g_k(a^k_{n_k}) \rangle,$$

where the domain of $g_i$ is $A_{\sigma(i)}$. Let $C(A)$ be the clone of operations on the set $A_1 \times \cdots \times A_k$ generated by the set of all such $k$-ary operations.

We call an algebra $B$ quasi-unary if for some $k$-sorted unary algebra $A$, $B$ is isomorphic to an algebra with universe $A_1 \times \cdots \times A_k$ and clone equal
to $C(A)$. It is not hard to show that $B$ is strongly Abelian. The first step in our characterization of the decidable locally finite strongly Abelian variety is to prove the following lemma.

**LEMMA 4.13** Let $S$ be a decidable locally finite strongly Abelian variety. Then every algebra in $S$ is quasi-unary.

A $k$-sorted unary algebra $A$ is called linear if for all nonconstant derived operations $f$ and $g$ of $A$ having the same domain, there is some other derived operation $h$ (having the appropriate domain and range) such that $A$ satisfies the equation $f(x) \approx hg(x)$ or the equation $g(x) \approx hf(x)$. We extend this definition to quasi-unary algebras in the natural way, i.e., a quasi-unary algebra $B$ is called linear if the multi-sorted unary algebra that is associated with it is linear.

**THEOREM 4.14** Let $S$ be a locally finite strongly Abelian variety of finite type. Then $S$ is decidable if and only if every algebra in $S$ is quasi-unary and linear. This is equivalent to having some generating algebra of $S$ of this form.

5 Conclusion

Theorem 2.14 reduces the study of decidable locally finite varieties to the examination of decidable locally finite varieties that fall into one of the three special cases. As we have noted, the decidable locally finite strongly Abelian varieties have been characterized.

For locally finite discriminator varieties, no criterion for decidability is known. We begin our list of open problems with this one.

**Problem 1:** Which locally finite discriminator varieties are undecidable?

S. Burris and H. Werner [6] proved that every finitely generated discriminator variety of finite type is decidable. Some more recent results on Problem 1 can be found in S. Burris [2] and S. Burris, R. McKenzie and M. Valeriote [4].

The decidability question for locally finite affine varieties is very interesting and seems to be very difficult. Corresponding to a locally finite affine
variety \( \mathcal{V} \) there is a finite ring \( R \) such that, according to Burris and McKenzie [3] Theorem 10.6, the variety of left unitary modules over \( R \) is decidable if and only if \( \mathcal{V} \) is decidable. Thus we have the next problem.

**Problem 2:** Which finite rings \( R \) with unit have the property that the variety of left unitary modules over \( R \) is decidable?


The methods used to prove Theorem 2.14 are clearly applicable to the next problems, although new methods may also be needed.

**Problem 3:** Which locally finite quasivarieties (universal Horn classes) are undecidable?

**Problem 4:** For which locally finite varieties is the class of finite members undecidable?

In A. P. Zamyatin [24], a list is given of all the varieties of rings whose class of finite members is decidable. Recently P. M. Idziak [10, 11, 12] has characterized those finitely generated congruence distributive varieties of finite type whose class of finite members is decidable. He proves that such a variety must be congruence permutable and the congruence lattice of every subdirectly irreducible algebra in the variety must be linearly ordered. If either of these conditions fail then the variety is shown to be \( \omega \)-unstructured.

In [22] it is shown that every locally finite, Abelian variety whose theory of its finite members is decidable is the varietal product of a strongly Abelian variety and an affine variety.

We close with a corollary of our structure theory.

**Corollary 5.1** There exists an algorithm which produces, given a finite algebra of finite type, a finite ring with unit such that the algebra generates a decidable variety iff the variety of left unitary modules over the ring is decidable.

**Proof.** Due to the restrictive nature of the structure of any finite algebra that generates a decidable variety, such an algorithm can be found. We can use the algorithm described in Burris, McKenzie (Theorem 11.3), adding to it the test for decidability of a strongly Abelian variety contained in our Theorem 4.14.
References


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