Model Theory of the Universal Covering Spaces of Complex Algebraic Varieties

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...on s'empressera de publier ses moindres observations pour peu qu'elles soient nouvelles, et on ajoutera: «je ne sais pas le reste»

Abstract

We study the interaction between the model theory, arithmetic and geometry of complex algebraic varieties. Our main results state that certain basic model-theoretic conditions do indeed hold. In general the proofs require some technical finiteness and compactness conditions and assume some complex-analytic and arithmetic conjectures, the Shafarevich conjecture on holomorphic convexity of universal covers first of all. For some classes of varieties, for example semi-Abelian varieties, the Shafarevich conjecture is known, so for these classes the results are unconditional. In particular we prove that there exists an $\aleph_1$-categorical $L_{\omega_1,\omega}$-axiomatisation of universal covering spaces in such classes.

Another important special case where we can say more is the class of elliptic curves. In this class we are able to prove that the **natural** $L_{\omega_1,\omega}$-axiomatisation of the universal cover of an elliptic curve is $\aleph_1$-categorical.

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Chapter I

Introduction

I.1 Introduction

I.1.1 General Framework

Is the notion of homotopy on a complex algebraic variety an algebraic notion? That is, can the notion of homotopy be characterised in a purely algebraic way, without reference to the complex topology?

We can restrict to 1-dimensional homotopies only: a 1-dimensional homotopy is a path, so the question is now whether the notion of a path on a complex algebraic manifold, up to fixed point homotopy, can be characterised in a purely algebraic way.

We provide a partial positive answer to the following more precise question. Assume that one has an abstract notion of a path up to homotopy, so that one is able to speak about homotopy classes of paths, their endpoints, liftings along topological coverings, paths lying in a subvariety. Can this notion be described without recourse to the complex topology?

Is it true that one can axiomatise this notion in such a way that any of its realisations comes from a choice of an embedding of the underlying field into $\mathbb{C}$, or equivalently, a choice of a locally compact Archimedean Hausdorff topology on the underlying field (if its cardinality is $2^{\aleph_0}$)?

Is the resulting formal theory "good" from a model-theoretic point of view?

Model theory allows a rigorous formulation of the question as the problem of proving categoricity of a structure related to the fundamental groupoid, or equivalently the universal covering space, of a complex algebraic variety. Such categoricity questions are extensively studied in model theory, specifically by Shelah [She83a, She83b] and a short list of conditions sufficient for categoricity of an

$L_{\omega_1,\omega}$-sentence is known (this is the notion of an excellent theory). Our model-theoretic analysis shows that the positive answer to our question is plausible and is essentially equivalent to deep geometric and arithmetic properties of the underlying variety. Some of the properties are known to hold, some others are conjectured.

We study the interaction between the model theory, arithmetic and geometry of complex algebraic varieties. Our main results state that certain basic model-theoretic conditions do indeed hold. In general the proofs require some technical finiteness and compactness conditions and assume some complex-analytic and arithmetic conjectures, the Shafarevich conjecture on holomorphic convexity of universal covers first of all. For some classes of varieties, for example semi-Abelian varieties, the Shafarevich conjecture is known, so for these classes the results are unconditional. In particular we prove that there exists an $\aleph_1$-categorical $L_{\omega_1,\omega}$-axiomatisation of universal covering spaces in such classes.

Another important special case where we can say more is the class of elliptic curves. In this class we are able to prove that the **natural** $L_{\omega_1,\omega}$-axiomatisation of the universal cover of an elliptic curve is $\aleph_1$-categorical.

Finally we would like to note that the model-theoretic analysis of universal covers falls very naturally into the framework of (analytic) Zariski geometries started by Hrushovski-Zilber in [HZ96] and further developed by Zilber and his collaborators [Zil05, Zilc] around an expectation that many basic mathematical structures may be considered as a model-theoretic structure with nice properties, above all categoricity. Importantly, it has been understood that the model theory relevant here is essentially non first-order. Our intermediate technical results characterising definable subsets can be used to check that the structures we consider are indeed analytic Zariski, thus providing a series of examples of analytic Zariski geometries.

I.1.2 An explicit example

Motivated by the belief that every structure naturally occurring in mathematics, is "logically perfect"([Zilc]), Zilber proves an $L_{\omega_1\omega}$-categoricity statement for the two-sorted structure

$$\mathbb{C}^{\mathrm{lin}}_{\exp} = ((\mathbb{C},+),(\mathbb{C}^* \cup \{\mathbf{0}\},+,\times),\exp:\mathbb{C}\to\mathbb{C}^*)$$

describing the exponential map $\exp:\mathbb{C}\to\mathbb{C}^*$. The language $L(\mathbb{C}^{\mathrm{lin}}_{\exp})$ separates the sorts $\mathbb{C}$ and $\mathbb{C}^*$ for the domain and the range of exp; and here the structure on sort $\mathbb{C}^*$ is that of an algebraic variety, while the structure on sort $\mathbb{C}$ is that of a $\mathbb{Q}$-vector space, together with the pull-back of the structure on sort $\mathbb{C}^*$. He also gives an explicit axiomatisation of the $L_{\omega_1\omega}(\mathbb{C}^{\text{lin}}_{\exp})$-theory.

The integral of $\frac{dz}{z}$ over a path in $\mathbb{C}^*$ does not change with a continuous transformation of the path fixing the ends and avoiding the singularity 0 of $\frac{dz}{z}$; in other words, the integral depends only on the homotopy class of the paths in $\mathbb{C}^*$, with homotopy fixing the ends. Thus, the map

$$\{\text{paths } [\gamma] \text{ in } \mathbb{C}^*, \gamma(0) = 1\} \longrightarrow \mathbb{C}$$

$$\gamma \mapsto \int_\gamma \frac{dz}{z} = \ln(\gamma(1)) + 2\pi i k, \text{for some } k \in \mathbb{Z}$$

identifies $\mathbb{C}$ with the homotopy classes $\gamma, \gamma(0) = 1$ of paths in $\mathbb{C}^*$.

The multiplication map $m : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^*$ induces a map on paths $m_* : (\gamma_1\gamma_2) \mapsto \gamma_1 \cdot \gamma_2$, where $(\gamma_1 \cdot \gamma_2)(t) = \gamma_1(t)\gamma_2(t)$ is the pointwise product of paths γ_1 and γ_2 as functions from $[0, 1]$; this allows us to express addition on $\mathbb{C}$ as

$$\int_{\gamma_1} \frac{dz}{z} + \int_{\gamma_2} \frac{dz}{z} = \int_{\gamma_1\cdot\gamma_2} \frac{dz}{z} = \ln(\gamma_1(1)) + \ln(\gamma_2(1)) + 2\pi i k, k \in \mathbb{Z}.$$

The above observations make it natural to think of $\mathbb{C}^{\text{lin}}_{\exp}$ as describing the homotopy classes of paths in $\mathbb{C}^*$, or indeed $\mathbb{C}^{*n}, n > 0$ in a language reflecting the behaviour of paths under morphisms and their concatenation properties. This approach easily generalases the question to arbitrary algebraic varieties.

It is natural not to restrict oneself to paths γ starting at $1, \gamma(0) = 1$, and consider integrals over arbitrary paths in $\mathbb{C}^*$.

The set of the homotopy classes of paths with a given starting point forms the universal covering space; this observation allows us to think instead of $\mathbb{C}^{\text{lin}}_{\exp}$ as a structure describing the universal covering space of $\mathbb{C}^*$, and reflecting the property of being a connected component of an algebraic closed subset.

We use the latter point of view to generalise the approach of Zilber [Zila]; the interpretation of $\mathbb{C}^{\text{lin}}_{\exp}$ in terms of paths allows one to formulate a categoricity question for an arbitrary variety defined over $\overline{\mathbb{Q}}$, or even a field of positive characteristic. The proofs of [Zila] use Kummer theory, as well as some other number-theoretic tools. Here we use similar tools, but in a rather different appearance; in particular, we have to use holomorphic convexity of the complex structure on the universal covering space to prove Kummer-theory type results over an algebraically closed field.

I.1.3 Technical summary of results

After some preparation in Chapter II we start Chapter III by studying universal covers in an informal geometric language. In III.3 this leads us to the definition of a natural formal countable language L_A associated with the universal covering space $p : U \to A(\mathbb{C})$ of a complex projective algebraic variety $A(\mathbb{C})$ defined over $\mathbb{Q}$ or $\overline{\mathbb{Q}}$. Assuming the conjecture of Shafarevich that the universal covering space U is holomorphically convex, we prove that *the positively definable sets in L_A form a topology analogous to the Zariski topology on the set of geometric points of a variety*. The properties of the topology on U are sufficient to imply that

the structure U^{L_A} is homogenous over countable submodels (ω-model homogeneity), and *realises countably many types over a countable submodel.*

We then consider in III.5 a fragment of the $L_{\omega_1\omega}(L_A)$-theory $\mathrm{Theory}^{L_A}_{\omega_1\omega}(U)$ of U^{L_A} and introduce a natural set of axioms $\mathfrak{X}$ of geometric, analytic Zariski flavour implying the properties of the topology on U. Then we show that

the class of models defined by $\mathfrak{X}$ is stable (in a non-elementary context) over countable models, and all its models are homogeneous over submodels.

These are prerequisites, by Shelah's theory, of categoricity in uncountable cardinals. Notice that some of the properties, e.g. atomicity of every model, could, by Shelah's theory, be obtained just by an $L_{\omega_1\omega}$-definable expansion of the theory $\mathfrak{X}$. This, by Shelah's theory, is enough to imply $\aleph_1$-categoricity of an $L_{\omega_1,\omega}$-class Φ containing U^{L_A}, for an arbitrary smooth projective variety A with a holomorphically convex universal cover and with certain conditions on the fundamental group. (Cf. Definition III.1.2.6 for the exact definition of the class of algebraic varieties).

Nevertheless, essentially for the applications, we want to stay with the natural axiomatisation. Because of this, it is important to know that standard minimal model $\mathbb{U}(\bar{\mathbb{Q}})$ is an atomic model for the natural axiomatisation.

In Chapter IV we analyse the example of $A = E$, an elliptic curve, and prove that in this case

the natural axiomatisation $\mathfrak{X}$ does have an atomic and prime model $\mathbb{U}(\overline{\mathbb{Q}})$. This implies the $\aleph_1$-categoricity of the natural axiomatisation.

We put the statement above in a rather *different* setting by reformulating it as a particular case of a conjecture about a universality property of the fundamental groupoid functor; we show that

$\mathbb{U}(\overline{\mathbb{Q}})$ being a prime model of the natural axiomatisation $\mathfrak{X}$ is equivalent to a universality property of the fundamental groupoid functor.

The proof of these facts for E depends on the Kummer theory of E and on the image of the Galois action on the torsion points E_{tors} of E; we think that our approach *explains the Kummer theory of E*, as well as *the results about the image of the Galois action.*

Finally we want to stress that our approach is essentially different from Zilber's of [Zild] since our language L_A is in general stronger than Zilber's. In fact L_A "adjusts" to the geometric properties of the covering of A, and is defined for any A whereas [Zild] is restricted to the class of Abelian varieties. Our language allows us to produce a sentence in all cases, conjecturally categorical for suitably "self-sufficient" A whereas [Zild] is restricted only to considering Abelian varieties, and those are sometimes obviously not "self-sufficient", say Abelian varieties of dimension greater than 1. In particular, to obtain a conjecturally categorical statement associated to an Abelian variety X, we consider the language L_A corresponding to an ample homogeneous $\mathbb{C}^*$-bundle $A = L^*$ over X, and show that L_A defines the 1st Chern class of $X(\mathbb{C})$ as an element $c_1 \in H^2(\pi_1(X(\mathbb{C}), 0), \mathbb{Z})$ or, equivalently, as an alternating bilinear Riemann form $\Lambda \times \Lambda \longrightarrow \mathbb{Z}$ (cf. §IV.6.1). In particular,

the first-order L_A-theory associated to an ample homogeneous $\mathbb{C}^*$-bundle over an Abelian variety is unstable.

I.2 Motivations and implications

In this section we discuss the motivations behind our choice of the language and explain our approach in greater detail. In our opinion the motivations here are more important than the proofs which will follow in the next chapters.

I.2.1 The Logic approach: What is an appropriate language to talk about paths?

Abstract algebraic geometry provides a language appropriate to talk about complex algebraic varieties; what language would be appropriate to talk about the homotopies on the algebraic varieties, in particular about paths, i.e. 1-dimensional homotopies? What is the right mathematical measure to judge appropriateness of the language for such a notion?

Abstract algebraic geometry over a field has no complete analogue of the notion. However, there is a strong intuition based on the naive notion of a path in complex topology; it is a well-known phenomenon that naive arguments based on the notion of a path quite often lead to statements which generalise, in one way or another, to, say, arbitrary schemes, but which are quite difficult to prove. There have been many attempts to develop substitute notions, starting from Grothendieck [SGA2,SGA4$\frac{1}{2}$] who developed for this purpose the notion of a finite *covering* in the category of arbitrary schemes (étale morphism); see Grothendieck [Gro] for an attempt to provide an algebraic formalism to express homotopy properties of topological spaces, and Voevodsky-Kapranov [VK91] for exact definitions.

Thus, from the point of view of philosophy of mathematics, it is natural to try to understand why the notion of a path is so fruitful and applicable, despite the fact that all attempts to generalise it to non-topological contexts have had only partial success.

We intend to propose in this work a model-theoretic structure which contains an abstract substitute for the notion of a path. The substitute must possess the familiar properties of paths appearing in the topological context, rich enough to imply a useful theory of paths; in particular *they must determinine the notion of a path on an abstract algebraic variety uniquely up to isomorphism.*

Note that Grothendieck [Gro], cf. also Voevodsky-Kapranov [VK91], provides a natural algebraic setup to talk about paths thereby rather directly leading to a choice of a language (of 2-functors). Our approach is in fact based on a similar idea.

Model theory provides a framework to formulate the uniqueness property in a mathematically rigoros fashion. Following [Zila, Zilc] we use the notion of *categoricity in uncountable cardinals* (of non-elementary classes). In his philosophy categoricity is a model-theoretic criterion for determining when an algebraic formalisation of an object, of perhaps geometric character, is canonical and reflects the properties of the object in a complete way.

In this work we introduce a language L_A which is appropriate for describing the basic homotopy properties of algebraic varieties in their complex topology, and prove some partial results towards categoricity and stability of associated structures in that language. The language L_A is capable of expressing properties of 1-dimensional homotopies, i.e. the properties of paths up to homotopies fixing the ends. We can speak in L_A in terms of lifting paths to a topological covering, paths lying in closed algebraic subvarieties (i.e. a homotopy class has a representative which lies in the subvariety), paths in direct products and so on. These properties are sufficient to carry out many basic 1-dimensional homotopy theory constructions. Most notably, following a construction in Mumford [Mum70] one can definably construct a bilinear form $\varphi_L : \pi_1(A(\mathbb{C}),0)\times\pi_1(A(\mathbb{C}),0) \to \pi_1(\mathbb{C}^*,1)$ in the second homology group $H^2(A(\mathbb{C}),\mathbb{Z}) \cong \bigwedge^2 H^1(A(\mathbb{C}),\mathbb{Z})$ associated to an algebraic $\mathbb{C}^*$-bundle L over a complex Abelian variety $X(\mathbb{C})$. Thus, generally the language has more expressive power than the one considered originally by Zilber in [Zild]; in particular, some Abelian varieties which are not categorical in Zilber's language of [Zild] are expected to be categorical in our language. It would be interesting to know whether our language can interpret Hodge decomposition on cohomology groups, using the isomorphism $H^n(A(\mathbb{C}),\mathbb{C}) \cong \bigwedge^n H^1(A(\mathbb{C}),\mathbb{C}) = \bigwedge^n \mathrm{Hom}\,(\pi_1(A(\mathbb{C}),0),\mathbb{C})$ (cf. [Mum70]).

The results which we prove towards categoricity in uncountable cardianlities are partial. We prove categoricity in cardinality $\aleph_1$ for some special classes of algebraic varieties, e.g. for elliptic curves. We also prove important necessary conditions, such as stability and homogeneity over models, for much wider classes.

I.2.2 The Geometric approach: Analytic Zariski structures

The universal covering of an algebraic variety is one of the simplest analytic structures associated to an algebraic variety and which is more than an algebraic variety itself; the universal covering space inherits all the local structure the base space possesses; and in particular, for a complex algebraic variety it is a complex analytic space. Thus it is natural to consider it in the context of Zariski geometries [Zil05]: one wants to define a Zariski-type topology on the universal covering space U of variety $A(\mathbb{C})$ reflecting the connection between U and A, and such that U possesses homogeneity, stability and categoricity properties, perhaps in a non-first order, $L_{\omega_1\omega}$, way, in a countable language related to the chosen topology on U.

For this, consider the universal covering space $p : U \to A(\mathbb{C})$ of an algebraic variety A. It is natural to assume that the covering map p and the full algebraic variety structure on $A(\mathbb{C})$ are definable. Then the analytic subsets of U which are the preimages $p^{-1}(Z(\mathbb{C}))$ of algebraic subvarieties Z of $A(\mathbb{C})$, are definable. It is natural to let the analytic irreducible components of such sets also be definable; one justification for this might be the desire for an irreducible decomposition.

The above considerations lead us to define a topology on U by proclaiming a set closed iff it is a union of analytic irreducible components of the preimages of finitely many closed algebraic subvarieties of $A(\mathbb{C})$.

It turns out that this topology is rather nice in that it (almost) admits quantifier elimination down to the level of closed sets, has DCC (the descending chain condition) for *irreducible* sets, and can be defined in a countable language (assuming that the universal covering space is holomorphically convex, which is a conjecture of Shafarevich). These properties of the topology are sufficient to imply the model homogeneity of the structure $p : U \to A(\mathbb{C})$, and, more generally, to construct an $L_{\omega_1\omega}$-class containing $p : U \to A(\mathbb{C})$ which is stable over models and whose models are model homogeneous. It also turns out that the language obtained in this way is the language appropriate for describing the paths, as explained in subsection above. We explain the connection in subsection §I.2.4.

I.2.3 The Category Theory approach: universality of the fundamental groupoid

The algebraic notion most commonly used to describe paths is that of the Poincaré fundamental groupoid; more generally, Grothendieck [Gro] tries to provide a natural algebraic setup to discuss homotopy properties of topological spaces, and paths in particular. He describes homotopies in terms of an *n-functor* from the category of topological spaces to sets; the notion of a 2-functor directly leads to the choice of a language equivalent to the languages described above. Then, in terms of category theory, what we ask is whether we can uniquely describe the fundamental groupoid functor, or 2-functor, on the category of complex algebraic varieties by some natural algebraic properties. However, the notion of isomorphism in model theory forces us to consider functors up to an equivalence which is weaker than natural equivalence of functors; namely, we consider natural equivalence but also *allow for an automorphism of the source category*. Therefore the question we ask is the following: *are there some algebraic conditions on a 2-functor on the category of complex algebraic varieties which force it to be naturally equivalent to the fundamental groupoid functor, perhaps combined with an automorphism of the source category?* We provide some partial positive answers to this question; we also remark that it is essential to allow for automorphism of the source category, as is shown by examples of Serre [Ser64] of algebraic varieties whose fundamental group depends on the embedding into $\mathbb{C}$.

I.2.4 Equivalence of analytic Zariski and paths approaches

One definition of the universal covering space $p : U \to A(\mathbb{C})$ (see e.g. [Nov86]) says that it is a set of homotopy classes of paths leaving a basepoint, with an induced topology. Thus, it is equivalent to talk about the universal covering spaces instead of homotopy classes of paths; this is also easier from the technical point of view.

The above two observations make it natural to consider the universal covering space $p : U \to A(\mathbb{C})$ in the context of analytic Zariski structures.

Recall we define a topology on U by proclaiming a set closed iff it is a union of analytic irreducible components of the preimages of finitely many closed algebraic subvarieties $W_i(\mathbb{C})$ of $A(\mathbb{C})$. The critical observation is that a connected component of the preimage of a normal subvariety is always irreducible, and that, if one thinks of U as a set of paths, then, forgetting technicalities, a connected component is a set of all paths lying in the underlying variety. The observation means that if subvarieties W_i were always normal, we would recover the interpretation of definable closed sets in terms of paths, i.e. we would see that each closed set is definable via the basic properties of paths, and algebraic subsets of $A(\mathbb{C})$. For W_i's not normal, it turns out that we may still interpret the irreducible component in a similar way, as a set of paths in a subvariety of *covering spaces* of $A(\mathbb{C})$ of *finite* degree, which are known always to be algebraic ([SGA1, Exp.XII,Th.5.1 and Cor.5.2]). However, this is not trivial and requires a geometric argument based on the assumption that U is a *holomorphically convex manifold*; $\mathbb{C}^n$ and its submanifolds are examples of such holomorphically convex manifolds (cf. Def. V.3.1.1). By a conjecture of Shafarevich (cf. Conj. V.3.1.3) this should hold for an arbitrary variety, say smooth and quasi-projective. In fact, we need the slightly weaker statement that a universal covering space satisfies the conclusions of Fact III.1.2.1.

Thus we see that analytic Zariski approach leads us back to the homotopy interpretation as above.

I.2.5 Countable language

In previous sections we have carefully avoided discussing the size of the language L_A: we have just discussed the definition of a topology on U, but not that of a language L_A. We have mentioned earlier that what we want is that the language L_A is to be able to define the closed sets in the chosen topology, and we want to consider U as an analytic Zariski structure in a *countable* language. This is desirable for several reasons; one is that if the language were too big, the notion of an isomorphism would be too strong, and we could not hope to have categoricity at all; another is that we want to be able to apply Shelah's theory of excellent classes, and this theory requires a countable language. In case of an algebraic variety, one takes the language to consist of all subvarieties defined over a sufficiently large number field; similarly, we define the language L_A by adding a relation to describe when a pair of points lies in the same connected component of each closed algebraic subvariety of A^n *defined over* $\mathbb{Q}$, or the field of definition of A. The fact that L_A is able to define connected components of the preimage of an *arbitrary* subvariety is a geometrically non-trivial result employing Stein factorisation for normal varieties; and again we use holomorphic convexity to reduce the general case to that of normal subvarieties.

Chapter II

$\aleph_1$-categoricity and model stability

II.1 Definitions and Examples

The goal of this chapter is to introduce the main objects, state precisely the main definitions and motivations, present the main results and give some examples.

II.1.1 The Goal

Let $p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})$ be the universal covering space of an algebraic variety $\mathbb{A}$. As explained in §I.2.2,§I.2.1, we have the following

Goal II.1.1.1. *The universal covering space* $\boldsymbol{U}$ *of a complex algebraic variety*

1. *is an* algebraic *object, by which we mean*
   (a) *there is a* countable *set L of distinguished subsets P_i of Cartesian powers of* $\boldsymbol{U}$
   (b) *there exists a* countable *axiomatisation $\mathfrak{X}$ of the object in terms of the language L expressible by "formulae of countable length and finite depth", that is an $L_{\omega_1,\omega}$-sentence*
   (c) *$\mathfrak{X}$ characterises the object uniquely, up to isomorphism preserving the distinguished relations.*
2. *the language L on* $\boldsymbol{U}$ *describes* some analytic or homotopic data*, say*
   (a) *the expressive power of L is sufficient to define connected components of the preimages of algebraic subvarieties of* $\mathbb{A}(\mathbb{C})^n$.
   (b) *the expressive power of L is sufficient to define analytic irreducible components of preimages of analytic subsets of* $\mathbb{A}(\mathbb{C})^n$

Examples of algebraic objects are a field, a ring, a module over a ring, and an algebraic variety defined over a countable ring, considered as the set of its points over an algebraically closed field; on the other hand, a priori a topological space or an analytic manifold are not algebraic objects in this sense: the topology consists of uncountably many sets, and a priori there is no way to choose a countable set of relations defining the topology. Thus, what we want is, in particular, to show that this is possible to do for (the analytic Zariski topology on) the universal covering space of some "good" and "simple" varieties.

In the model-theoretic jargon the above translates to

Subgoal II.1.1.2. *There is a countable language L_A on the universal covering space $p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})$, describing some analytic or homotopic data on U, such that $p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})$ in L_A admits a* natural *axiomatisation, perhaps non-first order, say in $L_{\omega_1\omega}(L_A)$, which is* uncountably categorical.

Following considerations of the introduction in Chapter I, the goal above may be translated to the following question of model theory; we explain our precise partial results towards Subgoal II.1.1.2 (Theorem III.5.4.7) after giving the definition of the language L_A in the next subsection.

In $3'$ below, $\pi(\boldsymbol{U})$ denotes the *group of deck transformations* of the covering $p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})$ consisting of the continuous analytic isomorphisms $\tau : \boldsymbol{U} \to \boldsymbol{U}$ which commute with the covering map p, $p \circ p = p$; cf. §V.1 for details and definitions. For $A(\mathbb{C})$ connected, the group $\pi(\boldsymbol{U})$ of deck transformations is isomorphic to the fundamental group $\pi_1(\mathbb{A}(\mathbb{C}), x)$; the isomorphism is well-defined up to conjugation.

Subgoal II.1.1.3. *There exist a countable* language *L_A, i.e. a countable collection $L_A = \{P_i\}$ of predicates (i.e. distinguished subsets of Cartesian powers of $\boldsymbol{U}$), and an* axiomatisation *$\mathfrak{X} = \mathfrak{X}(L_A, p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C}))$ of $p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})$ in $L_{\omega_1\omega}(L_A)$, i.e. a countable collection φ_i of sentences in $L_{\omega_1\omega}(L_A)$, such that*

1 *U admits an $L_{\omega_1\omega}(L_A)$-axiomatisation $\mathfrak{X}$:*
- $1'$ *all sentences $\mathfrak{X} = \{\varphi_i\}$ are valid on $\boldsymbol{U}$, in notation $\boldsymbol{U} \models \varphi_i$, $\varphi_i \in \mathfrak{X}$*

2 *the axiomatisation $\mathfrak{X}$ is uncountably categorical:*
- $2'$ *for any two uncountable models $\boldsymbol{U}_1, \boldsymbol{U}_2$ of the same cardinality, if $\boldsymbol{U}_1, \boldsymbol{U}_2 \models \varphi_i$, for all $\varphi_i \in \mathfrak{X}$, then $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ are isomorphic as L_A-structures, $\boldsymbol{U}_1 \cong_{L_A} \boldsymbol{U}_2$, i.e. there exists a bijection $\varphi : \boldsymbol{U}_1 \to \boldsymbol{U}_2$ preserving distinguished relations P_i's in L_A.*

3$\mathfrak{a}$ *the language L_A describes some of the analytic structure of $\boldsymbol{U}$ as a complex space:*
- $3'$ *closed analytic $\pi(\boldsymbol{U})$-invariant subsets of $\boldsymbol{U}^n$ are L_A-definable, with parameters, as well as the irreducible analytic components thereof.*

3$\mathfrak{b}$ *the language L_A describes some of topological and homotopy theory data on $\boldsymbol{U}$ as the universal covering space $\mathbb{A}(\mathbb{C})$.*
- $3''$ *the connected components of the preimage of an algebraic closed subset of $\mathbb{A}^n(\mathbb{C})$ are L_A-definable, with parameters.*

In addition to the precise statements above, we want that

1″ the sentences $\varphi_i \in \mathfrak{X}$ have a natural geometric meaning; there is an explicit description of the class of models satisfying axiomatisation $\mathfrak{X}$.

Remark II.1.1.4. The choice of 3𝔞, 3𝔟 is rather arbitrary, and is specific to this work; for example, in [Zilb] Zilber considers $\mathbb{A} = \mathbb{C}^*$ and replaces 3𝔞, 3𝔟 with

3‴ a closed analytic set definable by polynomial-exponential equations (i.e. those consisting of $+$, $\times$, exp) is $L_{\mathbb{C}^*}$-definable.

Accordingly, the richer structure of [Zilb] gives rise to deeper arithmetic and geometric conjectures, including a generalisation of Mordell-Lang.

II.1.2 L_A-structure on the universal covering space $p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})$

Item 3 leads us to introduce the following $\pi(\boldsymbol{U})$-invariant relations. For a closed subvariety $Z \subset \mathbb{A}(\mathbb{C})^n$, let $\sim_{Z,A}$ denote the relation on $\boldsymbol{U}^n$ given by

$x' \sim_{Z,A} y' \iff$ points $x' \in \boldsymbol{U}^n$ and $y' \in \boldsymbol{U}^n$ lie in the same (analytic) irreducible component of $p^{-1}(Z(\mathbb{C})) \subset \boldsymbol{U}^n$.

For $Z(\mathbb{C})$ smooth or even a *normal* subvariety, the connected components of $p^{-1}(Z(\mathbb{C}))$ are irreducible (as analytic closed sets). Thus, for such varieties, the relation $\sim_{Z,A}$ is an equivalence relation encoding topological data only, and data on $\mathbb{A}(\mathbb{C})$. For a normal subgroup $H \lhd \pi(\boldsymbol{U})^n$, let $x' \sim_H y'$ say that points $x', y' \in \boldsymbol{U}^n$ are conjugated by the action of H:

$x' \sim_H y' \iff \exists \tau \in \pi(\boldsymbol{U})^n : \tau x' = y'$ and $\tau \in H$.

Definition II.1.2.1. *We consider the structure $p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})$ in the language L_A which has the following symbols:*

symbols $\sim_{Z,A}$ for Z a closed subvariety defined over number field k, and,
symbols $\sim_H$, for each normal subgroup $H \lhd_{\text{fin}} \pi(\boldsymbol{U})^n$ of finite index

Note that we do not assume Z to be connected; that is important, because Z is defined over a small field. The group $\pi(\boldsymbol{U})$ acts on $\boldsymbol{U}$ by analytic isomorphisms; accordingly, we would want that $\pi(\boldsymbol{U})$ acts by L_A-automorphisms of $\boldsymbol{U}$ as an L_A-structure; this is the case.

II.1.3 Results

The main result of Chapter III is the following.

Theorem II.1.3.1 (Model Stability of $\mathfrak{X}(\mathbb{A}(\mathbb{C}))$). *Let $\mathbb{A}$ be a smooth projective algebraic variety defined over $\overline{\mathbb{Q}}$ such that the universal covering space $\boldsymbol{U}$ of $\mathbb{A}$ is holomorphically convex. Also assume that the deck transformation groups of $\boldsymbol{U}^n$ are residually finite and subgroup separable. (Those conditions are defined in Def. III.1.2.6, where a variety satisfying them is called a Shafarevich variety).*

Let L_A be the countable language defined in Def. II.1.2.1. Then $(\mathfrak{1}', \mathfrak{1}'', \mathfrak{3a}, \mathfrak{3b})$ *of Subgoal II.1.1.3 hold, and* $\mathfrak{2}'$ *is weakened to* $\mathfrak{2}'_{\aleph_0\to\aleph_1}$*:*

$\mathfrak{2}'_{\aleph_0\to\aleph_1}$ *Any two models $\boldsymbol{U}_1 \models \mathfrak{X}$ and $\boldsymbol{U}_2 \models \mathfrak{X}$ of axiomatisation $\mathfrak{X}$ which are of cardinality $\aleph_1$, such that*

there exist a common countable submodel $\boldsymbol{U}_0 \models \mathfrak{X}$, $\boldsymbol{U}_0 \subset \boldsymbol{U}_1$ and $\boldsymbol{U}_0 \subset \boldsymbol{U}_2$

are isomorphic, $\boldsymbol{U}_1 \cong_{L_A} \boldsymbol{U}_2$, and, moreover, the isomorphism is the identity on $\boldsymbol{U}_0$.

According to a conjecture of Shafarevich, the universal covering space of an arbitrary smooth projective variety is holomorphically convex, and thus Theorem II.1.3.1 should apply to arbitrary smooth projective varieties with the restrictions on the fundamental group. Examples of holomorphically convex spaces are compact complex spaces, $\mathbb{C}^n$, Gaussian half-plane $\mathbb{H}$, and closed analytic subsets thereof; thus the Shafarevich conjecture holds for $\mathbb{C}^*$, elliptic curves, and arbitrary curves.

The property that the fundamental group is residually finite and subgroup separable is known ([Sco85]) when variety $\mathbb{A}$ is of dimension 1, $\dim \mathbb{A} = 1$, and also holds for $\mathbb{A}$ an Abelian variety. However, it *does not* necessarily hold for the Cartesian powers of those groups; see §III.1.2 for a discussion.

The general theory of $L_{\omega_1\omega}$ by Shelah [She83a, She83b] implies that there exists an $\aleph_1$-categorical countable $L_{\omega_1\omega}$-axiomatisation $\mathfrak{X}' = \mathfrak{X}'(\mathbb{A}(\mathbb{C}))$ extending $\mathfrak{X}$.

Corollary II.1.3.2. *The model $\boldsymbol{U}$ belongs to an $\aleph_1$-categorical abstract elementary class of L_A-structures. Equivalently, $\boldsymbol{U}$ belongs to an $\aleph_1$-categorical class of atomic models of a first-order theory in a countable expansion L_A^* of the language L_A.*

In Chapter IV we analyse the case $\mathbb{A} = E$ is an elliptic curve defined over $\overline{\mathbb{Q}}$, and prove an $L_{\omega_1\omega}$-categoricity result for the natural axiomatisation.

Theorem II.1.3.3 ($\aleph_1$-categoricity of $\mathfrak{X}(\mathbb{A}(\mathbb{C}))$). *Let $\mathbb{A} = E$ be an elliptic curve defined over a number field.*

Let L_A be the countable language defined in Def. II.1.2.1. Then $(\mathfrak{1}', \mathfrak{1}'', \mathfrak{3a}, \mathfrak{3b})$ *of Goal II.1.1.3 hold, and* $\mathfrak{2}'$ *is weakened to* $\mathfrak{2}''_{\aleph_1}$*:*

$\mathfrak{2}''_{\aleph_1}$ *There exist only finitely many isomorphism classes of models of $\mathfrak{X}(E(\mathbb{C}))$ of cardinality $\aleph_1$*

If $\mathrm{End} E = \mathbb{Z}$*, then* $\mathfrak{2}'$ *is weakened to* $\mathfrak{2}'_{\aleph_1}$

$\mathfrak{2}'_{\aleph_1}$ *all models of cardinality $\aleph_1$ are isomorphic.*

Moreover, the axiomatisation can be made very explicit and so as to describe only the natural properties of $\boldsymbol{U}$; see §IV.1 for two explicit descriptions of the class of (atomic) models of $\mathfrak{X}(\mathbb{A}(\mathbb{C}))$.

II.1.4 Linear structure on the universal covering spaces of the field multiplicative group $\mathbb{C}^*$ and elliptic curves $E(\mathbb{C})$

Here we present some examples, and, for certain varieties A carrying Abelian group structure, make explicit a locally modular, linear structure contained in the L_A-structure on the universal covering space $\boldsymbol{U}$. In general, it is not clear whether the L_A-structure on $\boldsymbol{U}$ is a combination of a locally modular structure and the pull-backs to U of the algebraic variety structure on the variety $\mathbb{A}(\mathbb{C})$ as well as its finite covers, which are known to always be algebraic. In a few of the examples we present below, it is so; however, the proof requires a geometric argument, and we defer it until the last chapter where we actually have to use it to obtain categoricity.

Example II.1.4.1 ($A = \mathbb{G}_m$; $\mathbb{A}(\mathbb{C}) = \mathbb{C}^*$, the complex exponential map as the structure $\mathbb{C}^{\text{lin}}_{\text{exp}}$). *Let $\mathbb{A} = \mathbb{G}_m$ be the multiplicative group of a field. The universal covering space is $\mathbb{C} \xrightarrow{\exp} \mathbb{C}^*$: $\boldsymbol{U}_{\mathbb{G}m} = \mathbb{C}$ and $p = \exp$. The variety $\mathbb{G}_m$ is defined over $\mathbb{Q}$, as well as the multiplication morphism $m : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^*$ and monomial morphism $z^n : \mathbb{C}^* \to \mathbb{C}^*$ for $n \in \mathbb{Z}$, and the closed diagonal subvariety $\Delta \subset \mathbb{C}^* \times \mathbb{C}^*$. Thus their graphs are also $\mathbb{Q}$-defined, and we get that the following equivalence relations on $\mathbb{C}$ are in L_A:*

$$(x_1, y_1) \sim_\Delta (x_2, y_2) \Longleftrightarrow x_1 - y_1 = x_2 - y_2$$

$$(x_1, y_1) \sim_{\{z^n=y\}} (x_2, y_2) \Longleftrightarrow nx_1 - y_1 = nx_2 - y_2$$

$$(x_1, y_1, z_1) \sim_{\{xy=z\}} (x_2, y_2, z_2) \Longleftrightarrow x_1 + y_1 - z_1 = x_2 + y_2 - z_2$$

The fundamental group $2\pi i\mathbb{Z}$ acts by translations $z \mapsto z + 2\pi ik$, which are L_A-automorphisms of $\mathbb{C}^{\text{lin}}_{\text{exp}}$.

Complex conjugation $x \to \overline{x}$ provides a continuous L_A-automorphism of $\mathbb{C}^{\text{lin}}_{\text{exp}}$ which does not come from the fundamental group; it follows from the property

$$\exp(\bar{z}) = \overline{\exp(z)}, z \in \mathbb{C}.$$

In general, there are many such automorphisms, but they are not necessarily continuous or even measurable.

Thus, one can see that any automorphism of the kernel $2\pi i\mathbb{Z}$ as an Abelian group *can be extended to an automorphism of the whole structure. This observation is trivial in this case; we mention it because this is a property we want to keep in other examples; it is a homogeneity property of the L_A-structure.*

To conclude, we see that the L_A-structure on the $\mathbb{C}$-sort of $\mathbb{C}^{\text{lin}}_{\text{exp}}$ is the pull-back of algebraic structure on $\mathbb{C}^*$ enriched by an affine, locally modular structure on $\mathbb{C}$ itself.

The language L_A is rather robust under the change of $\mathbb{A}$; for example, if we take $\mathbb{A} = \mathbb{G}_m \times \mathbb{G}_m$, then the structure obtained is essentially equivalent to the structure $\mathbb{C}^{\text{lin}}_{\text{exp}}$.

Example II.1.4.2 ($\mathbb{A} = \mathbb{G}_m \times \mathbb{G}_m = \mathbb{C}^* \times \mathbb{C}^*$). *In this case the relations of II.1.4.1 concerning $\Delta_{\mathbb{C}^*}$ and $\{z^n = y\}$ are still definable, but as relations with two variables in $\mathbb{C} \times \mathbb{C}$ and not 4 variables in $\mathbb{C}$; the analogue of the relation concerning multiplication is also definable.*

The formulas are as follows (here (x_i, y_i) denote the coordinates of $z_i \in \boldsymbol{U}_{\mathbb{G}_m \times \mathbb{G}_m} \cong \boldsymbol{U}_{\mathbb{G}m} \times \boldsymbol{U}_{\mathbb{G}m}$):

$$z_1 \sim_\Delta z_2 \Longleftrightarrow x_1 - y_1 = x_2 - y_2$$

$$z_1 \sim_{\{x_1^n = y_1\}} z_2 \Longleftrightarrow nx_1 - y_1 = nx_2 - y_2$$

$$(z_1, z_2) \sim_{\{x_1 y_1 = x_2\}} (z_3, z_4) \Longleftrightarrow x_1 + y_1 - x_2 = x_3 + y_3 - z_4$$

We can repeat the previous example word-by-word to get an example about elliptic curves.

Example II.1.4.3 ($\mathbb{A} = E$ an elliptic curve; the Weierstrass function $\mathbb{C} \to E(\mathbb{C})$). *Take $\mathbb{A} = E$ be an elliptic curve defined over $\mathbb{Q}$ and possessing a $\mathbb{Q}$-rational point $O \in E(\mathbb{C})$; thus it carries a addition operation defined over $\mathbb{Q}$ where O is the zero point. The universal covering space is $\mathbb{C}$: $\boldsymbol{U}_E = \mathbb{C}$ and $p = \rho$ is the Weierstrass function. Thus, the morphisms $+ : E(\mathbb{C}) \times E(\mathbb{C}) \to E(\mathbb{C})$ and morphism $nz : E(\mathbb{C}) \to E(\mathbb{C})$, and the closed diagonal subvariety $\Delta \subset E(\mathbb{C}) \times E(\mathbb{C})$ are defined over $\mathbb{Q}$, and the following equivalence relations on $\boldsymbol{U}_E = \mathbb{C}$ are in L_A:*

$$(x_1, y_1) \sim_\Delta (x_2, y_2) \Longleftrightarrow x_1 - y_1 = x_2 - y_2$$

$$(x_1, y_1) \sim_{\{nz = y\}} (x_2, y_2) \Longleftrightarrow nx_1 - y_1 = nx_2 - y_2$$

$$(x_1, y_1, z_1) \sim_{\{x + y = z\}} (x_2, y_2, z_2) \Longleftrightarrow x_1 + y_1 - z_1 = x_2 + y_2 - z_2$$

The fundamental group $\Lambda = \mathrm{Ker}\rho$ is a 2-generated lattice in $\mathbb{C}$ acting by translations $z \mapsto z + \lambda, \lambda \in \Lambda$, which are L_A-automorphisms of $\boldsymbol{U}$.

Similarly to $\mathbb{C}^*$-case, it is also true that any automorphism of the kernel Λ as an End E-module *can be extended to an automorphism of the whole structure $\boldsymbol{U}$. However, this is a statement about arithmetic of the elliptic curve, and is more difficult to prove. We study the example of an elliptic curve in the last chapter IV while proving the existence and homogeneity of the prime model.*

We will also see that the $L_A(E)$-structure on the covering $\mathbb{C}$-sort contains the pull-back of algebraic structure on $E(\mathbb{C})$ enriched by an affine, locally modular End E*-module structure on $\mathbb{C}$ itself.*

II.1.5 $L_A(x'_0)$-definable subgroups $\pi(\boldsymbol{U}, x'_0)$ of $\boldsymbol{U}$

By abuse of notation, we will use $\boldsymbol{U}$ to refer to the covering map $p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})$.

Recall we denote

$$\begin{aligned}\pi(\boldsymbol{U}) = \pi(p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})) = & \quad \mathrm{Gal}(p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})) \\ & = \{\tau : \boldsymbol{U} \to \boldsymbol{U} \text{ continuous } : p \circ \tau = p\}\end{aligned}$$

and call it the *group of deck transformations*, or the *group of Galois transformations*, or the *Galois group* of covering $p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})$. For a point $x'_0 \in \boldsymbol{U}$, let the fibre of p above $p(x'_0) \in \mathbb{A}(\mathbb{C})$ be denoted by

$$\pi(\boldsymbol{U}, x'_0) = \{x' \in \boldsymbol{U} : p(x') = p(x'_0)\} = \{\tau x'_0 : \tau \in \pi(\boldsymbol{U})\} = \{x' \in \boldsymbol{U} : x' \sim_\pi x'_0\};$$

the fibre $\pi(\boldsymbol{U}, x'_0)$ acquires the group structure from $\pi(\boldsymbol{U})$ via the bijection

$$\tau x'_0 \mapsto \tau.$$

Via that bijection, the group $\pi(\boldsymbol{U}, x'_0)$ acts on $\boldsymbol{U}$: in particular, $x'_0 \in \pi(\boldsymbol{U}, x'_0)$ corresponds to the identity of $\pi(\boldsymbol{U})$ and acts trivially. Actions of $\pi(\boldsymbol{U}, x'_0)$ and $\pi(\boldsymbol{U}, x'_1)$ are conjugated by the unique element of $\pi(\boldsymbol{U})$ taking x'_0 into x'_1.

II.2 The analogue of the fundamental groupoid associated to the L_A-structure on $\boldsymbol{U}$

We show that the L_A-structure $\boldsymbol{U}$ can be equivalently thought of as the fundamental groupoid of $\mathbb{A}(\mathbb{C})$; we do so by interpreting the homotopy class of a path in $\mathbb{A}(\mathbb{C})$ as an equivalence class of $\boldsymbol{U} \times \boldsymbol{U}/\!\sim_\pi \,\&\, \sim_\Delta$. We also discuss the definability of the fundamental group.

II.2.1 Relation to the fundamental group $\pi_1(\mathbb{A}(\mathbb{C}), x_0)$ of $\mathbb{A}(\mathbb{C})$

For any normal subgroup $H \lhd \pi(\boldsymbol{U})$, the map $p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})$ factors as

$$\boldsymbol{U} \to^{p_H} A^H(\mathbb{C}) \to^{p^H} \mathbb{A}(\mathbb{C}),$$

where both $p_H : \boldsymbol{U} \to A^H(\mathbb{C})$ and $p^H : A^H(\mathbb{C}) \to \mathbb{A}(\mathbb{C})$ are covering maps, and

$$H = \pi(p_H : \boldsymbol{U} \to A^H(\mathbb{C})) \text{ and also } \pi(\boldsymbol{U})/H \cong \pi(p^H : A^H(\mathbb{C}) \to \mathbb{A}(\mathbb{C})).$$

For a subset $X' \subset \boldsymbol{U}$, let

$$\pi(X') = \{\tau \in \pi(\boldsymbol{U}) : \tau X' \subset X'\}$$

$$\pi(X', x_0') = \{\tau \in \pi(\boldsymbol{U}, x_0') : \tau x_0' \text{ and } x_0'$$
$$\text{lie in the same connected component of } X'\}.$$

For a topological space B, one defines *the fundamental group* $\pi_1(B, b)$ as the group of all loops starting and ending at point b, with the operation of concatenation; the loops are considered up to homotopy, i.e. a continuous transformation of one into another. Given a universal covering space U of B with a distinguished point b' above b, $p(b') = b$, there is a well-defined action of $\pi_1(B, b)$ on U via the path-lifting property, as explained below.

The path-lifting property of a covering $p : U \to B$ says that for every path $\gamma : [0, 1] \to \mathbb{A}(\mathbb{C})$, and every point a' with $p(a') = a = \gamma(0)$, there exists a unique path $\tilde{\gamma} : [0, 1] \to U$ such that $p(\tilde{\gamma}(t)) = \gamma(t), 0 \leqslant t \leqslant 1$ and $\tilde{\gamma}(t) = a'$; intuitively, one should think that we "lift a path γ to the covering space piece by piece", using the fact that p is a local isomorphism between the neighbourhoods in U and B implied by the definition of a covering.

In fact it can be shown that, given a point $b' \in \boldsymbol{U}$, the path-lifting property provides a bijective correspondence between $\boldsymbol{U}$ and paths in B leaving $b = p(b')$: take the end-point of the lifted path beginning at the point b'. If any two points in U can be joined by a path, this identifies U with the set of all paths in B leaving b. In general, if U is not connected, this identifies the set of points in a connected component of U containing b', and the set of paths in the connected component of B containing b, up to homotopy (continuous transformation) fixing the paths; cf. Appendix V.1.1 for generalities and §V.1.3 for particulars about the fundamental group and the path-lifting property.

Thus, in our situation there are well-defined isomorphisms

$$\pi(\boldsymbol{U}, x_0') \cong \pi_1(\mathbb{A}(\mathbb{C}), x_0), \ \ x_0' \mapsto x_0 \text{ as the trivial path}$$

and thus we canonically identify

$$\pi(\boldsymbol{U}, x_0') = \pi_1(\mathbb{A}(\mathbb{C}), x_0).$$

Note, however, that the isomorphism $\pi(\boldsymbol{U}) = \pi_1(\mathbb{A}(\mathbb{C}), x_0)$ is defined only up to a conjugation by an element of $\pi(\boldsymbol{U})$; this corresponds to choosing a point x_0' above x_0.

With an embedding $\iota : X \to \mathbb{A}(\mathbb{C})$, there is also a topological way to associate a subgroup $\iota_* \pi_1(X, x_0) \subset \pi_1(\mathbb{A}(\mathbb{C}), x_0)$; indeed, a continuous map $f : X \to \mathbb{A}(\mathbb{C})$

induces a well-defined map on the homotopy classes of paths taking $\gamma : [0,1] \to X$ into the composition $f \circ \gamma : [0,1] \to \mathbb{A}(\mathbb{C})$. It is not hard to check that this induces a group homomorphism

$$f_* \pi_1(X, x_0) \subset \pi_1(\mathbb{A}(\mathbb{C}), f(x_0)).$$

For a subspace $X \subset \mathbb{A}(\mathbb{C})$, there is a natural embedding of fundamental groups $\iota_* : \pi_1(X, x_0) \to \pi_1(\mathbb{A}(\mathbb{C}), x_0)$, which takes a path into itself. We denote its image by $\iota_* \pi_1(X, x_0)$. Then under identification $\pi(\boldsymbol{U}, x_0') = \pi_1(\mathbb{A}(\mathbb{C}), x_0)$ it holds that

$$\iota_* \pi_1(X, x_0) = \pi(X', x_0').$$

II.2.2 Interpretation of U as paths: groupoid structure on $\boldsymbol{U}$

The L_A-structure $\boldsymbol{U}$ interprets the paths in $\mathbb{A}(\mathbb{C})$ up to homotopy fixing the endpoints.

The lifting property says that, given a path $\gamma : [0,1] \to \mathbb{A}(\mathbb{C})$, there is a unique path $\gamma' : [0,1] \to \boldsymbol{U}$ above γ such that $p \circ \gamma'(t) = \gamma(t), t \in [0,1]$, and starting at any point $\gamma'(0) = a_0'$ where $p(a_0') = a_0 = \gamma(0)$. Moreover, the homotopy class of γ' depends only on the homotopy class of γ. On the other hand, by properties of the universal covering space, the homotopy class of a path in $\boldsymbol{U}$ is determined by its endpoints. Thus, we see that a path γ in $\mathbb{A}(\mathbb{C})$ gives rise to a pair of points $\gamma'(0), \gamma'(1)$ defined up to π-action.

This leads us to the following.

We think of an element of $\boldsymbol{U} \times \boldsymbol{U}/_{\sim_\pi \,\&\, \sim_\Delta}$ as *a path in* $\mathbb{A}(\mathbb{C})$ *up to fixed-point homotopy*. Namely, $(x_1, y_1), (x_2, y_2) \in \boldsymbol{U} \times \boldsymbol{U}$ are *(homotopy)* equivalent iff $x_1 \sim_\pi x_2 \;\&\; y_1 \sim_\pi y_2 \;\&\; (x_1, y_1) \sim_\Delta (x_2, y_2)$. The *endpoints* of a path (x, y) are the points x and y considered up to $\sim_\pi$, i.e., points $p(x), p(y) \in \mathbb{A}(\mathbb{C}) = \boldsymbol{U}/\sim_\pi$. Thus, two paths (x_1, y_1) and (x_2, y_2) *have the same end-points* iff $x_1 \sim_\pi x_2 \;\&\; y_1 \sim_\pi y_2$.

Let $\pi(\mathbb{A}(\mathbb{C})) = \boldsymbol{U} \times \boldsymbol{U}/_{\sim_\pi \,\&\, \sim_\Delta}$ denote the *set of paths in* $\mathbb{A}(\mathbb{C})$ *up to fixed-point homotopy*. The set $\pi(\mathbb{A}(\mathbb{C}))$ carries a groupoid structure; we concatenate path (x, y) and (y, z) to obtain $(x, z) = (x, y) \cdot (y, z)$. A *loop* is a pair (x, y) of $\sim_\pi$-equivalent points, $x \sim_\pi y$. The set of loops based at a given point $a_0/\sim_\pi$ carries a group structure. It is customary to represent loops by pairs (a_0, y) where $a_0 \in \boldsymbol{U}$ is fixed. With this choice, the concatenation is given by the formula

(II.2.1)

$$(a_0, a_1) \circ \ldots \circ (a_0, a_n) = (a_0, y_n) \iff$$

$$\exists x_1 \exists y_1 \ldots \exists x_n \exists y_n ((a_0, x_1) \sim_\Delta (a_0, a_1) \;\&\; (x_1, y_1) \sim_\Delta (a_0, a_2) \;\&\; \ldots \;\&\; (x_n, y_n) \sim_\Delta (a_0, a_n))$$

Note the essential use of existential quantifiers in the formula. Let γ lie in a subset Z of $\mathbb{A}(\mathbb{C})$; then any lifting γ' of γ to $\boldsymbol{U}$ lies in $p^{-1}(Z) \subset \boldsymbol{U}$; in particular, each path γ' has to lie in a connected component of $p^{-1}(Z)$. Thus, we say that *path* $(a, b) \in \pi(\mathbb{A}(\mathbb{C}))$ *lies in* Z iff $p(a), p(b) \in Z$ and $a \sim_Z b$. Then we may think of a connected component of $p^{-1}(Z)$ as the set of end-points, or fixed homotopy classes, of all paths lying in Z lifted from a particular fixed point in Z.

II.2.3 Topology on the universal covering space $p : \boldsymbol{U} \to \mathbb{A}(\mathbb{C})$

It is not hard to see that all sets defined by relations $\sim_\pi$, $\sim_H$ have the property that their π-invariant closure is closed in analytic topology. Provided A is Shafarevich, we may introduce the following topology on U:

Definition II.2.3.1. *A subset of U is* étale closed *iff either of the following two equivalent conditions holds*

1. *it is a union, possibly infinite, of irreducible components of a finite number of $\pi(\boldsymbol{U})$-invariant analytic closed subsets of $\boldsymbol{U}$,*
2. *it is a union, possibly infinite, of connected components of a finite number of analytic closed subsets of $\boldsymbol{U}$ invariant under action of a finite index subgroup of $\pi(\boldsymbol{U})$,*

We will see that these conditions are equivalent in Lemma III.1.4.1 and Corollary III.5.3.4, and that étale closed sets do form a topology, provided A is Shafarevich. We use this fact freely in this chapter to describe examples.

There is another characterisation of the étale topology in terms of the structure on factor-space $\mathbb{A}^H(\mathbb{C}) = \boldsymbol{U}/H$; indeed, a closed π-invariant subset of $\boldsymbol{U}$ is the preimage of its image in $\boldsymbol{U}/H$. The action of H on $\boldsymbol{U}$ is free and discrete, and thus factor-space $\mathbb{A}^H(\mathbb{C}) = \boldsymbol{U}/H$ inherits the analytic structure from $\boldsymbol{U}$; similarly the image of a π-invariant closed analytic set is an analytic subset of $\boldsymbol{U}$. Now, the analytic space morphism $\boldsymbol{U}/H \to \mathbb{A}(\mathbb{C})$ is a covering, and by the generalised Riemann existence theorem (Fact III.1.2.3), $\boldsymbol{U}/H = \mathbb{A}^H(\mathbb{C})$ carries the structure of an algebraic variety defined over the algebraic closure of k. Since A is assumed projective, $\mathbb{A}^H(\mathbb{C})$ is projective also. By Chow lemma, any analytic subset of $\mathbb{A}^H(\mathbb{C})$ is an algebraic subset of it, and thus we come to the following equivalent definition.

Definition II.2.3.2. *A subset of $\boldsymbol{U}$ is called étale closed iff it is a union of the connected components of the preimages of a finite number of closed algebraic subvarieties of $\mathbb{A}^H(\mathbb{C})$ where $H \triangleleft_{\mathrm{fin}} \pi(\boldsymbol{U})$ (defined over $\mathbb{C}$).*

The advantage of this as a definition is that it could be generalised to other fields instead of $\mathbb{C}$.

An important property of the étale topology we are after, is that any analytic irreducible component of an étale closed set is étale closed and L_A-definable.

The étale topology is not compact: the fibre $p^{-1}(x_0) = \pi(\boldsymbol{U}, x_0')$ is a countable discrete subset of U. The discrete group $\pi(\boldsymbol{U})$ acts on $\boldsymbol{U}$ by continuous transformations. On the other hand, the ètale topology is complete: the projection of a closed set is closed (§III.2.2).

II.3 Examples of subvarieties and their associated relations $\sim_Z$

The previous subsections provide us with the definition of the object we want to consider. Let us see the basic constructions which are possible in L_A.

II.3.1 Basic Examples of L_A-definable sets associated to normal subvarieties

In this subsection, to give explicit examples, we use the principles that for a large class of analytic sets defined by a local property, a connected component is always irreducible. Namely, if Z is a normal, say smooth, analytic set, then a connected component of Z is irreducible. Thus, if one considers relations $\sim_Z$ for Z normal, it has to be an equivalence relation.

We give examples of interpretations of basic L_A-predicates:

Example II.3.1.1 (A arbitrary, Z normal). *For normal Z, the connected components of $p^{-1}(Z)$ are analytically irreducible. Let Z' denote a connected component of $p^{-1}(Z)$, then*

$$x \sim_Z y \iff \exists \gamma \in \pi(\boldsymbol{U}) : x, y \in \gamma Z'$$

Thus, $\sim_Z$ is an equivalence relation.

We will later see that for arbitrary Z, $\sim_Z$ is a union of a finite number of equivalence relations (cf. Lemma III.1.4.1, III.5.3.2).

Example II.3.1.2 ($\mathbb{A}$ arbitrary, $Z = \Delta \subset \mathbb{A}^2$; L_A-definability of action of $\pi(\boldsymbol{U})$). *Let $Z = \Delta = \{(x, x) : x \in X\}$ be the* diagonal *subvariety of $\mathbb{A} \times \mathbb{A}$. Then $p^{-1}(\Delta) = \{(x', \gamma x') : x' \in U, \gamma \in \pi(\boldsymbol{U})\}$, and the connected components of $p^{-1}(\Delta)$ have form $\Delta_\gamma = \{(x', \gamma x') : x' \in U\}, \gamma \in \pi(\boldsymbol{U})$. Thus*

$$(x, y) \sim_\Delta (z, t) \iff \exists \gamma \in \pi(\boldsymbol{U})\, (x = \gamma y \;\&\; z = \gamma t)\,. \tag{II.3.1}$$

In particular, for any point $x_0' \in U$ the formula $(x_0, x_0) \sim_\Delta (z, t)$ defines the diagonal $\Delta' = \{(x', x') : x' \in U\}$, and, for a point $z_0' \in U$, the formula $(z_0', t) \sim_\Delta (z_0', t)$ defines the fibre $p^{-1}(p(z_0')) = \pi(\boldsymbol{U}) z_0'$.

In general, given a point $x_0' \in U$*, the predicate* $\sim_Z$ *defines the action* $\tau : p^{-1}(x_0) \times U \to U$ *of* $p^{-1}(x_0), x_0 = p(x_0')$ *on* U *by defining* $\tau_y(z)$ *as the unique element such that*

$$\tau : p^{-1}(x_0) \times U \to U$$

$$(x_0', y) \sim_Z (z, \tau_y(z)).$$

Thus, we have an étale continuous $L_A(x_0')$*-definable action*

$$\pi(\boldsymbol{U}, x_0') \times \boldsymbol{U} \to \boldsymbol{U}$$

and an $L_A(x_0')$*-definable group* $\pi(\boldsymbol{U}, x_0')$*.*

Example II.3.1.3 ($\mathbb{A}$ arbitrary, $Z = \Gamma_f$ is a graph of a morphism $f : A \to A$). *A connected component of* $\Gamma_f = \{(x, f(x)) : x \in \mathbb{A}(\mathbb{C})\}$ *has form* $\{(x, f'(x)) : x \in U$*, for some function* $f' : \boldsymbol{U} \to \boldsymbol{U}$*, and we have* $p^{-1}(\Gamma_f) = \{(x, \gamma f'(x)) : x \in U, \gamma \in \pi\}$*. Thus, the function* $f' : \boldsymbol{U} \to \boldsymbol{U}$ *is* L_A*-definable with parameters.*

Analogously, the induced homomorphism $f_* : \pi(\boldsymbol{U}, x_0) \to \pi(\boldsymbol{U}, f(x_0))$ *of the fundamental groups is also definable with parameters.*

An important observation is that the function f' is quite often bijective while f is not; an example is $\mathbb{A}(\mathbb{C}) = \mathbb{C}^*$, $U = \mathbb{C}$, $f = z^2$: f' is just a multiplication by 2 (plus an additive constant). This hints that the structure upstairs in $\boldsymbol{U}$ is simpler than the structure on $\mathbb{A}(\mathbb{C})$, at least in some respects.

Example II.3.1.4 ($\mathbb{A}$ arbitrary, $X, Y, Z \subset \mathbb{A}$, $Z \cong X \times Y$). *Let* $f : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ *induce an isomorphism* $X \times Y \cong Z$ *between* $X \times Y$ *and* Z*. Then* $\Gamma_{f'}$ *embeds into* $\mathbb{A} \times \mathbb{A} \times \mathbb{A}$ *and* f' *provides an isomorphism* $X' \times Y' \to Z'$*.*

The next example *suggests* that to get *all* the extra structure on U, we may restrict ourselves by considering only maximal subvarieties with a given fundamental group (as a subgroup of π^n). Sometimes such varieties admit a very clear description and form a locally modular geometry. For example, in case of $\mathbb{A} = \mathbb{C}^{*n}$ such varieties are given by $\mathbb{Z}$-linear equations. In case of elliptic curves $\mathbb{A} = E$ they are given by $\operatorname{End} E$-linear equations. In case of Abelian varieties $\mathbb{A} = A$, this class consists of Abelian subvarieties. In case $\pi_1(A(\mathbb{C}))$ is Abelian, the fibres of Shafarevich morphisms should be enough (cf. [Kol95] for relevant definitions and results).

Example II.3.1.5 ($\mathbb{A}$ arbitrary, $Z \subset Y \subset \mathbb{A}$ normal, $\pi(Z, z') = \pi(Y, z')$). *For* $Z \subset Y$*,* $\pi(Z, z') = \pi(Y, z')$*, we have* $\gamma Z' \cap Y' \neq \emptyset$ *implies* $\gamma Y' \cap Y' \neq \emptyset$*;* Y' *is a connected component of* $p^{-1}(Y) = \pi Y'$*, and thus* $\gamma Y' = Y'$*, i.e.* $\gamma \in \pi(Y', z') = \pi(Z', z')$*. Thus we see* $Z' = \pi Z' \cap Y' = p^{-1}(Z) \cap Y'$*.*

In general, copies of Z' *in* Y' *are indexed by* $\pi(Y')/\pi(X')$*.*

The language L_A contains relations $\sim_Z$ only for varieties Z defined over the field k of definition of $\mathbb{A}$. A priori this does not imply that $\sim_Z$ are definable for Z an arbitrary subvariety of $\mathbb{A}(\mathbb{C})$, but it is true and does require an argument. Are connected components of $p^{-1}(Z)$, Z not necessarily defined over the ground field k, L_A-definable?

The following example shows a geometric condition implying that fibres of a L_A-definable irreducible set are L_A-definable with parameters; that condition always holds for generic fibres, cf. Corollary III.2.2.2, Proposition III.2.1.1 for exact statement.

Example II.3.1.6 (A arbitrary, $Z = Y(x,\alpha)$ connected normal, $Y \subset A^2$ is normal and defined over k, α arbitrary). *Take $Z = Y(x,\alpha) = Y_\alpha$, Y/k, and consider fibre $Y'_{\alpha'} = Y' \cap U \times \{\alpha'\}$ where $p(\alpha') = \alpha$. Then, $Y'_{\alpha'}$ is a union of π-translates of Z', which do not intersect due to normality of Z, and thus it is connected iff $Y'_{\alpha'}$ is in fact just Z' itself. That is, $Y'_{\alpha'} = Z'$ is connected iff for any $\gamma \notin \pi(Z')$, we have $(\gamma, \mathrm{id}) \notin \pi(Y')$. Thus, we conclude that $Y'_{\alpha'} = Z'$ iff $\pi(Y') \cap \pi \times \mathrm{id} = \pi(Z') \times \mathrm{id}$.*

In general, similar considerations show that the translates of $Z' \times \{\alpha\}$ within Y are indexed by ${}^{\pi(Y')}\!/\!_{\pi(Z') \times \mathrm{id}}$.

In fact, we will later see in Lemma III.2.2.2, Lemma III.2.1.1 that the condition $\pi(Y') \cap \pi \times \mathrm{id} = \pi(Z') \times \mathrm{id}$ always holds for $\alpha \in \operatorname{pr} Y(\mathbb{C})$ generic, provided Y irreducible (assuming $\mathbb{A}$ is Shafarevich, as always); see the Lemma for exact formulation. This will imply that L_A is able to define generic fibres of closed sets, and in fact all étale closed sets.

Example II.3.1.7 ($A = X \times X$, $\boldsymbol{U}_A \cong \boldsymbol{U}_X \times \boldsymbol{U}_X$). *For $A = X \times X$, $Z = \Delta = \{(x,x) : x \in X\}$ the diagonal $\Delta' = \{(x',x') : x' \in U\}$ is a connected component of $p^{-1}\Delta$; other connected components are $p^{-1}(\Delta) = \{(x', \gamma x') : x' \in U\}$.*

Thus we see that in L_A, the decomposition $\boldsymbol{U}_A \cong \boldsymbol{U}_X \times \boldsymbol{U}_X$ is definable over a parameter.

Example II.3.1.8 ($\mathbb{A} = L^*$ is a homogenous $\mathbb{C}^*$-bundle over an Abelian variety A). *Let $U \to A$ be the universal covering space of A, let $\operatorname{pr} : L^* \to A$ be a homogeneous $\mathbb{C}^*$-bundle over A, and let $\boldsymbol{U}_{L^*} \to L^*$ be the universal covering space of $\mathbb{C}^*$-bundle L^*. We want to define* Chern class *of L^* in $H^2(\pi_1(A), \mathbb{Z}) \cong \wedge^2 H^1(\mathbb{A}(\mathbb{C}), \mathbb{Z})$. Take $\lambda'_1, \lambda'_2 \in \pi(U_L)$, and consider $\tau = [\lambda_1, \lambda_2] = \lambda_1\lambda_2\lambda_1^{-1}\lambda_2^{-1}$. Then $\operatorname{pr}[\lambda_1, \lambda_2] = [\operatorname{pr}\lambda'_1, \operatorname{pr}\lambda'_2] = 0$, and thus, for each fibre F_a of L, we have $[\lambda_1, \lambda_2] \in \pi(F_y) \cong \mathbb{Z}$. Thus we have a map $\pi(U_L) \times \pi(U_L) \to \pi(F_y)$. Moveover, a simple topological argument gives that $\gamma\lambda = \lambda\gamma$ for $\gamma \in \pi(F_y)$ and $\lambda \in \pi(L)$ arbitrary; using paths interpretation, this is because one can shift γ along λ in L^* by multiplicating it by $\gamma(t)$. Thus, $\pi(F_y)$ is central in $\pi(L)$, and the commutator* *map on $\pi(U_{L^*})$ descends to a map*

$$\pi(U_A) \times \pi(U_A) \to \pi(F_y),$$

i.e. a map $\Lambda \times \Lambda \to \mathbb{Z}$. This map could be checked to be bilinear and may be considered as an element of

$$\bigwedge^2 H^1(A(\mathbb{C}), \mathbb{Z}) \cong H^2(\Lambda, \mathbb{Z}).$$

The above construction corresponds to constructing group cohomology associated to short exact sequence of $\mathbb{Z}$-modules

$$0 \to \pi(F_y) \to \pi(U_{L^*}) \to \pi(U_A) \to 0$$

Cf. §IV.6.1 and [Mum70, p.239] for more details on line bundles and their bilinear forms associated to Abelian varieties.

To conclude, we see that in the case $\mathbb{A} = L^*$ there is a bilinear form definable on fibres of the covering map. Moreover, in §IV.6.1 we prove that if the form is non-degenerate then it allows one to interpret the ring of integers with addition and multiplication; this makes first-order theory of the structure unstable. Despite this, as was said before, we prove that it is model stable in $L_{\omega_1\omega}$; and in general conjecture it to be uncountably categorical in $L_{\omega_1\omega}$.

Example II.3.1.9 (A arbitrary, $\mathrm{pr}\, Z$). *We will see later that for every Z' irreducible in $\boldsymbol{U}$, the projection $\mathrm{pr}\, Z$ is étale closed, and there exists a finite index subgroup H such that $\pi(\mathrm{pr}\, Z') \cap H = \mathrm{pr}\,(\pi(Z') \cap [H \times H])$*

II.4 More examples of universal covering spaces

II.4.1 Examples of 1-dimensional universal covering spaces

Here we give a complete classification of the universal covering spaces of 1-dim complex algebraic curves. Such curves are also known as *Riemann surfaces*; they have two real dimensions.

The following are all simply connected (i.e. $\pi(X) = 0$ is trivial):

$\mathbb{CP}^1$ is the Riemann sphere

$\mathbb{C}$ is the Gaussian plane (i.e. the complex field viewed as a complex analytic space)

$\mathbb{H} = \{\boldsymbol{Im}\ z > 0\}$ is the upper half-plane, or Lobachevsky plane

$\mathbb{D} = \{|z| < 1\}$ is the unit disk

Their group of automorphisms as complex analytic spaces are:

$\mathrm{Aut}\mathbb{C} = \{z \mapsto az + b\}$
$\mathrm{Aut}\mathbb{CP}^1 = \mathrm{GL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$
$\mathrm{Aut}\mathbb{H} = SL(2,\mathbb{R})/\pm I$

Thus, $\Gamma = \pi_1(X(\mathbb{C}))$ is a discrete subgroup of $\mathrm{Aut}S$. The unit disk $\mathbb{D}$ and upper-half plane $\mathbb{H}$ are isomorphic as Riemann surfaces.

II.4.2 Classification of 1-dim Riemann surfaces and complete algebraic curves

Theorem II.4.2.1. *Any Riemann surface X (in particular, a complex algebraic curve) is isomorphic to a quotient $X = {}^{S}\!/_{\Gamma}$, where S is one of the simply connected* canonical regions $\mathbb{CP}^1, \mathbb{C}, \mathbb{H}$, *and the group Γ acts on S freely and discretely by automorphisms.*

Thus, $\Gamma = \pi_1(X)$ is the fundamental group of the Riemann surface X; for X a Riemann surface, such groups admit an explicit description in terms of generators and relations.

Γ_g : For $X(\mathbb{C})$ a complete algebraic curve, i.e. a compact Riemann surface, then the fundamental group of a complete algebraic curve $X(\mathbb{C})$ is

$$\Gamma_g =< a_1, b_1, \ldots, a_g, b_g : a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} > .$$

$\Gamma_{g,n}$: The fundamental group of a *punctured* compact Riemann surface $X(\mathbb{C}) - \{p_1, \ldots, p_n\}$ without n points, is

$$\Gamma_{g,n} =< a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n : a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} c_1 \ldots c_n > .$$

For $g = 0$, Γ_g is trivial, and $X = \mathbb{CP}^1$. For $g = 1$, $X = {}^{\mathbb{C}}\!/_{\Lambda}$ is an elliptic curve, and $\Gamma_1 = \mathbb{Z} \times \mathbb{Z}$. For $n > 1$, the groups Γ_n are non-commutative, hyperbolic.

II.4.3 Some other examples of covering spaces

Here are the basic examples of the universal covering spaces of complex manifolds. Incidently, this lists all universal covering spaces of 1-dim Riemann surfaces.

$\mathbb{CP}^1 \to \mathbb{CP}^1$ is the universal covering space of itself
$\mathbb{C} \to \mathbb{C}$ is the universal covering space of itself
$\mathbb{C} \xrightarrow{\exp} \mathbb{C}/2\pi i\mathbb{Z} = \mathbb{C}^*$ is the universal covering space of the multiplicative group $\mathbb{C}^*$
$\mathbb{C} \to \mathbb{C}/\Lambda = E_\Lambda(\mathbb{C})$ is the universal covering space of an elliptic curve, where Λ is a 2-dim discrete lattice in $\mathbb{C}$
$\mathbb{C}^{2g} \to \mathbb{C}^{2g}/\Lambda$ is the universal covering space of a complex torus $\mathbb{C}^{2g}/\Lambda$; for some choices of a $2g$-dimensional lattice Λ the complex torus $\mathbb{C}^{2g}/\Lambda = A(\mathbb{C})$ has the structure of an algebraic variety (*Abelian variety*)

$\mathbb{H} \to \mathbb{H}/\Gamma$ is the universal covering space of the quotient $\mathbb{H}/\Gamma$ where $\Gamma \subset SL(2,\mathbb{R})/\pm I$ is a discrete subgroup of $SL(2,\mathbb{R})/\pm I$; for some choices of $\Gamma \cong \Gamma_g$ one obtains the complete algebraic curves of genus $g > 1$.

The universal covering space of a direct product is the direct product of the universal covering spaces.

Chapter III

Geometric Case: model stability of the universal covering space of a variety

In this chapter we study the universal covering space $\boldsymbol{U}$ of a smooth projective algebraic variety $A(\mathbb{C})$; we assume that $\boldsymbol{U}$ is *holomorphically convex*, or rather, that the complex analytic space $\boldsymbol{U}$ satisfies the conclusions of Fact III.1.2.1. We also make an strong assumption that the Cartesian powers of the fundamental group of $A(\mathbb{C})$ are *subgroup separable*, or *locally extended residually finite* often abbreviated *lerf*. This assumption seems to be technical; it is not true even for complex curves. We then introduce a topology *Ét* on $\boldsymbol{U}$ which we like to call *an étale topology;* it has the property that projection of a closed set is closed; it inherits the descending chain condition and the existence of an irreducible decomposition from the analytic Zariski topology on $\boldsymbol{U}$; Proposition III.2.1.1 expresses an essential property of a more technical character relating the topology and the action of the fundamental group $\pi(\boldsymbol{U})$.

The properties of the étale topology *Ét* on $\boldsymbol{U}$ allow us to introduce a countable language L_A such that any étale irreducible closed subset of $\boldsymbol{U}$ is L_A-definable; we also introduce an analogue of Galois group $\mathrm{Aut}_{L_A}(\boldsymbol{U})$ for $\boldsymbol{U}$. Then we study $\boldsymbol{U}$ as an L_A-structure and prove that it is model homogeneous.

In the second half of the chapter we introduce an $L_{\omega_1\omega}(L_A)$-axiomatisation for $\boldsymbol{U}$ and prove that it describes a class of model homogeneous models which satisfies conclusion $2_{\aleph_0 \to \aleph_1}$ of Theorem III.5.4.7.

III.1 A Zariski-type topology on a universal covering space

We define here an étale topology *Ét* on the holomorphically convex universal covering space $\boldsymbol{U}$ of a projective complex algebraic variety $A(\mathbb{C})$ which is an analogue of Zariski topology on $A(\mathbb{C})$. The étale topology is substantially weaker than the analytic Zariski topology on $\boldsymbol{U}$, i.e. the topology given by closed analytic subsets of the complex space $\boldsymbol{U}$.

An important feature of the étale topology on $\boldsymbol{U}$ is that an analogue of Chevalley lemma holds for *Ét*; recall that Chevalley Lemma for a compact complex algebraic variety says that a projection of a closed set is closed; model-theoretically it is the quantifier elimination to the level of closed sets. We prove the analogous property of *Ét* in §III.2.2. Note that this property fails in general analytic context: there is an example of a closed analytic subset in $\mathbb{C}^3$ whose projection on $\mathbb{C}$ is contained in an open disk.

III.1.1 Definition of the étale topology on a holomorphically convex universal covering space U

Recall that $p : U \to A(\mathbb{C})$ denotes the universal covering space of a projective algebraic variety $A(\mathbb{C})$, and that we assume U to be a holomorphically convex (as a complex analytic space). We need the latter assumption in order for *Ét* to be a topology indeed. Recall $\pi = \pi(\boldsymbol{U})$ denotes the group of deck transformations of U.

Definition III.1.1.1 (Etale (pre)topology on holomorphically convex universal covering space U of an algebraic variety $A(\mathbb{C})$). *An analytic subset of* $\boldsymbol{U}$ *is called* étale closed *iff it is a union, possibly infinite, of the irreducible analytic components of finitely many π-invariant closed analytic subsets of* $\boldsymbol{U}$.

Similarly we extend the definition to $\boldsymbol{U}^n$, for any n. We denote the collection of étale closed subsets of $\boldsymbol{U}^n$ by Ét.

A π-invariant closed set of $\boldsymbol{U}$ covers a closed subset of $A(\mathbb{C})$. The covering map $p : \boldsymbol{U} \to A(\mathbb{C})$ being local isomorphism and analyticity being a local property, it implies that it covers a closed analytic subset. The ambient variety $A(\mathbb{C})$ is assumed projective, and by Chow Lemma, such a set is a Zariski closed algebraic subset of $A(\mathbb{C})$. Therefore, a π-invariant closed set of $\boldsymbol{U}$ covers a closed algebraic subset of $A(\mathbb{C})$.

It would be natural to consider the unions of *connected components* of such sets and not that of irreducible ones; this is natural if we try not to use the analytic structure of $\boldsymbol{U}$ but only the *topological* structure of $\boldsymbol{U}$ as the covering space of $A(\mathbb{C})$ as an algebraic variety. This seems plausible because for analytic sets good enough (smooth or even normal), the notions of a connected component and an irreducible component coincide (cf. §V.2.3), and indeed, that is possible, with a price:

An étale closed set is a union of connected components of finitely many H-invariant closed analytic sets, for a finite index subgroup $H \triangleleft_{\mathrm{fin}} \pi$.

We prove the equivalence of these two definitions in Decomposition Lemma III.1.4.1.

III.1.2 Normalisation and Local-to-global principles

The proof of Decomposition Lemma III.1.3.1 essentially uses the various local-to-global properties implied by homomorphic convexity: Local Identity Principle(Uniqueness of Analytic Continuation), and others. It also uses the properties of normalisation of algebraic varieties; we state them here, too.

We choose to list the properties here to put an emphasis on the properties of complex analytic space $\boldsymbol{U}$ which we use in the proof of the Decomposition Lemma. The exact formulation of those properties may be useful as a property one would expect from analytic Zariski structures.

Consequences of holomorphic convexity

To avoid confusion, below we say "an open ball" to mean a neighbourhood open in complex topology, not in the analytic Zariski topology.

Fact III.1.2.1. *Let U be a holomorphically convex space, and let $Y, Z \subset U$ be closed analytic subsets in U. Then*

1. *(analyticity is a local property) a set $X \subset U$ is analytic iff for all $x \in X$, there exists an open ball $x \in B_x$ such that $X \cap B_x$ is an analytic subset of B_x*
2. *(local identity principle) for an open ball $B \subset U$, if Y is irreducible and $Y \cap B \subset Z \cap B$ then $Y \subset Z$*
3. *(local identity principle; analytic continuation) for an open ball $B \subset U$, if Y and Z are irreducible, and $Y \cap B$ and $Z \cap B$ have a common irreducible component, then $Y = Z$*
4. *(density of smooth points) for an open ball $B \subset U$, if $Z_0 \subset Z \cap B$ is an irreducible component of $Z \cap B$, then there exist a point $z_0 \in Z_0$ and an open ball $z_0 \in B_0 \subset B$ such that $B_0 \cap Z \subset Z_0$*
5. *(local finiteness) a compact set $C \subset U$ intersects only finitely many irreducible components of a closed analytic set Z*
6. *(analyticity of a union of irreducible components) a union of, possibly infinitely many, irreducible components of an analytic set is analytic*
7. *(irreducible decomposition) if $Y \subset Z$ and Y is irreducible, then Y is contained in an irreducible component of Z*

Proof. Those are well-known properties of holomorphically convex spaces.

By Prop. 5.3 of [Č85], Theorem 5.1 [ibid.] states (6) and (5). Corollary 2 of Prop. 5.3 [ibid.] implies (2) and (3). Theorem 5.4 [ibid.] implies (4). (2,3,4) together imply (7). □

Fact III.1.2.2 (Chow Lemma). *A closed analytic subset of a complex projective algebraic variety is algebraic.*

Proof. This is a well-known fact in algebraic geometry, cf. Hartshorne [Har77]. □

We also need a generalised Riemann existence theorem.

Fact III.1.2.3 (Genenalised Riemann existence theorem). *Let $A(\mathbb{C})$ be a normal algebraic variety over $\mathbb{C}$. If $q : T \to A(\mathbb{C})$ is a covering of topological spaces, then T admits a structure of a complex algebraic variety such that $q_{\text{top}} : T \to A(\mathbb{C})$ becomes an algebraic morphism, i.e. there exists an algebraic variety $B(\mathbb{C})$ over $\mathbb{C}$, an algebraic morphism $q_{\text{alg}} : B(\mathbb{C}) \to A(\mathbb{C})$, and a homeomorphism $\varphi : T \to B(\mathbb{C})$ of topological spaces such that the diagramme of topological spaces commutes*

$$\begin{array}{ccc} T & \xrightarrow{q_{\text{top}}} & A(\mathbb{C}) \\ \varphi \downarrow & & \text{id} \downarrow \\ B(\mathbb{C}) & \xrightarrow{q_{\text{alg}}} & A(\mathbb{C}) \end{array}$$

Moreover, the homeomorphism $\varphi : T \to B(\mathbb{C})$ is well-defined up to an automorphism of B commuting with the covering morphism q_{alg}.

Proof. Our conventions (Appendix B, §V.2.1) imply a variety over $\mathbb{C}$ is a Noetherian scheme of finite type over $\mathbb{C}$. The existence of an analytic space B with the above properties follows from the fact that we may pull back the local analytic structure of $A(\mathbb{C})$ onto T; in 1-dim case this already implies that B would be an algebraic variety by Riemann existence theorem; the general case is done in Grothendieck [SGA1,Exp.XII,Th.5.1]; one may also look in [Har77, Appendix B,§3,Theorem 3.2] for some explanations. □

Fact III.1.2.4. *A closed analytic subset of a holomorphically convex set admits a unique decomposition into a countable union of analytic irreducible closed subsets.*

Proof. [Č85, §5.4,Theorem, p.49]. □

Fact III.1.2.5. *A connected component of a normal analytic set is irreducible.*

Proof. [Č85] □

We use the following fact as the defining property of an étale covering: the morphism $B(K) \to A(K)$ of varieties over an algebraically closed field K of char 0 is *étale* iff there exists an embedding $i : K \to \mathbb{C}$ of the field of definition of A and B into $\mathbb{C}$ such that the corresponding morphism $i(B)(\mathbb{C}) \to i(A)(\mathbb{C})$ is a covering of topological spaces.

Subgroup separability of $\pi(\boldsymbol{U})$

A group π is called *subgroup separable*, or *locally extended residually finite*, often abbreviated *lerf*, iff for any subgroup $G < \pi$ and an element $g \notin G$ there exists a *finite* index subgroup H such that $G < H$ and $g \notin H$. This is a non-trivial property rather hard to establish; it is known that the fundamental groups of complex curves ([Sco85]) and $\mathbb{Z}^n$, $\mathrm{SL}_2(\mathbb{Z})$ are subgroup separable; however, it is known that $F_2 \times F_2$ ([Mil71]) is not subgroup separable, and so in general the products of subgroup separable groups are not subgroup separable. This property may be reformulated topologically: the group $\pi = \pi_1(A)$ is subgroup separable if and only if for any finitely generated $G < \pi$ and any compact subset $C \subset A^G$ of the covering space A^G, there is a finite cover $X^H \to X$ of X such that the projection from X^G factors through X^H and such that C maps by a homeomorphism into X. In fact, we need this property only when G is the fundamental group of an algebraic subset of A, i.e. when C is an étale closed subset of X^G, not necessarily compact.

Shafarevich varieties

Definition III.1.2.6. *We call a smooth projective algebraic variety $A(\mathbb{C})$* Shafarevich *if*

1. *the universal covering space of $A(\mathbb{C})$ satisfies the conclusions of Fact III.1.2.1*
2. *the group of deck transformations $\pi(\boldsymbol{U})$ is subgroup separable, as well as all its Cartesian powers.*

Thus, the only explicit class of varieties to which the results of this section are applicable, is that of semi-Abelian varieties. However, we hope that the subgroup separability is only a technical property due to methods of the proofs of the thesis.

III.1.3 A geometric Decomposition Lemma; Noetherian property

The lemma states a finiteness property of the irreducible decomposition of the preimage of an algebraic subvariety in $A(\mathbb{C})$; it may be interpreted as saying that the irreducible components are not too far from being *connected* components of the preimage, *up to finite index*.

For a subset $Z \subset U$, let $\pi Z = \bigcup_{\gamma \in \pi} \gamma Z'$ denote the π-orbit of set Z.

For $H \lhd_{\mathrm{fin}} \pi$, let $p_H : \boldsymbol{U} \to \boldsymbol{U}/\sim_H$ be the factorisation map; by Fact III.1.2.3, we choose and fix isomorphisms $A^H(\mathbb{C}) \cong \boldsymbol{U}/_{\sim_H}$ where $A^H(\mathbb{C})$ is an algebraic variety; the deck group of covering $A^H(\mathbb{C}) \to A(\mathbb{C})$ is the finite group π/H.

Lemma III.1.3.1 (First Decomposition lemma; Noetherian property). *Assume A is Shafarevich.*

A π-invariant analytic closed set has an analytic decomposition of the form

$$W' = HZ'_1 \cup \ldots \cup HZ'_k,$$

where $H \lhd_{\mathrm{fin}} \pi$ is a finite index normal subgroup of π, the analytic closed sets $Z'_1, \ldots, Z'_k$ are irreducible, and for any $\tau \in H$ either $\tau Z'_i = Z'_i$ or $\tau Z'_i \cap Z'_i = \emptyset$.

Such decomposition also exists for closed analytic sets invariant under the action of a finite index subgroup of π.

Proof. Let us prove that (a) there exists a decomposition as above without the condition on intersections, and then prove (b) the irreducible components satisfy $\tau Z'_i = Z'_i$ or $\tau Z'_i \cap Z'_i = \emptyset$ for $\tau \in \pi$.

The proof of (a) is relatively simple, and follows from the Fact III.1.2.1 in a rather straightforward way; we do it first.

The proof of the second claim (b) uses rather more delicate local analysis of the structure, and several local-to-global properties of analytic subsets of holomorphically convex spaces as well as some finiteness properties of Zariski geometry of algebraic varieties.

So let us start to prove (a). Let Z' be an irreducible component of $p^{-1}(Z(\mathbb{C}))$; by π-invariance of $p^{-1}(Z(\mathbb{C}))$, for any $\gamma \in \pi$, the set $\gamma Z'$ is also an irreducible component of $p^{-1}(Z(\mathbb{C}))$, and so $\pi Z'$ is a union of irreducible components of $p^{-1}Z(\mathbb{C})$; thus, by Fact III.1.2.1 above, $\pi Z' \subset p^{-1}(Z(\mathbb{C}))$ is analytic.

The covering morphism $p : U \to A(\mathbb{C})$ is a local isomorphism, and analyticity is a local property; by π-invariance of $\pi Z'$, it implies $p(\pi Z')$ is analytic. For different irreducible components $Z'_1 \neq Z'_2$ of $p^{-1}(Z(\mathbb{C}))$ it can not hold that $p(Z'_1) \subsetneq p(Z'_2)$; indeed, then $\pi Z'_1 = p^{-1}p(Z'_1) \subset \pi Z'_2 = p^{-1}p(Z'_2)$, and so $Z'_1 = \bigcup (Z'_1 \cap \gamma Z'_2), \gamma \in \pi$; thus, Z'_1 can not be irreducible unless $Z'_1 \subset \gamma Z'_2$, for some $\gamma \in \pi$, which is impossible by π-invariance of $p^{-1}Z(\mathbb{C})$. To conclude, closed sets $p(Z')$, Z' vary among irreducible components of an algebraic subvariety $Z(\mathbb{C})$, cover the whole of $Z(\mathbb{C})$; they are also irreducible. Thus they are the analytic irreducible components of Z. By [GR65], the analytic irreducible components of an algebraic set are algebraic and irreducible, and thus they are the algebraic irreducible components; in particular there are only finitely many of them. That gives the required decomposition.

Now let us start to prove (b). First of all, note that we may suppose Z to be irreducible.

Let $Z'^{(n)} = \bigcup Z'_{i_1} \cap \ldots \cap Z'_{i_n}$ be the union of all intersections of n-tuples of different irreducible components of $p^{-1}(Z(\mathbb{C}))$.

Claim III.1.3.2. *The set $p(Z'^{(n)})$ is an algebraic subset of $Z(\mathbb{C})$, for $n > 0$. For n sufficiently large, $Z'^{(n)}$ is empty.*

Proof. By the local finiteness (Fact III.1.2.1) a compact subset intersects only finitely many of the irreducible components $\gamma Z'_i$'s; thus $Z'^{(n)}$ is locally a finite union of intersections of analytic sets, and therefore is analytic. By the π-invariance of $\gamma Z'_i$'s it is π-invariant, and thus p provides a local isomorphism of $Z'^{(n)}$ and its image; therefore the image $p(Z'^{(n)})$ is analytic. By Chow Lemma III.1.2.2 this implies it is in fact algebraic. If n is greater then the number of local irreducible components at a point of Z in A, then by Fact III.1.2.1(local identity principle) $Z'^{(n)}$ has to be empty. $\square$

The claim above implies $Z'^{(n)}$ are étale closed, for any n. By Claim (a) of Lemma, we may choose finitely many points z'_i's so that any irreducible component of $Z'^{(n)}$, for each $n > 0$, contains a π-translate of one of z'_i's.

Choose $Z'_1, \ldots, Z'_k$ to be the irreducible components of $p^{-1}(Z(\mathbb{C})$ containing any of the points z'_i's: by Fact III.1.2.1(5) there are only finitely many irreducible components of $p^{-1}(Z(\mathbb{C}))$ containing each point z'_i. The group $\pi(Z'_i)$ is finitely generated because it is the image in π of the fundamental group of the interior of the closed algebraic subset $p(Z'_i)$. By subgroup separability of π we see that there exists a finite index subgroup $H \subset \pi$ such that $HZ'_i \neq HZ'_j$ for $i \neq j$, i.e. $p_H(Z'_i) \neq p_H(Z'_j)$.

Consider $Z'_i \cap hZ'_i, h \in H$ and assume its non-empty. Then there exists $\gamma^{-1} \in \pi$ such that $\gamma^{-1} z'_j \in Z'_i \cap hZ'_i$, i.e. $z'_j \in \gamma Z'_i \cap \gamma h Z'_i = \gamma Z'_i \cap h'\gamma Z'_i$. Both $\gamma Z'_i$ and $h'\gamma Z'_i$, $h' \in H$ are connected components containing z'_j and by definition we have chosen H small enough so that $H\gamma Z'_i \neq Hh'\gamma Z'_i$, a contradiction.

In other words, we have proven that there exists a finite index subgroup $H < \pi(A(\mathbb{C})$ such that Z'_i is a connected component of $p_H^{-1} p_H(Z'_i)$, i.e. the connected components of the preimages of the irreducible components of $p_H p^{-1}(Z(\mathbb{C}))$ are irreducible. $\square$

The next corollary allows for an equivalent definition of the étale topology.

Notice that the notion of an H-invariant set is essentially algebraic: an H-invariant set is a preimage of a closed algebraic subset in finite cover $A^H(\mathbb{C})$. Thus, the meaning of the next corollary that in fact étale closed sets encode a mix of algebraic data and *topological, homotopical* data, not of analytic one.

Corollary III.1.3.3. *A set is étale closed iff it is a union of* connected *components of a finite number of H-invariant sets, for some $H \lhd_{\text{fin}} \pi$ a finite index subgroup of π.*

Proof. Lemma III.1.3.1 above implies that each étale closed set can be represented in such a form.

On the other hand, the lemma implies that each H-invariant set is a finite union of sets of the form HZ_i' where Z_i' are irreducible. Then, $\pi Z_i'$ is also closed analytic as a finite union of translates of HZ_i', and moreover, each translate of Z_i' is an irreducible component of $\pi Z_i'$ and thus étale closed. This implies the converse of the corollary. □

The Lemma has the following algebraic consequence. All the notions mentioned in the Corollary are preserved under replacing the ground field by another algebraically field; thus it holds for any characteristic 0 algebraically closed field instead of $\mathbb{C}$. One may think of this property as a rather weak property of irreducible decomposition for the *étale* topology; it is also a statement about a resolution of non-normal singularities.

Corollary III.1.3.4. *Let A be Shafarevich. Then for any closed subvariety $Z \subset A(\mathbb{C})$, there exists a finite étale cover $q : A^H(\mathbb{C}) \to A(\mathbb{C})$ such that, for any further étale cover $q' : A^G(\mathbb{C}) \to A^H(\mathbb{C})$, the connected components of ${q'}^{-1}(Z_i) \subset A^G(\mathbb{C})$ are irreducible, where Z_i's are the irreducible components of $q^{-1}(Z)$.*

Proof. Indeed, it is enough to take H as in Decomposition Lemma. □

Note that when Z is normal, the corollary is a well-known geometric fact.

III.1.4 Decomposition Lemma for the étale topology

Recall notation $\pi Z' = \bigcup_{\gamma\in\pi} \gamma Z'$. Note we do not yet known that the intersection of two étale closed sets is étale closed.

Corollary III.1.4.1 (Decomposition Lemma). *Assume A is Shafarevich. The collection Ét of étale closed subsets of $\mathbf{U}$ forms a topology with a descending chain* *conditions on irreducible sets. An étale closed set possesses an irreducible decomposition as a union of a finite number of étale closed sets whose étale connected components are étale irreducible. A union of irreducible components of an étale closed set is étale closed.*

That is,

1. *the collection of étale closed subsets on $\boldsymbol{U}^n, n > 0$ forms a topology. The projection and inclusion maps* $\mathrm{pr} : \boldsymbol{U}^n \to \boldsymbol{U}^m, (x_1, \ldots, x_n) \mapsto (x_{i_1}, \ldots, x_{i_m})$ *and* $\iota : \boldsymbol{U}^n \hookrightarrow \boldsymbol{U}^m, (x_1, \ldots, x_n) \mapsto (x_{i_1}, \ldots, x_{i_{m'}}, c_{m'}, \ldots, c_m)$ *are continuous.*
2. *There is no infinite decreasing chain* $.. \subsetneq U_{i+1} \subsetneq U_i \subsetneq \ldots \subsetneq U_0$ *of closed étale irreducible sets.*
3. *A union of irreducible components of an étale closed set is étale closed.*
4. *A set is étale closed iff it a union of* connected *components of a finite number of H-invariant sets, for some $H \lhd_{\mathrm{fin}} \pi$ a finite index subgroup of π.*
5. *Each étale closed set is a union of a finite number of étale closed sets whose étale connected components are étale irreducible. Moreover, those sets may be taken so that their connected components within the same set are translates of each other by the action of a finite index subgroup $H \lhd_{\mathrm{fin}} \pi$.*

The following property is "ideologically" important, and is the main property in proving the properties of the collection *Ét*. An analogue of this property should also hold in other examples, say full exponentiation; there it says that a *definably* irreducible set is analytically irreducible.

Lemma III.1.4.2. *An étale irreducible closed set is analytically irreducible, i.e. it is irreducible as an analytic subset of $\boldsymbol{U}$.*

Proof of Lemma. By definition III.1.1.1, an étale irreducible étale closed set W' is a countable union of irreducible component of π-invariant closed analytic sets. Those components are étale closed by definition, and thus étale irreducibility implies the union is necessarily trivial. Thus, the set is an analytic irreducible component of a π-invariant set, i.e. in particular irreducible as an analytic set. □

Proof of Corollary III.1.4.1. We defer the proof of (1) until we prove (4), (2), (3).

By Lemma III.1.4.2, a decreasing chain of étale irreducible sets is a decreasing sequence of closed analytic irreducible sets, and this implies (2) immediately.

Property (3) is immediate from the same property of the irreducible decomposition of analytic sets (Fact III.1.2.1(6)).

A finite union of étale closed sets satisfying (4) also satisfies (4), and thus it is enough to prove that a union V' of the irreducible components of a π-invariant set $W' = \pi V'$ satisfies (4). By Decomposition Lemma III.1.3.1, set $W' = \pi V'$ admits a decomposition

$$W' = HZ'_1 \cup \ldots \cup HZ'_k,$$

where $H \lhd_{\mathrm{fin}} \pi$ is a finite index normal subgroup of π, the analytic closed sets $Z'_1, \ldots, Z'_k$ are irreducible, and for any $\tau \in H$ either $\tau Z'_i = Z'_i$ or $\tau Z'_i \cap Z'_i = \emptyset$.

First note that by the definition its irreducible components Z'_i's are étale closed. By assumption, the analytic irreducible decomposition of set V' thus has form

$$V' = V' \cap W' = (HZ'_1 \cap V') \cup \ldots \cup (HZ'_k \cap V') =$$
$$\bigcup_{h \in H : hZ'_1 \subset V'} hZ'_1 \cup \ldots \cup \bigcup_{h \in H : hZ'_k \subset V'} hZ'_k.$$

The sets $\bigcup_{h \in H : hZ'_i \subset V} hZ'_i$'s are étale closed, hZ'_i's are their connected components by the second claim of Decomposition Lemma III.1.3.1. Thus this is the étale irreducible decomposition required in (4).

Let us prove that $\acute{E}t$ is a topology, the most difficult property. A union of a finite number of étale closed set is closed by the definition. Let us prove the intersection of two étale closed set Z'_i and Y'_i is étale closed.

Assume W' and V' are unions of connected component of H-invariant sets HW' and HV'. The intersection $HW' \cap HV'$ is H-invariant and the set $W' \cap V'$ is a union of the connected components of $HW' \cap HV'$. The H-invariant intersection $HW' \cap HV'$ is étale closed by Corollary III.1.3.3, and thus its connected components are also étale closed. By definition this implies $W' \cap V'$ is étale closed.

To prove that an infinite intersection is closed, it is sufficient to prove that the intersection of a decreasing sequence of étale closed sets is étale closed. Use Koenig lemma, Fact III.1.2.1(7) and the fact that a sequence of decreasing étale irreducible sets stabilises.

The argument is as follows. Let $\ldots \subset X_i \subset X_{i-1} \subset ..$ be a decreasing sequence of étale closed sets, and let Z^i_j's be the (representatives of the) irreducible components of X_i up to $\pi(\boldsymbol{U})$-action. Let us make a tree whose vertices are sets Z^i_j, and Z^i_j and Z^{i-1}_k are joined by an edge iff $Z^i_j \subset \gamma Z^{i-1}_k$ for some $\gamma \in \pi$. The number of vertices in each level is finite, and thus the tree has finite branching; on the other hand, each branch has finite depth as it consists of irreducible sets. Thus, the tree has to be finite by Koenig's Lemma. This means that for i large, the intersection of first i sets is a union of translates of a fixed finite number of étale irreducible sets. As any such union is étale closed, this concludes the argument.

This completes the proof of the corollary. □

Let us state some more specific properties; property (3) below does require subgroup separability of π to be valid.

Corollary III.1.4.3. *1. For any étale closed set W', the set $\pi W'$ is étale closed.*

2. An irreducible étale closed set W' is a connected component of an H-invariant set HW' for some $H \triangleleft_{\mathrm{fin}} \pi$ a finite index normal subgroup of π.

3. For any two irreducible étale closed sets V', W' if for any finite index normal subgroup $H \triangleleft_{\mathrm{fin}} \pi$ it holds $HW' = HV'$, then $W' = V'$.

Proof. (1,2) trivially follows from the definitions and the Decomposition Lemma; (3) is slightly more difficult and requires an additional assumption on $A(\mathbb{C})$. We may assume H be so that V' and $W' = hV'$ are connected components of $HV' = HW'$, correspondingly. Take $x \in V'$, $y = hx \in W'$. Subgroup separability of π implies we may take H sufficiently small of finite index that $Hx \neq H\pi(V')y$ and therefore $HV' \neq HW'$. □

III.1.5 Θ-definable sets, Θ-generic points and Θ-definable closure

Recall that $\boldsymbol{U}/\pi \cong A(\mathbb{C})$ has the structure of an algebraic variety over $\mathbb{C}$ and that the π-invariant sets are in a bijective correspondence with the algebraic subvarieties of $A^H(\mathbb{C})$. Thus suggests us that we may try to pull back to $\boldsymbol{U}$ the notion of a generic point in $A(\mathbb{C})$.

The following definition behaves well only for $\Theta \subset \mathbb{C}$ algebraically closed.

Definition III.1.5.1. *We say that a π-invariant étale closed subset $W' \subset \boldsymbol{U}$ is* defined over an algebraically closed subfield $\Theta \subset \mathbb{C}$ *iff $p(W') \subset A(\mathbb{C})$ is a subvariety defined over Θ.*

An étale closed set is defined over a subfield $\Theta \subset \mathbb{C}$ *iff it is a countable union of irreducible components of π-invariant étale closed subsets defined over Θ.*

Definition III.1.5.2. *For a set $V \subset U^n$, let $\mathrm{Cl}_\Theta V$ be the intersection of all closed Θ-definable sets containing V:*

$$\mathrm{Cl}_\Theta(V) = \bigcap_{V \subset W, W/\Theta \textit{ is } \Theta\textit{-definable closed}} W$$

A point $v \in V$ is called Θ-generic *iff* $V = \mathrm{Cl}_\Theta(v)$, *i.e. there does not exist a closed Θ-definable proper subset of V containing v.*

The following finiteness property will be needed to show that a variant of Chevalley lemma implies a variant of ω-homogeneity.

Lemma III.1.5.3. *(a)* $\mathrm{Cl}_\Theta(V)$ *is Θ-definable*
(b) $\mathrm{Cl}_\Theta(V) = \bigcup_{v \in V} \mathrm{Cl}_\Theta(v) = \bigcup_{S \subset_{fin} V} \mathrm{Cl}_\Theta(S)$ *(union over all finite subsets)*

Proof. (a) : By Decomposition Lemma, it is sufficient to consider only irreducible V. However, for irreducible V we may assume that all sets appearing in the definition of $\mathrm{Cl}_\Theta(V)$ are again irreducible and therefore the intersection is finite. It is immediate that a finite intersection of Θ-definable sets is Θ-definable.

(b) : This follows from the Decomposition Lemma. If V is irreducible, then $V = \mathrm{Cl}_\Theta(v)$ for v a Θ-generic point of V. If not, by Decomposition Lemma, V decomposes as a union of translates of irreducible sets $V_1, \ldots, V_n$. Thus the union $\bigcup_{v \in V} \mathrm{Cl}_\Theta(v)$ is the union of the corresponding translates of the closures $\mathrm{Cl}_\Theta(V_1)$,...,$\mathrm{Cl}_\Theta(V_n)$ of the irreducible components $V_1, \ldots, V_n$. By Lemma III.1.4.1, $\mathrm{Cl}_\Theta(V_i)$ being closed implies any union of translates of $\mathrm{Cl}_\Theta(V_i)$ is closed; and thus $\bigcup_{v \in V} \mathrm{Cl}_\Theta(v)$ is a finite union of closed sets, therefore closed itself. But obviously $V \subset \bigcup_{v \in V} \mathrm{Cl}_\Theta(v)$ and therefore $\mathrm{Cl}_\Theta(V) \subset \bigcup_{v \in V} \mathrm{Cl}_\Theta(v)$. On the other hand, for any $v \in V$ $\mathrm{Cl}_\Theta(v) \subset \mathrm{Cl}_\Theta(V)$, and thus $\mathrm{Cl}_\Theta(V) \supset \bigcup_{v \in V} \mathrm{Cl}_\Theta(v)$. This implies the lemma. □

Lemma III.1.5.4. *If a set $W' \subset \boldsymbol{U}$ is defined over $\overline{\mathbb{Q}} \subset \mathbb{C}$ then $W' \subset \boldsymbol{U}$ is L_A-defined with parameters from $p^{-1}(A(\overline{\mathbb{Q}}))$.*

Proof. An irreducible component of the preimage of an algebraic variety $W(\mathbb{C}) \subset A(\mathbb{C})$ defined over $\overline{\mathbb{Q}}$ is an irreducible component of the preimage of the variety

$$\bigcup_{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/k)} \sigma W(\mathbb{C})$$

defined over k. In order for the union to be finite, we use that W is defined over $\overline{\mathbb{Q}}$, i.e. over a finite degree subfield of $\overline{\mathbb{Q}}$. The relation $\sim_W$ is in $L_A(A)$, and W' can be defined by $x \sim_W a_1$ & ... & $x \sim_W a_k$, for some set of $\overline{\mathbb{Q}}$-rational points $a_1, \ldots, a_k \in W'(\overline{\mathbb{Q}})$. □

Recall we assume Θ to be algebraically closed.

Lemma III.1.5.5. *For every finite index subgroup $H \lhd_{\mathrm{fin}} \pi$, if W' is irreducible étale closed, then $w' \in W'$ is Θ-generic iff $w = p_H(w') \in W = p_H(W')$ is Θ-generic in W.*

Proof. The point $w' \in W'$ is not Θ-generic iff there exists a Θ-defined irreducible set $w' \in V' \subsetneq W'$; by Corollary III.1.4.3 the latter is equivalent to $p_H(V') \neq p_H(W')$. □

We would rather avoid using this Corollary due to its non-geometric character, but unfortunately we do use it.

Lemma III.1.5.6. *A connected component of a Θ-generic fibre of a etale closed irreducible set defined over Θ contains a Θ-generic point. That is, if $W' \subset \boldsymbol{U} \times \boldsymbol{U}$ is etale irreducible and* $\mathrm{pr} : W' \to \boldsymbol{U}$ *is the projection, and $g' \in \mathrm{Clpr}\, W'$ is a Θ-generic point of the etale closed set $V' = \mathrm{Clpr}\, W'$, then the* Θ-generic fibre $W'_{g'} = \mathrm{pr}^{-1}(g')$ *contains a Θ-generic point of W'.*

Proof. The property holds for algebraic varieties (Corollary V.3.3.3 of Stein factorisation). Let $W'^{c}_{g'}$ be a connected component of a fibre of W' over a Θ-generic point g' of $\mathrm{Clpr}\, W'$. Then $p(W'^{c}_{g'})$ is a connected component of the fibre W_g, where $W = p_H(W'), g = p(g')$ is such that W' is a connected component of $p_H^{-1}(W)$; this may be seen with the help of path-lifting property, for example. Genericity of $g' \in \mathrm{Clpr}\, W'$ implies that the point $g \in \mathrm{Clpr}\, W$ is Θ-generic, and, as a connected component of the fibre W_g of an algebraic variety, $p(W'^{c}_{g'})$ contains a Θ-generic point, and then its preimage in $W'^{c}_{g'}$ is also Θ-generic. □

III.2 Main property of the group action

The following property is specific to algebraic geometry; it fails in a general topological situation. It is this property which allows us to connect the Zariski topology on $A(\mathbb{C})$ and the étale topology on the universal covering space U. This is the property used to imply Chevalley Lemma for étale closed sets.

Roughly, the property of group action is a corollary of what is know as Stein factorisation. Stein factorisation says that every algebraic morphism can be decomposed as a finite morphism and a morphism with connected fibres. Moreover, on a Zariski open subset, over $\mathbb{C}$, any morphism can be decomposed as a finite étale morphism, i.e. a topological covering in complex topology, and a fibre bundle in complex topology, cf. Lemma V.3.4.1. Thus, an algebraic morphism has a rather transparent *topological* structure on a *Zariski open* subset.

Holomorphic convexity is used deal with non-normal case.

III.2.1 Stabilisers of irreducible closed subsets

Recall notation $\pi(V') = \{\gamma \in \pi : \gamma V' \subset V'\}$.

Proposition III.2.1.1 (Action of $\pi(\boldsymbol{U})$ on $\boldsymbol{U}$). *Let W' and $V' = \operatorname{Clpr} W'$ be étale irreducible closed sets. Then there is a finite index subgroup $H \lhd_{\text{fin}} \pi$ such that*

1. $\pi(W') \cap H = \{\gamma \in H : \gamma W' \subset W'\} = \{\gamma \in H : \gamma W' \cap W' \neq \emptyset\} = \{\gamma \in H : \gamma x'_0 \in W'\}$, *for any point* $x'_0 \in W'$
2. $\operatorname{pr}[\pi(W') \cap H] = \pi(V') \cap H$.
3. *for an open subset $V^{0\prime} \subset V'$ it holds that for arbitrary connected component $W'^c_{g'}$ of fibre $W'_{g'}$ over $g' \in V^0$ there is a sequence exact up to finite index*
$$\pi(W'^c_{g'}) \longrightarrow \pi(W') \xrightarrow{\operatorname{pr}_*} \pi(V') \longrightarrow 0,$$
i.e. there exists a finite index subgroup $H \lhd_{\text{fin}} \pi$ independent of g and $W'^c_{g'}$ such that the sequence is exact:
$$\pi(W'^c_{g'}) \cap H \longrightarrow \pi(W') \cap [H \times H] \xrightarrow{\operatorname{pr}_*} \pi(V') \cap H \longrightarrow 0,$$
Moreover, if W' and V' are defined over an algebraically closed field Θ, so is $V - V^0$. In particular, the above sequence is exact for g a Θ-generic point of $V' = \operatorname{Clpr} W'$.

Proof of Proposition. To prove (1), apply Decomposition Lemma to the étale closed set $\pi W'$; by Decomposition Lemma, take $H \lhd_{\text{fin}} \pi$ to be such that the set $\pi W'$ decomposes as a union of a finite number of H-invariant sets whose connected components are irreducible, and therefore they are translates of W'. This implies (1). The item (2) is implied by (3).

Let us now prove item (3). Let H be such that W' and V' are connected components of $p_H^{-1} W(\mathbb{C})$, $p_H^{-1}(V(\mathbb{C}))$, respectively, where $W(\mathbb{C}) = p_H(W'), V(\mathbb{C}) = p_H(V')$. Consider projection morphism $\operatorname{pr} : A \times A \to A$; it induces a morphism $\operatorname{pr} : W(\mathbb{C}) \to V(\mathbb{C})$. By Lemma V.3.4.1 it gives rise to a sequence exact up to finite index:

$$\iota_* \pi_1(W^c_g(\mathbb{C}), w) \to \iota_* \pi_1(W(\mathbb{C}), w) \to \iota_* \pi_1(V(\mathbb{C}), \operatorname{pr} w) \to 0$$

where W^c_g is a connected component of a fibre of W over $g \in V$, and g varies in an open subset V^0 of V, and w varies in W^c_g. The index depends only on the Stein factorisation of the projection, and is therefore independent of g and fibre $W'^c_{g'}$.

Recall by §II.2.1 we may identify $\pi(W')$ and $\iota_* \pi_1(W(\mathbb{C}), w)$, and $\iota_* \pi_1(W^c_g(\mathbb{C}), w)$ and $\pi(W'^{c\prime}_{g'})$, etc. This proves Proposition. □

III.2.2 Corollaries: Chevalley Lemma and Finiteness of Generic Fibres

Corollary III.2.2.1 (Chevalley Lemma). *For the étale topology , it holds:*

1. *Projections of closed sets are closed.*
2. *The projection of a set open in its closure is a set open in its closure.*

Proof. The projection of an H-invariant closed set is closed; indeed, say $H = \pi$, then note $\mathrm{pr}\, p(\pi W') = p\mathrm{pr}\,(W')$, and thus $\mathrm{pr}\,\pi W' = p^{-1}p(\mathrm{pr}\, W') = p^{-1}p(V)$, where $V = \mathrm{pr}\, p(W')$. As $A(\mathbb{C})$ is projective, V is a closed algebraic subset of $A(\mathbb{C})$, and thus $p^{-1}p(V)$ is a π-invariant closed subset of $\boldsymbol{U}$. By definition of $\acute{E}t$, it is étale closed.

Let now W' be an étale irreducible closed set which is a connected component of HW'. Let V' be the closure of $\mathrm{pr}\, W'$; we intend to apply item (3) of Proposition above.

The set $\mathrm{pr}\, HW'$ is closed, and thus $V' \subset \mathrm{pr}\, HW'$. The set V' is closed, and thus it is contained in a connected component V_1' of $\mathrm{pr}\, HW'$.

Take $v' \in V' \subset V_1'$, and find $w' \in W'$ such that $\mathrm{pr}\,(hw') = v'$; this is possible due to $V' \subset \mathrm{pr}\, HW'$. Also $\mathrm{pr}\, W' \subset V'$, and thus $\mathrm{pr}\,(w') \in V', \mathrm{pr}\,(h)\mathrm{pr}\,(w') = v' \in V'$. Then $v' \in \mathrm{pr}\,(h)V_1' \cap V_1'$. We may further take H sufficiently small so that

$$\pi(V_1') \cap H = \{\tau \in \pi : \tau(V_1') \cap V_1' \neq \emptyset\} = \{\tau \in \pi : \tau V_1' = V_1'\}.$$

Then $\mathrm{pr}\,(h) \in \pi(V_1')$, and Proposition III.2.1.1(2) implies there exists an element $h_1 \in \pi(W') \cap [H \times H]$ such that $\mathrm{pr}\,(h) = \mathrm{pr}\, h_1$. Then, $h_1 W' = W'$, and thus $\mathrm{pr}\,(h_1 w') = \mathrm{pr}\,(h)\mathrm{pr}\, w' = v'$, as required.

This argument can be given topologically. We prove the second claim topologically.

First, we may assume that W' is a connected component of $p_H^{-1}p_H(W') = HW'$, and by Chevalley Lemma for algebraic varieties there is a set $V^0 \subset \mathrm{pr}\, p_H(W') \subset V$ such that $V^0 \subsetneq V$ is open in V. Let V' be the connected component of $p_H^{-1}(V)$ containing $\mathrm{pr}\, W'$. Take $V^{0\prime} = V' \cap p_H^{-1}(V^0)$; then $V^{0\prime} \subset V'$ is open in V' as an intersection with an open set.

Take $v' \in V^{0\prime}$, and take $w' \in W'$, $\mathrm{pr}\, p_H(w') = p_H(v') \in V^0 \subset \mathrm{pr}\, W$; such a point w' in W' exists by what we call the covering property of connected components. Now, $\mathrm{pr}\, w' \in V'$, and thus $\gamma_0 \in \pi(V')$ where γ_0 is defined by $v' = \gamma_0 \mathrm{pr}\, w'$. Condition $\mathrm{pr}\, p_H(w') = p_H(v') \in A^H(K)$ implies $\gamma_0 \in H$. Thus the inclusion $\mathrm{pr}\,\pi(W') \cap H = \pi(V') \cap H$ implies there exists $\gamma_1 \in \pi(W')$, $\mathrm{pr}\,\gamma_1 = \gamma_0$, and thus $v' = \gamma_0 \mathrm{pr}\, w' = \mathrm{pr}\,(\gamma_1 w')$, and the Chevalley lemma is proven. $\square$

Let $\pi_0(W')$ denote the set of irreducible components of W'.

Corollary III.2.2.2 (Generic Fibres). *In notation of Proposition above, for a Θ-generic point $g' \in V' = \operatorname{Clpr} W'$, the fibre $W'_{g'}$ has finitely many connected components and for any connected component $W'^c_{g'}$ of $W'_{g'}$, it holds*

$$W' \cap g' \times HW'^c_{g'} = g' \times W'^c_{g'},$$

$$W' \cap g' \times HW'_{g'} = g' \times W'_{g'}.$$

Moreover, the formulae above hold for g' not contained in some proper Θ-definable closed subset of V.

Proof. Let H be as in Proposition III.2.1.1. The fibre $W'_{g'}$ is the intersection of W_g with a coordinate plane, and therefore is étale closed. By Decomposition Lemma, the fibre $W'_{g'}$ is a union of H-translates of a finite number of irreducible sets $Z'_1, \ldots, Z'_k$.

To prove the claim, take $h \in H$ such that $Z'_i, hZ'_i \subset W'_{g'}$. Then $(\mathrm{id}, h) \in H \times H$, and $(\mathrm{id}, h^{-1})W' \cap W' \supset g' \times Z'_i \neq \emptyset$, and by Proposition III.2.1.1(1) this implies $(\mathrm{id}, h^{-1})W' = W'$ and $(\mathrm{id}, h^{-1}) \in \pi(W')$. However, by Proposition III.2.1.1(2) $\pi(W'^c_{g'}) \cap H = \ker(\mathrm{pr}_*(\pi(W') \to \mathrm{pr}\,(V')) \cap H)$, and thus $h \in \pi(W'^c_{g'}), hW'^c_{g'} = W'^c_{g'}$ for any connected component $W'^c_{g'}$ of fibre $W'_{g'}$.

To prove $W' \cap g' \times HW'^c_{g'} = g' \times W'^c_{g'}$, take $h \in H$ such that $g' \times hW'_{g'} \cap W \neq \emptyset$. Then $(\mathrm{id}, h) \in H \times H$ and

$$(\mathrm{id}, h)W' \cap W' \supset g' \times hW'_{g'} \cap W_g \neq \emptyset,$$

by Proposition III.2.1.1(1) this implies $(\mathrm{id}, h)W' = W'$ i.e. $(\mathrm{id}, h) \in \pi(W')$. Now Proposition III.2.1.1(2), $\pi(W'^c_{g'}) \cap H = \ker(\mathrm{pr}_*(\pi(W') \to \mathrm{pr}\,(V')) \cap H)$ gives $hW'_{g'} = W'_{g'}$, i.e $h \in \pi(W'_{g'})$, as required.

In particular, $W' \cap HW'_{g'} = W'_{g'}$ and $W' \cap W'^c_{g'} = g' \times W'^c_{g'}$ □

III.3 A language for the étale topology: k-definable sets

So far we have defined a topology on U (and its Cartesian powers U^n's) whose closed sets are rather easy to understand. Now, to put the considerations above in a framework of model-theory, we want to define a *language* able to define closed sets in the étale topology. From an algebraic point of view, that corresponds to defining an automorphism group of U with respect to the étale topology. The automorphism group is to be an analogue of a Galois group.

Let us draw an analogue to the action of Galois group on $\overline{\mathbb{Q}}$ as an algebraic variety defined over $\mathbb{Q}$ endowed with Zariski topology. The Galois group may not be defined as the group of bijections continuous in Zariski topology: for example, all polynomial maps are continuous in Zariski topology; linear and affine maps $x \to ax + b$ are such continuous bijections.

Thus we distinguish certain $\mathbb{Q}$-definable subsets among Zariski closed subsets of $\overline{\mathbb{Q}}^3$, and then define Galois group as the group of transformation (of $\overline{\mathbb{Q}}$) *preserving the distinguished* $\mathbb{Q}$*-defined subsets (of* $\overline{\mathbb{Q}}^3$*)*; in this case the graphs of addition and multiplication. It is *then* derived, rather trivially, that this implies that Galois group acts by transformation continuous in Zariski topology.

Recall the way this is derived: the $\mathbb{Q}$-definable subsets are given *names*, in this case addition and multiplication, and then each closed set (subvariety) is given a name by the equations defining the set of its points; in fact, in algebraic geometry the word variety means rather the *name*, the set of equations, rather that the set of points the equations define.

In order to define a useful automorphism group of the étale topology, we follow the same pattern.

Model theory provides us with means to give precise meaning to the argument above, and to define mathematically what is it exactly that we want. In these terms, the distinguished subsets form a *language*, and the Galois group is the group of automorphisms of *the structure in that language*. Model theory studies that group via the study of the structure.

III.3.1 Definition of a language L_A for universal covers in the étale topology

In this §, it becomes essential that A is defined over an algebraic field $k \subset \overline{\mathbb{Q}} \subset \mathbb{C}$ embedded in $\mathbb{C}$.

We consider $p : U \to A(\mathbb{C})$ as a structure in the following language.

Definition III.3.1.1. *We consider the universal covering space* $p : \boldsymbol{U} \to A(\mathbb{C})$ *as a one-sorted structure* $\boldsymbol{U}$*, in the language* L_A *which has the following symbols:*

> *the symbols* $\sim_{Z,A}$ *for* Z *a closed subvariety of* $A(\mathbb{C})^n$ *defined over number field* k*, and,*
>
> *the symbols* $\sim_H$*, for each normal subgroup* $H \lhd_{\mathrm{fin}} \pi(\boldsymbol{U})^n$ *of finite index*

The symbols are interpreted as follows:

> $x' \sim_{Z,A} y' \iff$ *points* $x' \in \boldsymbol{U}^n$ *and* $y' \in \boldsymbol{U}^n$ *lie in the same (analytic) irreducible component of the* π*-invariant closed analytic set* $p^{-1}(Z(\mathbb{C})) \subset \boldsymbol{U}^n$*.*

$$x' \sim_H y' \iff \exists \tau \in \pi(\boldsymbol{U})^n : \tau x' = y' \text{ and } \tau \in H.$$

Note that we do not assume Z to be connected.

Note that by Decomposition Lemma it is enough to introduce predicates for *connected* components of closed analytic sets which are invariant under the action of *finite index* subgroup of the fundamental group. To such sets, Chow Lemma still applies.

Thus we may use an alternative definition by considering predicates for each $Z \subset A(\mathbb{C})$ defined over $\mathbb{Q}$.

> $x' \sim^c_{Z,A^H} y'$ iff x' and y' lie in the same *connected* component of the preimage $p_H^{-1}(Z_i(\mathbb{C}))$, $Z_i \subset A^H(\mathbb{C})^n$ an irreducible component of algebraic variety $p_H^{-1}(Z(\mathbb{C})) \subset A^H(\mathbb{C})^n$.

Let us state the above observation due to its importance and for future reference:

Corollary III.3.1.2. *For every closed π-invariant analytic subset Z' of $\boldsymbol{U}^n$, there exist closed analytic subsets $Z'_1, \ldots, Z'_n$ invariant under action of a finite index subgroup H of π, such that*

$$x \sim_{Z'} y \iff x \sim^c_{Z'_1} y \vee x \sim^c_{Z'_2} y \vee \ldots \vee x \sim^c_{Z'_n} y.$$

Consequently, for every closed subvariety Z of A, there exist subvarieties $Z_1, \ldots, Z_n$ of a finite étale cover A^H such that

$$x \sim_Z y \iff x \sim^c_{Z_1} y \vee x \sim^c_{Z_2} y \vee \ldots \vee x \sim^c_{Z_n} y.$$

Proof. Take H and $Z'_1, \ldots, Z'_n$ as in Decomposition Lemma III.1.3.1. □

Note that the language L_A is countable. This is an essential property, from model-theory point of view; in technical, down-to-earth terms it is useful to make inductive constructions.

Let us use this opportunity to remind that we use symbols $\sim_Z$ rather abusively to mean "lie in the same irreducible component of" either πZ, $p_H^{-1}(Z)$, etc.

III.3.2 L_A-definability of $\pi(\boldsymbol{U})$-action etc

In the next lemma, a closed set means an étale closed set.

Lemma III.3.2.1. *For any normal finite index subgroup $H \lhd_{\text{fin}} \pi$ it holds*

1. *the relation*

$$\text{Aff}_H(x, y, z, t) = \exists \gamma \in H : \gamma x = y \;\&\; \gamma z = t$$

is $L_A(\emptyset)$-definable

2. *An H-invariant closed set is L_A-definable with parameters.*
3. *A connected component of a generic fibre of an L_A-definable irreducible closed set is uniformly L_A-definable; the definition is valid over an over an open subset of the projection, definable over the same set of parameters.*
4. *Any étale closed irreducible set is a connected component of a fibre of an L_A-definable set.*
5. *An irreducible closed set is L_A-definable.*

Proof. To prove (1), note that

$$p^{-1}(\Delta(\mathbb{C})) = \bigcup_{\gamma\in\pi}\{(x,\gamma x) : x \in U\}$$

where $\Delta = \{(x, x) : x \in A\}$ is an algebraic closed subvariety defined over k. The connected components $\{(x, \gamma x) : x \in U\}, \gamma \in \pi$ are the equivalence classes of the relation $\sim_\Delta$, and thus are definable with parameters.

Evidently $\mathrm{Aff}_\pi(x, y, s, t)$ iff $(x, y) \sim_\Delta (s, t)$ lie in the same connected component of $p^{-1}(\Delta(\mathbb{C})) \subset U \times U$.

To prove (2), we consider two cases. **$\overline{\mathbb{Q}}$-case:** An irreducible closed subvariety $Z_{/\overline{\mathbb{Q}}} \subset A$ defined over $\overline{\mathbb{Q}}$ is an irreducible component of subvariety

$$Z_k = \bigcup_{\sigma:k_Z\hookrightarrow\mathbb{C}} \sigma(Z)$$

of A, where k_Z is the field of definition of Z of finite degree. The formula implies Z is L_A-definable with parameters with the help of symbol $\sim_{Z_k,A}$; the parameters may be taken to lie in $A(\overline{\mathbb{Q}})$ but not necessarily in $A(k_Z)$. A slightly more complicated argument could give a construction defining Z as a connected component.

For an analytic étale closed irreducible set $Z' \subset \boldsymbol{U}$, it holds that Z' is an irreducible component of $\pi Z'$, i.e. it is an irreducible component of $p^{-1}(Z) = p^{-1}p(Z')$. Thus the above argument gives that every étale irreducible subset of $\boldsymbol{U}$ defined over $\overline{\mathbb{Q}}$ is L_A-definable with parameters.

$\overline{\mathbb{Q}(t_1,\ldots,t_n)}$-case: Thus we have to deal with the case when $p(Z)$ is not $\overline{\mathbb{Q}}$-definable. Our strategy is to show that any such set is a connected component of a $\overline{\mathbb{Q}}$-generic fibre of a $\overline{\mathbb{Q}}$-definable set, and then show that such connected components are uniformly definable. Uniformity will be important for us later in axiomatising $\boldsymbol{U}$.

Let us see first that each étale closed irreducible set is a connected component of a fibre of an étale closed irreducible set defined over $\overline{\mathbb{Q}}$.

Take an étale irreducible set Z' and take $H \lhd_{\mathrm{fin}} \pi$ such that Z' is a connected component of $HZ' = p_H^{-1}(Z)$, for an irreducible algebraic closed set $Z = p_H(Z')$. By the theory of algebraically closed field, we know that Z can be defined as a Boolean combination, necessarily a positive one, of $\overline{\mathbb{Q}}$-definable closed subsets and their fibres; by passing to a smaller subset if necessary, we see that the irreducibility of Z implies that algebraic subset $Z \subset A(\mathbb{C})$ is a connected component of a $\overline{\mathbb{Q}}$-generic fibre of a $\overline{\mathbb{Q}}$-definable closed subset $W \subset A(\mathbb{C})^n$. Then HZ' is the corresponding fibre of $p_H^{-1}(W)$. The closed set Z' is a union of the corresponding fibres of the irreducible components of $p_H^{-1}(W)$, and irreducibility of Z' implies that union is necessarily trivial. Thus, we have that Z' is a connected component of a fibre of an irreducible étale closed set defined over $\overline{\mathbb{Q}}$. We may also ensure that Z' is a connected component of a $\overline{\mathbb{Q}}$-generic fibre of W' by intersecting W' with the preimage of an irreducible $\overline{\mathbb{Q}}$-definable set containing $\operatorname{pr} Z'$, and repeating the process if necessary.

Let us now prove that the connected components of the $\overline{\mathbb{Q}}$-generic fibres of an irreducible $\overline{\mathbb{Q}}$-definable set are $\overline{\mathbb{Q}}$-definable.

Let $W' \subset A(\mathbb{C})^2$, and let $V' = \operatorname{Clpr} W'$ be as in Proposition III.2.1.1 and Corollary III.2.2.2. The morphism $\operatorname{pr} : W \to V$ admits a Stein factorcisation (Fact V.3.3.4) $\operatorname{pr} = f_0 \circ f_1$ as a composition of a finite morphism $f_0 : W \to V_1$ and a morphism with connected fibres $f_1 : V_1 \to V$. In particular, two points $x_1, x_2 \in W_g$ lie in the same connected component of fibre W_g iff $f_0(x_1) = f_0(x_2)$.

Now set

$$\text{(III.3.1)}\quad x' \sim^c_{W_g} y' \iff x' \sim_W y' \;\&\; \&\; \operatorname{pr} x' = \operatorname{pr} y' \;\&\; f_0(p_H(x')) = f_0(p_H(y'))$$

(here subscript g is a part of the notation, and does not denote an element of $\boldsymbol{U}$).

In notation of Corollary III.2.2.2, we have

Corollary III.3.2.2. *If $\operatorname{pr} x' = g' \in V'^0$, then the formula $x' \sim^c_{W_g} y'$ holds iff $x' \sim_H y'$ and x' and y' lie in the same connected component of fibre $W'_{g'}$ of W. If W, V are Θ-definable, so is V'^0. The parameters needed to define $\sim^c_{W_g}$ live in $\boldsymbol{U}/H$.*

Proof. This is a reformulation of the formula $W' \cap g' \times HW'^c_{g'} = g' \times W'^c_{g'}$. Indeed, $\operatorname{pr} x' = \operatorname{pr} y' \;\&\; f_0(p_H(x')) = f_0(p_H(y'))$ holds iff $x', y' \in g' \times HW'^c_{g'}$ for $g' = \operatorname{pr} x' = \operatorname{pr} y'$ and some $W'^c_{g'}$ a connected component of fibre of W' above g'. The relation of lying in the same connected component of a fibre being translation invariant, we may as well assume $x', y' \in W'$ if $x' \sim_W y' \in W'$ lie in the same connected component of W'. Then the formula means that x', y' lie in the same connected component of fibre $g' \times W'_{g'}$.

The claim that the formula holds for $g' \in V'^0$ in an open subset is Θ-definable is a part of the conclusion of Corollary III.2.2.2. □

The claim above implies (3); (4) and (3) imply (5) and (2). □

Corollary III.3.2.3. *Let* $\mathrm{Aut}_{L_A}(U)$ *be the group of bijections* $\varphi : U \to U$ *preserving relations* $\sim_{Z,A} \in L_A$*; then* $\mathrm{Aut}_{L_A}(U)$ *acts by transformations continuous in étale topology .*

Proof. Immediate by previous results. □

The results above justify thinking of $\mathrm{Aut}_{L_A}(U)$ as a *Galois group of* U.

Remark III.3.2.4. Via identifications $\boldsymbol{U}/H \cong A^H(\mathbb{C})$, there is a natural inclusion of a subgroup of $\mathrm{Aut}_{L_A}(U)$ into $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$; what can one say about the common subgroup of $\mathrm{Aut}_{L_A}(U)$ and $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, or rather a conjugacy class of such subgroups? Is there any relations between $\mathrm{Aut}_{L_A}(\boldsymbol{U})$ and the Grothendieck's fundamental group $\hat{\pi}_1(A_{\mathbb{Q}}, 0)$?

III.4 Model homogeneity: an analogue of n-transitivity of $\mathrm{Aut}_{L_A}(U)$-action.

Now we want to study the action of $\mathrm{Aut}_{L_A}(U)$ on U, and analyse orbits of its action on U and $U^n, n > 1$. In model theory one would hope that the aforementioned orbits can be analysed in terms of the language; in presence of a nice topology with a Chevalley property we may hope to analyse orbits in terms of closed sets.

The situation when this is possible is called homogeneity; Property III.4.0.8 below states *model homogeneity* of U. Model homogeneity says, roughly, that two tuples of points lie in the same orbit (of the action fixing an algebraically closed subfield) iff there are no obvious obstructions, i.e. iff they lie in the same closed sets (defined over an algebraically closed subfield which we assume fixed).

Definition III.4.0.5. *We say that* W *is a* Θ-constructible set *iff*

1. *the closure* $\mathrm{Cl}W$ *is defined over* Θ
2. W *contains all* Θ*-generic points of the irreducible components of* $\mathrm{Cl}W$.

An irreducible *constructible set is a set whose closure is irreducible.*

Lemma III.4.0.6. *A projection of a* Θ*-constructible set is* Θ*-constructible.*

Proof. Let $W \subset U \times U$ be an irreducible set defined over Θ, and let W_0 be the set of all Θ-generic points of W; generally speaking, W_0 is not definable. We need to prove that $\operatorname{pr} W$ is also Θ-constructible. Let g be a Θ-generic point of the closure of $\operatorname{pr} W$; we know $g \in \operatorname{pr} W$ by Chevalley Lemma. By Lemma 1.5.6 we know that the (non-empty) fibre W_g contains a Θ-generic point of W, and thus $g \in \operatorname{pr} W_0$, as required. □

The set of realisations of a complete quantifier-free syntactic type p/Θ with parameter set Θ is Θ-constructible; and conversely, every Θ-constructible set can be represented in this form.

Thus, the above lemma is equivalent to ω-homogeneity for such types.

Definition III.4.0.7. *We say that U is* homogeneous for closed sets over Θ*, or* homogeneous for syntactic quantifier-free complete types over Θ*, or* model homogeneous *iff either of the following equivalent conditions holds*

1. *the projection of a Θ-constructible set is Θ-constructible;*
2. *for any tuples $a, b \in U^n$ and $c \in U^m$ if $\operatorname{qftp}(a/\Theta) = \operatorname{qftp}(b/\Theta)$ then there exists $d \in U^m$ such that $\operatorname{qftp}(a, c/\Theta) = \operatorname{qftp}(b, d/\Theta)$*

To see that the conditions are equivalent, note that the set of realisations of a complete quantifier-free type $\operatorname{qftp}(a, c/\Theta)$ is Θ-constructible; its projection contains a and also is Θ-constructible; a is its Θ-generic point; then $\operatorname{tp}(a/\Theta) = \operatorname{tp}(b/\Theta)$ implies b is also Θ-generic, i.e. belongs to the projection.

The above proves the following result.

Property III.4.0.8. *The standard model $p : U \to A(\mathbb{C})$ in language L_A is model homogenous, i.e. it is ω-homogeneous for closed sets over arbitrary algebraically closed subfield $\Theta \subset \mathbb{C}$.*

Proof. Follows directly from Def. III.4.0.7 and Lemma III.4.0.6. □

Corollary III.4.0.9. *The set of realisations of a quantifier-free type* $\operatorname{qftp}(x/\Theta)$ *over $p^{-1}(A(\Theta))$ consists of Θ-generic points of some étale irreducible closed subset of $\boldsymbol{U}$.*

Proof. Follows from the previous statements. □

III.5 An $L_{\omega_1\omega}$-axiomatisation $\mathfrak{X}(A(\mathbb{C}))$ and stability of the corresponding $L_{\omega_1\omega}$-class.

In this § we introduce an axiomatisation $\mathfrak{X}(A(\mathbb{C}))$ for $L_{\omega_1\omega}(L_A)$-class which contains the standard model $p : U \to A(\mathbb{C})$, and is stable over models and all models in it are model homogeneous. We then show that the class of models satisfies $2_{\aleph_0 \to \aleph_1}$ of Theorem III.5.4.7.

III.5.1 Algebraic $L_A(G)$-structures

We know that $\boldsymbol{U}/G = A^G(\mathbb{C})$ carries the structure of an algebraic variety over field $\mathbb{C}$. The covering $A^G(\mathbb{C}) \to A(\mathbb{C})$ carries a structure in a reduct $L_A(G)$ of language L_A. In fact, similar interpretation works for an arbitrary algebraically closed field K instead of $K = \mathbb{C}$.

For every finite index subgroup $G \triangleleft_{\text{fin}} \pi$, there is a well-defined covering $A^G \to A$ of finite degree. The space $A(\mathbb{C})$ is projective, and thus $A^G(\mathbb{C})$ is also a complex projective manifold. By Fact III.1.2.3, A^G has the structure of an algebraic variety.

Recall that we use the following fact as the defining property of an étale covering: the morphism $B(K) \to A(K)$ of varieties over an algebraically closed field K of char 0 is *étale* iff there exists an embedding $i : K' \to \mathbb{C}$ of the field K' of definition of A and B into $\mathbb{C}$ such that the corresponding morphism $i(B)(\mathbb{C}) \to i(A)(\mathbb{C})$ is a covering of topological spaces.

Definition III.5.1.1 (Finitary reducts of L_A). *Let $p_G : A^G(K) \to A(K)$ be a finite étale morphism. Let $L_A(G) \subset L_A$ be the language consisting of all predicates of L_A of form $\sim_Z$ and symbols $\sim_H$ for $G \subset H$. Then $A^G(K) \to A(K)$ carries an $L_A(G)$-structure as follows:*

1. $x' \sim_Z y' \iff$ *points $x', y' \in A^G(K)^n$ lie in the same irreducible component of algebraic closed subset $p_G^{-1}(Z(K))$ of $A^G(K)^n$.*
2. $x' \sim_H y' \iff$ *there exist an algebraic morphism $\tau : A^G \to A^G$ and an étale covering morphism $q : A^G \to A^H$ such that $\tau(x') = y'$ and $\tau \circ q = q$:*

$$\begin{array}{ccc} A^G & \xrightarrow{\tau} & A^G \\ \Big\downarrow q & \textit{étale cover} & \Big\downarrow q \\ A^H & \xrightarrow{\text{id}} & A^H \end{array}$$

For $G = e$ the trivial group and $K = \mathbb{C}$, the construction above degenerates into the interpretation of $\boldsymbol{U} \to A$ if it were well-defined.

For $G = \pi$, $A^G = A$, and thus $L_A(\pi)$ is just a form of the language for the algebraic variety A; here the point is that we have predicates for the relations for irreducible components of k-definable closed subsets only.

In general, the above is just a variation of an ACF structure on A. In particular, all Zariski closed subsets of $(A^G)^n(K)$ are $L_A(G)$-definable.

III.5.2 Axiomatisation $\mathfrak{X}(A(\mathbb{C}))$ of the universal covering space $\boldsymbol{U}$

We define the axiomatisation $\mathfrak{X} = \mathfrak{X}(A(\mathbb{C}))$ to be an $L_{\omega_1\omega}(L_A)$-sentence corresponding to Axiom III.5.2.1 and Axioms III.5.2.2-III.5.2.6 below.

Basic Axioms

These axiom describe quotations $U/\sim_H$ for $H \lhd_{\mathrm{fin}} \pi$, and some properties of $U \to \boldsymbol{U}/\sim_H$.

Axiom III.5.2.1. *All first-order statements valid in $\boldsymbol{U}$ and expressible in terms of L_A-interpretable relations*

$$x' \sim_{Z,A^G} y' := \exists x'' \exists y'' (x'' \sim_Z y'' \ \&\ x'' \sim_G x' \ \&\ y'' \sim_G y'), G \lhd_{\mathrm{fin}} \pi$$

and $\sim_G, G \lhd_{\mathrm{fin}} \pi$.

Note that we do not allow $\sim_{Z,A}$ by itself and the Axioms essentially describe $\boldsymbol{U}/G$, which is an algebraic variety.

Path-lifting Property Axiom, or the covering property Axiom

Axiom III.5.2.2 (Path-lifting Property for W; Covering Property for W). *For every L_A-predicate $\sim_W$ and all $G \lhd_{\mathrm{fin}} \pi$ small enough, we have an axiom*

$$x' \sim_{W,A^G} y' \Longrightarrow \exists y'' (y'' \sim_G y' \ \&\ x' \sim_W y'')$$

We also have a stronger axiom for *fibres* of W; here we use that the relation "to lie in the same connected component of a fibre of a variety" is algebraic and therefore the corresponding G-invariant relation is L_A-definable.

Axiom III.5.2.3 (Lifting Property for fibres). *For all $G \lhd_{\mathrm{fin}} \pi$ sufficiently small, we have an axiom*

$$(x'_0, x'_1) \sim^c_{W_g,A^G} (y'_0, y'_1) \Longrightarrow \exists y''_1 [y'_0 \sim_G x'_0 \ \&\ y''_1 \sim_G y'_1 \ \&\ (x'_0, x'_1) \sim_W (x'_0, y''_1)]$$

in a slightly different notation

$$x' \sim^c_{W_g,A^G} y' \Longrightarrow \exists y'' (y'' \sim_G y' \ \&\ \mathrm{pr}\, x' = \mathrm{pr}\, y'' \ \&\ x' \sim_W y'')$$

The relation $x' \sim^c_{W_g,A^G} y'$ is defined by the formula (III.3.1) (cf. Claim III.3.2.2).

Axiom III.5.2.4 (Fundamental group is residually finite).

$$\forall x' \forall y' (x' = y' \iff \bigwedge_{H \lhd_{\mathrm{fin}} \pi} x' \sim_H y')$$

Thus, it says that two elements $\sim_H$-close for every $H \lhd_{\mathrm{fin}} \pi$, have to be equal.

The next property is strengthening of the previous one; namely, if an element b is $\sim_H$-equivalent to an element of a group generated by $a_1, \ldots, a_n$, then it is actually in the group. In terms of paths, this has the following interpretation: take loops $\gamma_1, \ldots, \gamma_n$ and a loop λ. If for every $H \lhd_{\mathrm{fin}} \pi$ it holds that λ is $\sim_H$-equivalent to some concatenation of paths $\gamma_1, \ldots, \gamma_n$, then it is actually a concatenation of these paths.

Axiom III.5.2.5 ("Translations have finite length", subgroup separability). *For all $N \in \mathbb{N}$ we have an $L_{\omega_1\omega}$-axiom*

$$\forall b \forall a_1 \ldots \forall a_N .$$

$$\bigwedge_{H \lhd_{\mathrm{fin}} \pi} \bigvee_{n \in \mathbb{N}} \exists h_1 \ldots h_n \left(b \sim_H h_n \ \& \ h_1 = a_1 \ \& \bigwedge_{1 \leqslant i \leqslant n} \bigvee_{1 \leqslant j < N} (h_i, h_{i+1}) \sim_\Delta (a_j, a_{j+1}) \right)$$

$$\implies \bigvee_{n \in \mathbb{N}} \exists h_1 \ldots h_n \left(b = h_n \ \& \ h_1 = a_1 \ \& \bigwedge_{1 \leqslant i \leqslant n} \bigvee_{1 \leqslant j < N} (h_i, h_{i+1}) \sim_\Delta (a_j, a_{j+1}) \right)$$

The next axiom is needed to apply the principles above. It reflects the fact that the fundamental groups of varieties are finitely generated, cf. Arapura [Ara95]; this can also be obtained from the fact that topologically a variety can be split into finitely many contractible pieces nicely glued together (CW-complex).

Axiom III.5.2.6 (Groups $\pi(W_g)$ are finitely generated). *For every symbol $\sim_W$ and for each H small enough we have an $L_{\omega_1\omega}$-axiom:*

$$\bigvee_{N \in \mathbb{N}} \exists a_1 \ldots \exists a_N \forall b .$$

$$\bigwedge_{1 \leqslant i \neq j \leqslant N} (a_i \sim_W a_j \ \& \ a_i \sim_H a_j \ \& \ \mathrm{pr}\, a_i = \mathrm{pr}\, a_j) \ \& \ \left(\bigwedge_{i=1}^{N} (b \sim_W a_i \ \& \ \mathrm{pr}\, b = \mathrm{pr}\, a_i) \implies \right.$$

$$\left. \bigvee_{n \in \mathbb{N}} \exists h_1 \ldots h_n \left(b = h_n \ \& \ h_1 = a_1 \ \& \bigwedge_{j=1}^{N} \bigvee_{j=1}^{N-1} (h_i, h_{i+1}) \sim_\Delta (a_j, a_{j+1}) \ \& \ \mathrm{pr}\, h_i = \mathrm{pr}\, h_{i+1} \right) \right)$$

In fact, we may combine the two axioms above into one weaker axiom which would require *subgroup separability with respect to the subgroups $\pi(W)$.*

Standard model $\boldsymbol{U}$ is a model of $\mathfrak{X}$

The universal covering space $p : \boldsymbol{U} \to A(\mathbb{C})$ satisfies the axioms Axiom III.5.2.1 by definition.

To prove $\boldsymbol{U}$ satisfies Axiom III.5.2.2, note that for $G \lhd_{\mathrm{fin}} \pi$ small enough, the relations $x' \sim_{W,G} y'$ means that $p_G(x')$ and $p_G(y')$ lie in the same irreducible component W_i of the preimage of $W \subset A(\mathbb{C})^n$ in $A^G(\mathbb{C})^n$. Take a path γ connecting $\gamma(0) = p_G(x')$ and $\gamma(1) = p_G(y')$ lying in W_i; by the lifting property it lifts to a path $\gamma', \gamma'(0) = x'$ such that $p_G(\gamma'(t)) = \gamma(t)$, $0 \leqslant t \leqslant 1$. Then, $p_G(\gamma'(1)) = p_G(y')$, and thus $\gamma'(1) \sim_G y'$. On the other hand, $\gamma'(1)$ and x' lie in the same connected component of the preimage of the irreducible component W_i in $\boldsymbol{U}$. Now note that by Decomposition Lemma III.1.4.1 for G small enough such a connected component has to be irreducible, and thus Axiom III.5.2.2 holds.

The Axiom III.5.2.3 has a similar geometric meaning as Axiom III.5.2.2; the assumption is that $p_G(x')$ and $p_G(y')$ lie in the same connected component of a fibre W_g; it is enough to take γ to lie in fibre W_g to arrive to the conclusion of Axiom III.5.2.2.

Let us see Axiom III.5.2.4 follows from the condition 2 of the definition of a Shafarevich variety.

Axioms III.5.2.5 is condition 2 from the definition of a Shafarevich variety.

The geometric meaning of $(h_i, h_{i+1}) \sim_\Delta (a_i, a_{i+1})$ is as follows. The pair of points a_i, a_{i+1} determines a path γ in $A(\mathbb{C})$, $\gamma(0) = \gamma(1) = p(a_i) = p(a_{i+1})$. For points h_i, h_{i+1} such that $p(h_i) = p(h_{i+1})$, they can be joined by a lifting of γ iff $(h_i, h_{i+1}) \sim_\Delta (a_i, a_{i+1})$. . . . Thus the assumption in the axiom says that if any two points of fibre above $p(b) = p(a_1)$ can be joined by a concatenation of liftings of finitely many paths γ_i's in $A(\mathbb{C})$, *up to a translate by an element of H*, then they can in fact be just joined by such a sequence. In a way, this can be thought of as disallowing paths of infinite length.

On the other hand, the condition $(h_i, h_{i+1}) \sim_\Delta (a_i, a_{i+1})$ can be interpreted as $h_{i+1} = \tau_i h_i$ where τ_i is the deck transformation taking a_i into a_{i+1}, $\tau_i a_i = a_{i+1}$. Then, the assumption says that if $b \in \pi(\boldsymbol{U})$ belongs to the group generated by τ_i's, up to $\sim_H$, then b does belong to the subgroup generated by τ_i's.

The last remaining Axiom III.5.2.6 means that the fundamental groups $\pi(W_g)$ is finitely generated, which is a well-known fact, see for example Arapura [Ara95].

III.5.3 Analysis of models of $\mathfrak{X}$

Models $U/_{\sim_H}$ as algebraic varieties

Let $U \models \mathfrak{X}$ be an L_A-structure modelling axiomatisation $\mathfrak{X}(A(\mathbb{C}))$, and let $\boldsymbol{U}$ be the standard model, i.e. the universal covering space of $A(\mathbb{C})$ considered as an L_A-structure.

We know that $\boldsymbol{U}/_{\sim_H} \cong A^H(\mathbb{C})$ for some algebraic varieties $A^H(\mathbb{C})$ defined over $\mathbb{C}$. The relations $\sim_H, \sim_{Z,H}$ are essentially relations on $\boldsymbol{U}/_{\sim_H}$, and thus Axiom III.5.2.1 says that the first-order theories of $\boldsymbol{U}/_{\sim_H}$ and standard model $\boldsymbol{U}/_{\sim_H}$

in the languages $L_A(H) = \{\sim_H, \sim_{Z,H}$: Z varies$\}$ coincide. We know by properties of analytic covering maps that an irreducible étale closed subset of $\boldsymbol{U}$ covers an irreducible Zariski closed subset of $A^H(\mathbb{C})$, and thus the relation $\sim_{Z,H}$ on $\boldsymbol{U}/\sim_H$ interpreters as $x, y \in A^H(K)$ lie in the same (Zariski) irreducible component of the preimage of $Z(K)$ in $A^H(K)$.

Thus, by Lemma III.1.5.4 any algebraic subvariety defined over $\overline{\mathbb{Q}}$ of $A^H(\mathbb{C})$ is $L_A(H)$-definable. Thus, full theory of an algebraically closed field is reconstructible in $L_A(H)$ on $U/\sim_H$; and thus, there is an algebraically closed field $K = \bar{K}, \mathrm{char} K = 0$ such that $U/\sim_H \cong A^H(K)$; here $A^H(K)$ corresponds to $A^H(\mathbb{C})$ with a different ground field.

Fix these isomorphisms $U/\sim_H \cong A^H(K)$, and let $p_H : U \to A^H(K)$ be the projection morphism. Then the above considerations say

$x' \sim_{W,H} y' \iff p_H(x') \sim_{W,H} p_H(y') \iff x'$ and y' lie the same (Zariski) irreducible component of the preimage of $Z(K)$ in $A^H(K)$.

$x \sim_G y' \iff$ there exist an algebraic morphism $\tau : A^G \to A^H$ and an étale covering morphism $q : A^H \to A^G$ such that $\tau(x') = y'$ and $\tau \circ q = q$:

$$\begin{array}{ccc} A^H & \xrightarrow{\tau} & A^H \\ \big\downarrow q \text{ étale cover} & & \big\downarrow q \\ A^G & \xrightarrow{\mathrm{id}} & A^G \end{array}$$

An important corollary of above considerations is that any set of form $p_H^{-1}(Z(K)), Z(K) \subset A^H(K)$ is L_A-definable.

Notation III.5.3.1. Let us introduce new relations on U; eventually we will prove that they are first-order definable. We introduce the relations below for every closed subvariety of $A(K)$, not necessarily defined over k (those would be in L_A)

$$x' \sim_W y' \iff p_H(x') \sim_{W,H} p_H(y') \text{ for all } H \lhd_{\mathrm{fin}} \pi.$$

An *irreducible component* of relation $\sim_W$ is a maximal set of points in U pairwise $\sim_W$-related. A subset of U is *closed* iff it is a union of irreducible components of relations $\sim_{W_1}, \ldots, \sim_{W_n}$, for some $W_1, \ldots, W_n$. An *irreducible closed set* is an irreducible component of a relation $\sim_W$ for some closed subvariety W. Let us call a subset of U *étale closed* iff it is a union of irreducible components of a finite number of relations $\sim_{W_1}, \ldots, \sim_{W_n}$. This defines an analogue of the étale topology on $\boldsymbol{U}$.

Group action of fibres of $p : U \to A(K)$ on U

For a point $x_0 \in U$, let $\pi(U, x_0') = \{y : y \sim_\pi x_0'\} = p^{-1}p(x_0')$ be the fibre of $p : U \to A(K)$. For every point $z' \in U$ and every point $y' \sim_\pi x_0'$, there exists a point $z'' \in U$ such that $p_G(z', z') \sim_\Delta p_G(x'_0, y')$; this follows from Axiom III.5.2.1. Then, by lifting property for $\Delta \subset A^2(K)$, there exists $z''' \in U$ such that $z''' \sim_G z''$ and $(z', z''') \sim_\Delta (x'_0, y')$. Moreover, such a point z''' is unique. Indeed, by Axiom III.5.2.1 the conditions $p_H(z''') \sim_H p_H(z'')$ and $(z', z''') \sim_{\Delta,H} (x'_0, y')$ determine $p_H(z''')$ uniquely for every $H \lhd_{\mathrm{fin}} G$. This implies that z''' is unique by Axiom III.5.2.4.

The above construction defines an action σ of $\pi(U, x'_0) = \{y : y \sim_\pi x'_0\} = p^{-1}p(x'_0)$ on U: a point $y' \sim_\pi x'_0$ sends z' into z''', $\sigma_{y'} z' = z'''$. Axiom III.5.2.1 and Axiom III.5.2.2 imply that it is in fact a group action.

Let $\pi(U)$ be the group of transformations of U induced by $\pi(U, x'_0)$; the group does not depend on the choice of x'_0. We refer to $\pi(U)$ as the *group of deck transformations*, or *the fundamental group* of U. This terminology is justified by the fact that $\tau \circ p = p$, for $p : U \to A(K)$ the covering map.

For a subset $W \subset U^n$, let $\pi(W) = \{\tau : U^n \to U^n : \tau(W) \subset W, \tau \in \pi(U)^n\}$.

Decomposition Lemma for U

We use a Corollary to Lemma III.1.3.1.

Lemma III.5.3.2 (Decomposition lemma; Noetherian property). *Assume A is Shafarevich.*

A subset $p^{-1}(W), W \subset A(K)$ has a decomposition of the form

$$W' = HZ'_1 \cup \ldots \cup HZ'_k,$$

where $H \lhd_{\mathrm{fin}} \pi$ is a finite index normal subgroup of π, the étale closed sets $Z'_1, \ldots, Z'_k$ are irreducible components of relations $\sim_{Z_i}$, for some algebraic subvarieties Z_i of $A(K)$, and for any $\tau \in H$ either $\tau Z'_i = Z'_i$ or $\tau Z'_i \cap Z'_i = \emptyset$.

Proof. By a corollary to Decomposition Lemma III.1.3.1 we may choose $H \lhd_{\mathrm{fin}} \pi$ with the following property.

Let $Z_i \subset A^H(K)$'s be the irreducible components of $p_H p^{-1}(W)$. Then, they have the property that the connected components of $p_G p_H^{-1}(Z_i) \subset A^G(K)$ are irreducible. Choose Z'_i to be an irreducible components of relations $\sim_{Z_i}$, i.e. the closed sets $p_H^{-1}(Z_i)$. We claim that these Z'_i's give rise to a decomposition as above.

Before we are able to prove this, let us prove the *lifting property for* $\sim_{Z_i}$, namely that the map $p_H : Z'_i \to Z_i(K)$ is surjective. For convenience, we drop the index i below.

By passing to a smaller subgroup if necessary we may find a variety $V \subset A^H(K)^n$ defined over $\overline{\mathbb{Q}}$ such that for some $g \in A^n(K)$, Z_i is a connected component of fibre V_g of V over g, and it holds that if points x', y' are such that $p_H(x'), p_H(y') \in Z_i$ and $x' \sim_W y', p_H(\operatorname{pr} x') = p_H(\operatorname{pr} y') = g'$ lie in the same connected component of V' over $g, p_H(g') = g$, then in fact x' and y' lie in the same connected component of the preimage of $g \times Z_i$, $x' \sim_Z y'$.

Consider Axiom III.5.2.3 for all $G \triangleleft_{\text{fin}} \pi$ sufficiently small

$$x' \sim^c_{V_g, A^G} y' \Longrightarrow \exists y''(y'' \sim_G y' \;\&\; \operatorname{pr} x' = \operatorname{pr} y'' \;\&\; x' \sim_V y'')$$

Now take any point $z' \in Z' \subset U$ and a point $y \in Z(K)$. We want to prove $p_H(Z') \supset Z(K)$, and thus it is enough to prove there exists $y_1 \in U$, $p_H(y_1) = y, z' \sim_Z y_1$. We know that there exist $y_2 \in U$, $z' \sim^c_{Z,A^G} y_2$, due to Axiom III.5.2.1. Since $Z = V_g$ for some $g \in U^{n-1}$, we also have $(g', z') \sim^c_{V_g, A^G} (g', y_2)$, and taking $p_H(g') = g$, $x' = (g', z'), y' = (g', y_2)$, Axiom III.5.2.3 gives the conclusion

$$\exists y''(y'' \sim_G y' \;\&\; \operatorname{pr} x' = \operatorname{pr} y'' \;\&\; x' \sim_V y'').$$

The conclusion says points $x', y'' \in U^n, p_H(x'), p_H(y') \in Z_i$ lie in the same connected component of $p_H^{-1}(V)$, are $\sim_G$-equivalent, and lie above the same point $g', p_H(g') = g$. Then by Lemma III.3.2.2 we know that $p_H(x'), p_H(y')$ lie in the same connected component of the corresponding preimage of Z_i. By definition of Z', this means $\operatorname{pr}_2 y' \in Z'$. Thus, we have proved that $p_H(Z') = Z(K)$ is surjective.

Now the following by now standard argument concludes the proof.

The the covering property implies that

$$p_H^{-1}(Z(K)) = \bigcup_{h \in H} hZ' = HZ';$$

indeed, by properties of Z we know that the relations $x' \sim_{Z,G} y'$ are equivalence relations for all $G \triangleleft_{\text{fin}} H$. Moreover, we know that any two equivalence classes are conjugated by the action of an element of H; this is so because the covering property implies that there is an element of each of the classes above each element of $Z(K)$. This implies the lemma. $\square$

We single out the following part of the proof as a corollary.

Recall that $\sim^c$ means "to lie in the same connected component of".

Corollary III.5.3.3 (the covering property). *For a subvariety $Z \subset A(K)$, $x' \sim^c_{Z,G} y' \Longrightarrow \exists y''(y'' \sim_G y' \;\&\; x' \sim^c_Z y'')$.*

Proof. The proof of the lifting property above proves the corollary for $Z \subset A^H(K)$ if the relations $\sim_Z^c$ and $\sim_Z$ are equivalent. However, by Decomposition Lemma any set $p_H^{-1}(Z)$ can be decomposed into a union of such sets; then going from one irreducible component to another one intersecting it gives the corollary. $\square$

Corollary III.5.3.4 (Topology on U). *The collection of étale closed subsets of U forms a topology with a descending chain conditions on irreducible sets. An étale closed set possesses an irreducible decomposition as a union of a finite number of étale closed sets whose étale connected components are étale irreducible. A union of irreducible components of an étale closed set is an étale closed.*

That is,

1. *the collection of étale closed subsets on $\boldsymbol{U}^n, n > 0$ forms a topology. The projection and inclusion maps* $\mathrm{pr} : \boldsymbol{U}^n \to \boldsymbol{U}^m, (x_1, \ldots, x_n) \mapsto (x_{i_1}, \ldots, x_{i_m})$ *and* $\iota : \boldsymbol{U}^n \hookrightarrow \boldsymbol{U}^m, (x_1, \ldots, x_n) \mapsto (x_{i_1}, \ldots, x_{i_{m'}}, c_{m'}, \ldots, c_m)$ *are continuous.*
2. *There is no infinite decreasing chain* $.. \subsetneq U_{i+1} \subsetneq U_i \subsetneq \ldots \subsetneq U_0$ *of closed étale irreducible sets.*
3. *A union of irreducible components of an étale closed set is étale closed.*
4. *A set is étale closed iff it a union of* connected *components of a finite number of H-invariant sets, for some $H \lhd_{\mathrm{fin}} \pi$ a finite index subgroup of π.*
5. *Each étale closed set is a union of a finite number of étale closed sets whose étale connected components are étale irreducible. Moreover, those sets may be taken so that their connected components within the same set are translates of each other by the action of a finite index subgroup $H \lhd_{\mathrm{fin}} \pi$.*

Proof. The last item is just a reformulation of Decomposition Lemma. All the items trivially follow from it but (1).

Let us prove the intersection of two étale closed set Z'_i and Y'_i is étale closed.

Assume W' and V' are unions of connected component of H-invariant sets HW' and HV'. The intersection $HW' \cap HV'$ is H-invariant and the set $W' \cap V'$ is a union of the connected components of $HW' \cap HV'$. The intersection $HW' \cap HV' = p_H^{-1}(p_H(W') \cap p_H(V'))$ is étale closed by definition, and thus its connected components are also étale closed. By definition this implies $W' \cap V'$ is étale closed.

To prove that an infinite intersection is closed, it is sufficient to prove that the intersection of a decreasing sequence of étale closed sets is étale closed. Use Koenig lemma and the fact that a sequence of decreasing étale irreducible sets stabilises.

The descending chain condition follows from the fact that an irreducible subset of an irreducible set necessarily have lesser dimension. □

Chevalley Lemma

Let $W' \subset U$ be an *irreducible closed subset* of U, i.e. a subset of U defined by

$$x \sim_W a_1 \ \& \ \dots \ \& \ x \sim_W a_n$$

where $a_1, \dots, a_n \in U$ are such that

$$\forall y \forall z \left(\bigwedge_{1 \leqslant i \leqslant n} y \sim a_i \ \& \bigwedge_{1 \leqslant i \leqslant n} z \sim a_i \Longrightarrow y \sim_W z \right).$$

Such a set W' we call *an irreducible component of closed set defined by* $x \sim_W x$, or simply *an irreducible component of relation* $\sim_W$.

Lemma III.5.3.5 (Chevalley Lemma). *A projection of an étale irreducible closed set is étale closed.*

Proof. Let W' be such an irreducible set, and let $V' = \mathrm{Clpr}\, W'$ be the least closed set containing its closure. By definition of V' $p_H(\mathrm{pr}\, W') \subset p_H(V')$; and by definition of closure $V' \subset \mathrm{pr}\, HW' = p_H^{-1}(\mathrm{pr}\, p_H(W'))$; the set $\mathrm{pr}\, p_H(W')$ is closed by Chevalley Lemma for projective algebraic varieties. The inequalities imply $p_H(\mathrm{pr}\, W') = p_H(V')$ for every subgroup $H \triangleleft_{\mathrm{fin}} \pi$.

A deck transformation leaving W' invariant, also leaves V' invariant, i.e. $\mathrm{pr}\, \pi(W') \subset \pi(V')$. On the other hand, the equality $p_H(\mathrm{pr}\, W') = p_H(V')$ implies for any $H \triangleleft_{\mathrm{fin}} \pi$, $\mathrm{pr}\, \pi(W')/H = \pi(V')/H$.

Let us now use Axiom III.5.2.5 to show that this implies that $\mathrm{pr}\,(\pi(W) \cap [H \times H]) = \pi(V') \cap H$.

Let us now prove that $\pi(W') \cap H \times H$ is finitely generated for some $H \triangleleft_{\mathrm{fin}} \pi$.

We know by Corollary to Lemma III.2.2.2 that $W' = Y'_{g'}$ is a fibre of a $\overline{\mathbb{Q}}$-defined set Y' over a point g' such that $p_H(g') \in \mathrm{pr}\, p_H(Y')$ $\overline{\mathbb{Q}}$-generic.

We know that for every $G \triangleleft_{\mathrm{fin}} H$, for a connected component Y_G of $p_G p_H^{-1}(Y)$, the intersection $Y_G \cap g' \times p_G p_H^{-1}(Y_g)$ is connected; geometrically, that means that a lifting of $W = Y_g \subset Y$ along the covering map $Y_G \to Y$ is a fibre of Y. This holding for every $G \triangleleft_{\mathrm{fin}} H$, it implies that for Y' a connected component of $p_H^{-1}(Y)$, the intersection ${Y'_{g'}}^c = Y' \cap g' \times p_H^{-1}(Y_g)$ is connected, and therefore it coincides with a connected component of $p_H^{-1}(Y_g) = p_H^{-1}(W)$. Moreover, this implies that if $h \in H$

is such that $hY'^{\,c}_{g'} \subset p_H^{-1}(Y_g)$ then $hY'^{\,c}_{g'} \subset Y'^{\,c}_{g'}$, i.e. $h \in \pi(Y'^{\,c}_{g'}) \cap H = \pi(Y'_{g'}) \cap H$. Thus, to prove that $\pi(W) \cap H = \pi(Y'^{\,c}_{g'}) \cap H$ is finitely generated, it is enough to prove that $\pi(Y'_{g'}) \cap H$ is finitely generated. However, the latter is claimed by Axiom III.5.2.5 for every variety Y defined over $\overline{\mathbb{Q}}$.

Let $g_1, \ldots, g_n$ be the generators of $\pi(W') \cap [H \times H]$. Now take $\tau \in \pi(V') \cap H, \tau(V') = V'$. We know that $\tau/G \in \mathrm{pr}\, \pi(W')/G$, for every $G \lhd_{\mathrm{fin}} H$, and therefore τ, up to $\sim_G$, is expressible as a product of $g_1, \ldots, g_n$. In other words, that means that x' and $\tau x'$ can be joined by a sequence of points $x' = h_1, h_2, \ldots, h_n = \tau x'$ such that $h_{i+1} = g_{j_i} h_i$ for all $1 \leqslant i \leqslant n$, and here $n = n(G)$ depends on subgroup G. By Axiom 5 there is a uniform bound on such $n = n(G)$, and τ is expressible as a product of $g_1, \ldots, g_n$, and therefore belongs to $\mathrm{pr}\, \pi(W')$.

Now we finish the proof by the covering property argument similar to the topological proof of Chevalley Lemma in complex case.

Let $V_0 \subset \mathrm{pr}\, p_H(W') \subset V$ where $V_0 \subsetneq V$ is open in V; then V is irreducible. Recall $V' = \mathrm{Clpr}\, W'$ and take $V'_0 = V' \cap p_H^{-1}(V_0)$; we know $V'_0 \subset V'$ is open in V'. We also know $V'_0 \subset \mathrm{Clpr}\, W'$.

Take $v' \in V'_0$, and take $w' \in W'$, $\mathrm{pr}\, p_H(w') = p_H(v') \in V_0 \subset \mathrm{pr}\, W$; such a point w' in W' exists by the covering property. Now, $\mathrm{pr}\, w' \in V'$, and thus $\gamma_0 \in \pi(V')$ where γ_0 is defined by $v' = \gamma_0 \mathrm{pr}\, w'$. Condition $\mathrm{pr}\, p_H(w') = p_H(v') \in A^H(K)$ implies $\gamma_0 \in H$. Thus the inclusion $\mathrm{pr}\, \pi(W') \cap H = \pi(V') \cap H$ implies there exists $\gamma_1 \in \pi(W')$, $\mathrm{pr}\, \gamma_1 = \gamma_0$, and thus $v' = \gamma_0 \mathrm{pr}\, w' = \mathrm{pr}\,(\gamma_1 w')$, and the Chevalley lemma is proven. □

Corollary III.5.3.6. *A projection of a set open in its irreducible closure contains an open subset of the closure of the projection.*

Proof. Let $\emptyset \neq W^{0\prime} \subset W'$ be an open subset of an irreducible closed set W'. Then $W'_1 = W' \setminus W^{0\prime} \subsetneq W'$ is a closed set. Consider $W' \setminus HW'$. If $W' \subset HW'_1$, then by irreducibility of W' it holds $W' \subset hW'_1 \subset hW'_1$, for some $h \in H$. This forces $hW' = W'$, and also $W'_1 = W'$, which constricts the assumption $\emptyset \neq W^{0\prime}$. Thus $W' \setminus HW'_1 \neq \emptyset$, and $\mathrm{pr}\, p_H(W') \supsetneq p_H(W')$. Now, $p_H(W')$ is an irreducible closed subset of $A^H(K)$, and therefore $p_H(W')$ is of smaller dimension then $p_H(W')$. Now we may apply Chevalley Lemma for algebraic varieties to get the conclusion. □

III.5.4 Homogeneity and stability over models

In the §§ above we have established the main properties of the étale topology on U (and its Cartesian powers U^n). That allows us to define and prove the basic properties of Θ-generic points, for Θ an algebraically closed subfield of K.

The notion of a Θ-generic point extends to U in a natural way. Recall that for a closed Θ-defined set V', the set $\mathrm{Cl}_\Theta V'$ is the set of all Θ-generic points of V'. Recall also that a set of Θ-generic points of a Θ-defined set is called Θ-constructible.

Lemma III.5.4.1 (Homogeneity). *Any structure $\boldsymbol{U} \models \mathfrak{X}$ is model homogeneous, i.e. the projection of a Θ-constructible set is Θ-constructible, for any algebraically closed subfield Θ of the ground field.*

Proof. First note that a point $w' \in W'$ in an irreducible set W' is Θ-generic iff $p(w') \in p(W')$ is Θ-generic. By Chevalley Lemma, the fibre $W'_{g'}$ is non-empty for $g' \in \mathrm{pr}\, W'$ Θ-generic. Moreover, by Lemma III.1.5.5 a connected component of fibre $W_g, g = p(g')$ always contains a Θ-generic point $w \in W$ of W. The lifting $w', p(w') = w$ is always Θ-generic, and we may find such a lifting in any connected component of a fibre over a generic point. This implies the lemma. $\square$

Definition III.5.4.2. *Let $\boldsymbol{U}, \boldsymbol{U}_1, \boldsymbol{U}_2 \models \mathfrak{X}$ be L_A-models of $\mathfrak{X}(A(\mathbb{C}))$ and $\boldsymbol{U} \subset \boldsymbol{U}_1 \cap \boldsymbol{U}_2$. We say that tuples $a \in \boldsymbol{U}_1^n$ and $b \in \boldsymbol{U}_2^n$* have the same syntactic quantifier-free type over $\boldsymbol{U}$ in class $\Re$ *if a and b satisfy the same quantifier-free L_A-formulae with parameters in $\boldsymbol{U}$.*

Definition III.5.4.3. *A class $\Re$ of L_A-structures is* syntactically stable over countable submodels *iff for any countable structure $\boldsymbol{U} \in \Re$, the set of complete L_A-types over a structure $\boldsymbol{U}$ realised in a structure $\boldsymbol{U}' \in \Re$ is at most countable.*

Definition III.5.4.4. *A class $\Re$ of L_A-structures is* quantifier-free syntactically stable over countable submodels *iff there are only countably many quantifier-free syntactic types in class $\Re$ over any countable model $\boldsymbol{U} \in \Re$.*

Lemma III.5.4.5 (Stability over submodels). *Assume A is Shafarevich. The class of L_A-models of $\mathfrak{X}(A(\mathbb{C}))$ is quantifier-free syntactically stable over submodels.*

Proof. If $\boldsymbol{U} \prec \boldsymbol{U}'$ is an elementary substructure, then $\boldsymbol{U} = \boldsymbol{U}'(\Theta) = \{u \in \boldsymbol{U}' : p(u) \in A(\Theta)\}$, for some algebraically closed subfield Θ.

Every positive quantifier-free L_A-formula over $\boldsymbol{U}$ determines a closed set defined over Θ. For every tuple $v' \in \boldsymbol{U}'$, there is a least closed set $V' = \mathrm{Cl}_\Theta(v')$ containing v' and defined over Θ; it is irreducible, and is a connected component of an algebraic subvariety V/Θ of A^H defined over Θ, for some $H \lhd_{\mathrm{fin}} \pi$. Moreover, $\mathrm{Cl}_\Theta(v')$ has a Θ-point v'_Θ. Thus, the quantifier-free L_A-type of tuple v' is determined by the point $v'_\Theta \in \boldsymbol{U}$ and a subvariety V/Θ. Therefore, there are only countable number of such types, which implies that class $\Re$ is quantifier-free syntactically stable over submodels. $\square$

Property III.5.4.6 (Homogeneity and Stability of class $\Re$). *Assume A is Shafarevich.*

All structures L_A-models of $\mathfrak{X}(A(\mathbb{C}))$ are model homogeneous. The class of L_A-models of $\mathfrak{X}(A(\mathbb{C}))$ is syntactically quantifier-free stable over countable submodels.

Proof. Implied by preceeding two lemmata. □

Finally, we may state Theorem III.5.4.7, which was the goal of the chapter.

Theorem III.5.4.7 (Model Stability of $\mathfrak{X}(\boldsymbol{U})$). *Let A be a smooth projective algebraic variety which is Shafarevich. That is, we assume the universal covering space $\boldsymbol{U}$ of A is holomorphically convex, and such that the product of the fundamental groups of connected components of $A(\mathbb{C})$ is subgroup separable. Let L_A be the countable language defined in Def. II.1.2.1. Then $(\mathfrak{1}', \mathfrak{3a}, \mathfrak{3b})$ of Subgoal II.1.1.3 hold, and $\mathfrak{2}'$ is weaken to $\mathfrak{2}'_{\aleph_1}$:*

$\mathfrak{2}'_{\aleph_0 \to \aleph_1}$ *Any two models $\boldsymbol{U}_1 \models \mathfrak{X}$ and $\boldsymbol{U}_2 \models \mathfrak{X}$ of axiomatisation $\mathfrak{X}$ and of cardinality $\aleph_1$, such that*

there exist a common countable submodel $\boldsymbol{U}_0 \models \mathfrak{X}$, $\boldsymbol{U}_0 \subset \boldsymbol{U}_1$ and $\boldsymbol{U}_0 \subset \boldsymbol{U}_1$

are isomorphic, $\boldsymbol{U}_1 \cong_{L_A} \boldsymbol{U}_2$, and, moreover, the isomorphism φ is identity on $\boldsymbol{U}_0$.

Proof. This is closely related to Proposition III.5.4.6; however, let us prove this directly in an explicit manner; in this argument we try to put an emphasis on the properties of the topology, although this could also be treated as a very common model-theoretic argument.

We will prove that every partial L_A-isomorphism $f : \boldsymbol{U}_1 \dashrightarrow \boldsymbol{U}_2, f(a) = b, a \in \boldsymbol{U}_1^n, f_{|\boldsymbol{U}_0} = \mathrm{id}_{|\boldsymbol{U}_0}$, $n \in \mathbb{N}$ finite, defined on $\boldsymbol{U}_0 \cup \{a_1, \ldots, a_n\}$, can be extended to $\boldsymbol{U}_0 \cup \{a_1, \ldots, a_n\} \cup \{c\}$, $f(c) \in \boldsymbol{U}_2$ for any element $c \in \boldsymbol{U}_1$. This allows to extend a partial L_A-isomorphism from a *countable* model to its *countable* extension. This is enough: by taking unions of chains of countable submodels we get isomorphism between models of cardinality $\aleph_1$. Note that one cannot get isomorphism between models of cardinality $\aleph_2$ in this way.

Let $V_1 = \mathrm{Cl}_{\boldsymbol{U}_0}(a), W_1 = \mathrm{Cl}_{\boldsymbol{U}_0}(a, c)$ be the minimal closed irreducible subsets containing points $a \in \boldsymbol{U}_1^n$ and $(a, c) \in \boldsymbol{U}_1^{n+1}$; let $V_2 = \mathrm{Cl}_{\boldsymbol{U}_0}(f(a))$ be the corresponding subset of $\boldsymbol{U}_2$. Since f is an L-isomorphism, sets V_1 and V_2 are defined by the same L-formulae with parameters in $\boldsymbol{U}_0$.

Take a subgroup $H \triangleleft_{\mathrm{fin}} \pi$ sufficiently small such that V_1, V_2, W_1, W_2 are connected components of $p_H^{-1} p_H(V_1), p_H^{-1} p_H(V_2), p_H^{-1} p_H(W_1), p_H^{-1} p_H(W_2)$, respectively. Pick points $v_1, w_1 \in \boldsymbol{U}_0$ such that $v_1 \in V_1, V_2$ and $w_2 \in W_1, W_2$.

Now, by definition of W_2 we have $\mathrm{pr}\, p_H W_2 = p_H V_2$, and also $\mathrm{pr}\, w_2 \in V_2$; choose $c' \in \boldsymbol{U}_2$ such that $(p_H(b), p_H(c')) \in p_H(W_2)$ is a $\boldsymbol{U}_0$-generic point of $p_H(W_2)$. Then by the lifting property for W_2 there exists a point $(b', c'') \in W_2$ such that $p_H(b') = p_H(b), p_H(c'') = p_H(c')$. However, this implies that $b' \in \operatorname{pr} W_2 \subset V_2$ is a $\boldsymbol{U}_0$-generic point of V_2. Therefore by the homogeneity properties in Lemma III.5.4.1, or equivalently because the projection $\operatorname{pr} W_2$ is a closed set definable over $\boldsymbol{U}_0$, this implies $V_2 \subset \operatorname{pr} W_2$, and, in particular, there exists $d \in \boldsymbol{U}_1$ such that $(b, d) \in W_2$ is a $\boldsymbol{U}_0$-generic point. Now set $f(c) = d$. By construction, the points $(a, c) \in \boldsymbol{U}_1$ and $(b, d) \in \boldsymbol{U}_2$ lie in the same $\boldsymbol{U}_0$-definable closed sets, and, since every basic relation of L_A defines a closed $\boldsymbol{U}_0$-defined set, this implies that f is indeed an L_A-isomorphism, as required. $\square$

Chapter IV

Atomicity: relations to arithmetics

We present a model theory conjecture that the universal covering space U_A described in Chapter III contains an atomic submodel, and for $A = E$ an elliptic curve over $\overline{\mathbb{Q}}$, we reformulate and prove the conjecture (Proposition IV.1.0.8) in a very explicit way free from model-theoretic terminology. The proof admits a geometric interpretation as a slightly unusual universality property of the fundamental groupoid functor (Proposition IV.1.0.11) or an equivalent statement (Proposition IV.1.0.9) describing an orbit of the Galois action on End$E$-module extensions $\mathrm{Ext}^1_{\mathrm{End}E\text{-mod}}(E(\overline{\mathbb{Q}}), \Lambda)$ of $E(\overline{\mathbb{Q}})$ by Λ. The proofs use essentially the Kummer theory of the underlying elliptic curve as well as the description of the image of the Galois representation; to a model-theorists mind, these statements *explain* why those deep facts of algebraic geometry are natural and true.

The chapter is fully self-contained apart from motivation, and we repeat most definitions used; see Chapter II and I for motivations.

In §IV.1 we present three equivalent Propositions IV.1.0.8, IV.1.0.9 and IV.1.0.11 mentioned above. We prove the equivalence of Propositions IV.1.0.9 and IV.1.0.11 in §IV.4.2. Propositions IV.1.0.9 and IV.1.0.11 require no model theory for either statement or the proof, and could be read independently of the rest of the thesis. In §IV.2 we introduce the necessarily terminology and several equivalent reformulations of the atomicity conditions of Propositions IV.1.0.8 which apply to any underlying variety. We prove Proposition IV.1.0.9 in §IV.3.3.

In §IV.7 we state some conjectures.

IV.1 Introduction and Results

In §IV.1 we quickly state three equivalent Propositions IV.1.0.8, IV.1.0.9, IV.1.0.11; we state them in detail and prove them later in §IV.3.3, §IV.5, §IV.4, respectively.

The L_A-structure $U_E(\overline{\mathbb{Q}})$ is atomic

Let E be an elliptic curve over a number field $k \subset \mathbb{C}$, and let $U_E(\mathbb{C})$ be the universal covering space of $E(\mathbb{C})$. In Chapter III, Definition II.1.2.1 we have defined a countable language L_A and its interpretation on the universal covering space $U(\mathbb{C})$ of an arbitrary variety $A(\mathbb{C})$ defined over a number field. Recall that positively L_A-definable sets are unions of connected and irreducible components of the preimages of algebraic subvarieties of $A^n(\mathbb{C})$.

Proposition IV.1.0.8. *The substructure $U_E(\overline{\mathbb{Q}})$ of the universal covering space $U_E(\mathbb{C})$ in language $L_A(E)$ is atomic, and is elementarily equivalent to $L_A(E)$-structure $U_E(\mathbb{C})$.*

Due to nature of language L_A, the Proposition above may be seen as a precise mathematical formalisation, and a rather straightforward one at that, of the idea that "complex-analytic or homotopy properties of a variety determine its arithmetic".

In §IV.3.1, we show that an elementary substructure of $U_E(\mathbb{C})$ can be viewed equivalently as an extension of $E(\overline{\mathbb{Q}})$ of a particular kind. It turns out that one can prove categoricity for that kind of extensions as well; see the next subsection for details.

Uniquely divisible extensions of Abelian groups

The universal covering space of an elliptic curve $A = E$ is just $\mathbb{C}$; the linear structure on $\mathbb{C}$ plays an important rôle in the theory of (complex) elliptic curves as algebraic varieties: it defines the group operation, the properties thereof, etc. This suggests that the relevant linear structure on $\mathbb{C}$ is determined up to isomorphism by the curve itself. An analysis of L_A based on these observations shows that L_A is a language describing the linear structure on $\mathbb{C}$, and the pull-back of the algebraic structure on the elliptic curve. The next proposition confirms these suggestions. We conjecture it holds for the group $E(\mathbb{C})$ and in general, the group of points over any algebraically closed field of zero characteristic; the action of Galois group is then replaced by the action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ or the automorphism group of the field.

In terms of model theory, the Proposition describes explicitly a class of structures, namely that of extensions of $E(\overline{\mathbb{Q}})$ of a particular kind, and then proves a categoricity result for the class. It could be seen that the class consists of atomic models of certain first-order axioms. For an elliptic curve without complex multiplication, the Proposition proves the class is just categorical. What we conjecture above is that if we appropriately expand the class to contain structures of arbitrary cardinalities, the class remains categorical in every uncountable cardinality. To prove the conjecture, it is sufficient to prove that the class is excellent

[She83a, She83b]; in particular, one of the conditions, that of $(2,\aleph_0)$-*existence*, required is

$$\mathrm{div}E(L_1 \otimes_k L_2) = \mathrm{div}E(L_1) \times_{E(k)} \mathrm{div}E(L_2),$$

for $k \subset L_1, L_2$ algebraically closed countable fields of characteristic 0. Here $\div H = \{h \in H : \forall N \exists n > N, h_n \in H\ nh_n = h\}$.

Proposition IV.1.0.9. *Let Λ be an $\mathrm{End}E$-module isomorphic to $\mathbb{Z}^2$ if $\mathrm{End}E = \mathbb{Z}$, or, if $\mathrm{End}E \neq \mathbb{Z}$, to an order in the ring of integers in the ring $\mathbb{Q}(\mathrm{End}\,E)$ of fractions of the ring $\mathrm{End}E$ of endomorphisms of an elliptic curve E defined over a number field k. Assume further that all the endomorphisms of E are definable over k.*

1. There exists a uniquely divisible extension in $\mathrm{Ext}^1_{\mathrm{End}E\text{-}mod}(E(\overline{\mathbb{Q}}), \Lambda)$.

2. (a) if $\mathrm{End}E = \mathbb{Z}$, then the Galois group acts transitively on the set of uniquely divisible extensions $\mathrm{Ext}^1_{AbGroups}(E(\overline{\mathbb{Q}}), \Lambda)$.

(b) the action of the Galois group has only finitely many orbits on the set of uniquely divisible extensions in $\mathrm{Ext}^1_{\mathrm{End}E\text{-}mod}(E(\overline{\mathbb{Q}}), \Lambda)$.

In other words,

1. Then there exists a uniquely divisible $\mathrm{End}E$-module V and a short exact sequence of $\mathrm{End}E$-modules

$$0 \longrightarrow \Lambda \longrightarrow V \longrightarrow E(\overline{\mathbb{Q}}) \longrightarrow 0$$

2. (a) If E has no complex multiplication and $\mathrm{End}E = \mathbb{Z}$, then for any uniquely divisible $\mathbb{Z}$-module extensions W, V of $E(\overline{\mathbb{Q}})$ by Λ, fitting into the short exact sequences as above, there exists a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Lambda \arrow[r] \arrow[d, "\exists h_{|\Lambda}"'] & V \arrow[r, "\varphi"] \arrow[d, "\exists"', "h\in\mathrm{Hom}\,(V,W)"] & E(\overline{\mathbb{Q}}) \arrow[r] \arrow[d, "\exists\sigma\in Gal(\overline{\mathbb{Q}}/k)"] & 0 \\
0 \arrow[r] & \Lambda \arrow[r] & W \arrow[r, "\psi"] & E(\overline{\mathbb{Q}}) \arrow[r] & 0
\end{tikzcd}$$

(b) If $\mathrm{End}E \neq \mathbb{Z}$, then there exist finitely many uniquely divisible $\mathrm{End}E$-module extensions $W_1, ..., W_n$ of $E(\overline{\mathbb{Q}})$ by Λ, fitting into the short exact sequences as above, such that for any uniquely divisible $\mathrm{End}E$-module extension V there exists a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Lambda \arrow[r] \arrow[d, "\exists h_{|\Lambda}"'] & V \arrow[r, "\varphi"] \arrow[d, "\exists"', "h\in\mathrm{Hom}\,(V,W_i)"] & E(\overline{\mathbb{Q}}) \arrow[r] \arrow[d, "\exists\sigma\in Gal(\overline{\mathbb{Q}}/k)"] & 0 \\
0 \arrow[r] & \Lambda \arrow[r] & W_i \arrow[r, "\psi_i"] & E(\overline{\mathbb{Q}}) \arrow[r] & 0
\end{tikzcd}$$

An isomorphism $h : V \to W$, $h(\Lambda) = \Lambda$ induces an End$E$-linear map $\tau : E(\overline{\mathbb{Q}}) \to E(\overline{\mathbb{Q}})$; the proposition (item (a)) says we may modify h to have τ induced by a Galois automorphism. In fact, we show we can extend any h, σ defined on *finite* dimensional submodule of V to a total map $h : V \to W, \sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.

By the classical theory the conditions on Λ guarantee that there exists a covering

$$0 \longrightarrow \Lambda \longrightarrow \mathbb{C} \xrightarrow{p} E(\mathbb{C}) \longrightarrow 0.$$

Restricting the ground field $\mathbb{C}$ to $\overline{\mathbb{Q}}$ in the short exact sequence above gives us a sequence as required; we prove (2) in §IV.5.

In terms of Ext-functor classifying extensions, Proposition IV.1.0.9 says the Galois action has finitely many orbits on the set of uniquely divisible extensions in $\mathrm{Ext}^1_{\mathrm{End}E\text{-mod}}(E(\overline{\mathbb{Q}}), \Lambda)$: the set of non-equivalent extensions $\mathrm{Ext}^1_{\mathrm{End}E\text{-mod}}(E(\overline{\mathbb{Q}}), \Lambda)$ is of cardinality $2^{\aleph_0}$. Also note that the injectivity of the profinite completion $\widehat{\Lambda}$ of the kernel Λ implies $\mathrm{Ext}^1_{\widehat{\mathrm{End}E}\text{-mod}}(E(\overline{\mathbb{Q}}), \widehat{\Lambda}) = 0$ making the proposition trivially true in this case. Model-theoretically, the question with the kernel profinitely completed would correspond to treating the *homogeneous case*: namely, it says that there exists a unique homogeneous model in the class of structures elementary equivalent to the standard model provided by the exponential function, with the additional property that the kernel is isomorphic to $\hat{\mathbb{Z}}^2$.

A way to think of it is that the proposition claims that *it is possible to describe the universal covering space of an elliptic curve in a purely algebraic way*, admittedly with respect to a rather weak, linear structure on it.

Remark IV.1.0.10. We believe that our treatment of elliptic curves with complex multiplication is incomplete; the formulation of $2(b)$ that an extension is equivalent to one of finitely many is ugly. It should be possible to prove a full analogue of $1(b)$, after perhaps some suitable reformulation. This is particularly apparent as there is a complete description of Galois action on E_{tors} provided by the theory of complex multiplication on elliptic curves.

Universality of the fundamental groupoid functor

We ask whether *the notion of paths (up to fixed point homotopy) on an elliptic curve $E(\mathbb{C})$ may be described by its natural algebraic properties.* "Paths up to fixed point homotopy" are usually thought of in the context of the *Poincaré's fundamental groupoid*, which can be thought of as a 2-functor. Hence, we may reformulate the question as: *Is the fundamental groupoid functor on the complex algebraic varieties determined by its natural algebraic properties up to natural equivalence and an automorphism of the source category?* The Proposition IV.1.0.11 is a partial positive answer to this question. Let us now remind the notations related to groupoid and 2-functors.

For a topological space T, let $\pi_1^{\rm top}(T)$ denote the set of all paths in T up to fixed point homotopy; that set of paths has a groupoid structure with respect to concatenation of paths. This defines the *Poincaré fundamental groupoid functor* $\pi_1^{\rm top}(T) : \mathfrak{Top} \to \mathfrak{Groupoids}$ from topological spaces to groupoids. It is convenient to think of $\pi_1^{\rm top}$ as of a 5-tuple $(\mathrm{Pt}, \Omega, s, t, \cdot)$ consisting of two functors $\mathrm{Pt}, \Omega : \mathfrak{Top} \to \mathfrak{Sets}$, two natural transformations $s, t : \Omega \to \mathrm{Pt}$ and a functorial operation $\cdot_V$ on $\Omega(V)$; here $\mathrm{Pt}(T) = T$ (as a set) is *the functor of points*, $\Omega(T) = \pi_1^{\rm top}(T)$ (as a set) is the *functor of paths*, and the *source* and *target* maps s_T, t_T give the beginning and ending point of a paths in $\Omega(T)$; partial operation $\cdot_T : \Omega(T) \times \Omega(T) \to \Omega(T)$ is concatenation of paths in T. To such data, one refers as a *2-functor* (Grothendieck [Gro]). A *natural transformation* of 2-functors $\underline{\Omega} = (\mathrm{Pt}, \Omega, s, t, \cdot)$ and $\underline{\Omega}' = (\mathrm{Pt}', \Omega', s', t', \cdot')$ is defined naturally as a pair of natural transformation of functors $h_{\mathrm{Pt}} : \mathrm{Pt} \to \mathrm{Pt}'$, $h_\Omega : \Omega \to \Omega'$ respecting the structure given by $s, t, \cdot$ and $s', t', \cdot'$, cf. commutative diagrams (IV.4.1),(IV.4.2),(IV.4.3) and some others.

We consider the restriction of the functor $\pi_1^{\rm top}$ to the following category. Let $\mathfrak{E} \subset \mathrm{Var}/\overline{\mathbb{Q}}, \mathcal{O}b\,\mathfrak{E} = \{E^n : n \geqslant 0\}, \mathcal{M}or_{\mathfrak{E}}(X,Y) = \mathcal{M}or_{\mathrm{Var}/\overline{\mathbb{Q}}}(X,Y)$ be the full subcategory of the category of varieties whose objects are the Cartesian powers of E including the trivial Cartesian power $E^0 = 0$. The morphisms are morphisms of varieties between its objects, not necessarily preserving 0; this is the definition of a *full* subcategory. The Galois group $Gal(\overline{\mathbb{Q}}/k)$ acts on the category $\mathrm{Var}/\overline{\mathbb{Q}}$, and its action restricts to the category $\mathfrak{E}$: it leaves each object of $\mathfrak{E}$ invariant and permutes the morphisms. Note that $Gal(\overline{\mathbb{Q}}/k)$ acts on the set of 2-functors from $\mathfrak{E}$ via its action on the category $\mathfrak{E}$ by automorphisms. Recall a groupoid $\Omega(E)$ is *connected* iff for every $x, y \in \mathrm{Pt}(E)$ there exists $\gamma \in \Omega(E)$ "going from point x to point y", i.e. $x = s(\gamma), y = t(\gamma)$.

The following proposition is derived from Proposition IV.1.0.9 in IV.4.2.

Proposition IV.1.0.11 (Universality of fundamental groupoid functor). *Assume that E defined over a number field k is an elliptic curve without complex multiplication. Let $\mathfrak{E}$ be the category of Cartesian powers of E as above.*

Let $\underline{\Omega} = (\mathrm{Pt}, \Omega, s_V, t_V, \cdot_V)$ be a functor from category $\mathfrak{E}$ to $\mathfrak{Groupoids}$ (i.e. a 2-functor) such that

1. *$\underline{\Omega}$ preserves direct product*
2. *$\underline{\Omega}$ has the unique path-lifting property along étale morphisms (Def. IV.4.1.1).*

Assume further that

1. *$\mathrm{Pt}(E^n)$ is the functor of $\overline{\mathbb{Q}}$-points: $\mathrm{Pt}(E^n) = E^n(\overline{\mathbb{Q}}) = \mathcal{M}or_{\mathfrak{E}}(0, E)$*
2. *$\Omega(E)$ is a connected groupoid*

3. there is an isomorphism

$$\Omega_{0,0}(E) \cong \mathbb{Z}^2 \textit{ as } \mathrm{End}E\textit{-modules.}$$

Then there exists an automorphism $\sigma \in Gal(\overline{\mathbb{Q}}/k)$ *such that the 2-functors* $\underline{\Omega}, \pi_1^{\mathrm{top}}{}_{|\overline{\mathbb{Q}}} \circ \sigma : \mathfrak{E} \to \mathfrak{Sets}$ *are naturally equivalent (as 2-functors):*

$$\underline{\Omega} \cong \pi_1^{\mathrm{top}}{}_{|\overline{\mathbb{Q}}} \circ \sigma$$

$$\begin{tikzcd} \mathfrak{E} \arrow[r, "\underline{\Omega}"'] \arrow[d, "\exists\, \sigma\in Gal(\overline{\mathbb{Q}}/k)"] \arrow[dr, phantom, "\exists\Downarrow \text{natural transformation}"] & \mathfrak{Groupoids} \\ \mathfrak{E} \arrow[r, "\pi_1^{\mathrm{top}}{}_{|\overline{\mathbb{Q}}}"] & \mathfrak{Groupoids} \end{tikzcd}$$

Here $\mathrm{Pt}_{\pi_1^{\mathrm{top}}{}_{|\overline{\mathbb{Q}}}}(V) = V(\overline{\mathbb{Q}})$ and $\Omega_{\pi_1^{\mathrm{top}}{}_{|\overline{\mathbb{Q}}}}(V) = \{\gamma \in \pi_1^{\mathrm{top}}(V(\mathbb{C})) : s(\gamma), t(\gamma) \in V(\overline{\mathbb{Q}})\}$ is the restriction of π_1^{top} to $\overline{\mathbb{Q}}$-points.

Similarly to Proposition IV.1.0.9, for E possessing complex multiplication, there exists finitely many σ_i's such that any 2-functor on $\mathfrak{E}$ as above, is equivalent to one of $\pi_1^{\mathrm{top}}{}_{|\overline{\mathbb{Q}}} \circ \sigma_i$'s.

The condition in the definition of an abstract fundamental groupoid functor are somewhat reminiscent of the conditions defining the scheme-theoretic algebraic fundamental group π_1^{alg} [SGA4$\frac{1}{2}$]; however, there π_1^{alg} takes values in the category of *profinite* groups, in particular $\pi_1^{\mathrm{alg}}(\mathbb{C}^*, 1) = \hat{\mathbb{Z}}$, $\pi_1^{\mathrm{alg}}(E(\mathbb{C}), 0) = \hat{\mathbb{Z}}^2$. In terms of model theory, this corresponds to considering *homogeneous models*.

It is a natural to consider whether a path lies in an algebraic subvariety; thus our notion of a path given via $\underline{\Omega}$ *restricted* to $\mathfrak{E}$ should be able to express when (a representative of the homotopy class of) a path lies in an *arbitrary* algebraic subvariety. This is indeed the case:

Remark IV.1.0.12 (Recovering $\Omega(Z)$ for arbitrary closed subvariety Z of E^n). The information contained in the functor $\pi_1^{\mathrm{top}}|\mathfrak{E}$ restricted to the full subcategory $\mathfrak{E}$ of Cartesian powers of an elliptic curve is enough to determine whether a path lies in a closed subvariety. The key fact here is for a normal subgroup $H \lhd \pi_1(E^n(\mathbb{C}))$, there exists an *H-Shafarevich* morphism $\mathrm{Sh}_H : E^n \to E^m$ such that for an arbitrary irreducible $Z \subset E^n(\mathbb{C})$, it holds $Z \subset \ker f$ iff the image $Im\,[\pi_1(\hat{Z}, z) \to \pi_1(E^n(\mathbb{C}), z)]$ has a finite index subgroup contained in H. Lemma IV.3.1.1 states this in a rather different language; cf. also Remark IV.3.1.3 for a technical explanation of the connection between the reformulations.

A homotopy class of paths has a representative lying in an open algebraic subvariety iff it has one lying in its closure; this holds at least for normal varieties. An extension of this observation based on Fact V.3.3.1 and considerations in §V.3.4 shows that the functor $\underline{\Omega}$ is enough to determine whether the homotopy class of a path has a representative lying in an arbitrary constructible subset.

IV.2 Atomicity of $U_A(\overline{\mathbb{Q}})$: Definitions

Definitions and terminology

We recall the main definitions. Let $U \longrightarrow^p A(\mathbb{C})$ be the universal covering space of a smooth complex algebraic variety A defined over a number field $k \subset \mathbb{C}$ embedded in $\mathbb{C}$, not necessarily connected; we assume A is Shafarevich (cf. Def. III.1.2.6). Let $\pi \times U \to U$ be the action of the group of continuous deck transformations of U. For every normal finite index subgroup $H \triangleleft_{\text{fin}} \pi$ of π, define the relation $x \sim_H y$ iff $x \in Hy$, i.e. there exists $h \in H$ taking x into y, $x = hy, h \in H$. For every π-invariant closed analytic subset $Z = \pi Z$ of U^n, define the relation $x \sim_Z y$ iff x, y lie in the same *irreducible* component of $Z = \pi Z$. For a closed analytic subset $Z = HZ$ invariant under the action of a finite index subgroup H of π, define a relation $\sim_Z^c$ iff points x, y lie in the same *connected* component of $Z = HZ$.

Recall that in Chapter III we defined an étale closed set as a union of components of finitely many relations $\sim_Z$, or equivalently $\sim_Z^c$, and proved that étale closed sets indeed form a topology (under the assumptions on A) which enjoys the property that the projection of an étale closed set is étale closed.

Let $u_1, \ldots, u_n$ be free variables ranging over U, and let $b_1, \ldots, b_m \in U$ be arbitrary elements of U.

We say that *a formula* $\varphi(u_1, \ldots, u_n, b_1, \ldots, b_n)$ *is a formula in language* $\mathrm{L}^{\mathrm{an}}, \mathrm{L}^{\mathrm{top}}$ iff φ is a formula of first-order logic in language $\mathrm{L}^{\mathrm{an}}, \mathrm{L}^{\mathrm{top}}$, i.e. if $\varphi(u, b)$ is a formula of finite length well-built from *free variables* $u_1, \ldots, u_n$, *bounded variables* $v_1, \ldots, v_n$, *parameters* $b_1, \ldots, b_n \in U$, logic connectives conjunction &, disjunction $\vee$, negation $\neg$, quantifiers $\forall$ and $\exists$ applied to variables $v_1, ..., v_n$, equality symbol $=$, and binary relation symbols $\sim_H, \sim_Z \in \mathrm{L}^{\mathrm{an}}$ or $\sim_H, \sim_Z^c \in \mathrm{L}^{\mathrm{top}}$, respectively. For some Z, we have the property that $p(Z)$ is an algebraic closed subset of $A(\mathbb{C})^n$ defined over a field k of definition of A; if we restrict Z above to be such, then we obtain languages denoted $L_A = \mathrm{L}^{\mathrm{an}}(k), \mathrm{L}^{\mathrm{top}}(k)$; we use a similar notation $\mathrm{L}^{\mathrm{an}}(K), \mathrm{L}^{\mathrm{top}}(K)$ for any field $K \subset \mathbb{C}$. One of the main results of Chapter III is that the language L^{an} is interpretable in $L_A = \mathrm{L}^{\mathrm{an}}(k)$, under the assumption that A is Shafarevich, e.g. if A is a semi-Abelian variety.We also use the notation $L_A(A)$ when we wish to stress the underlying variety.

We say that an L^{an}-formula $\varphi(b_1, \ldots, b_n)$ without free variables is *valid, holds in* L^{an}*-structure* $U = U(\mathbb{C})$, in notation $U(\mathbb{C}) \models \varphi(b_1, \ldots, b_n)$ if it expresses a correct statement about the parameters $b_1, \ldots, b_n \in U$ as interpreted in $U(\mathbb{C})$; this can be easily defined rigourously by induction on the length of the formula. We can extend this definition to define validity of an $\mathrm{L}^{\mathrm{an}}(K)$-formula in $U(K') = \{x \in U : p(x) \in A(K')\}$, for $K' \supset K$.

For languages L' and L, we say that an L'-formula $\varphi(u_1, \ldots, u_n)$ *isolates the complete* L*-type of a tuple* $(b_1, \ldots, b_n) \in U^n$ iff

1. $\varphi(b_1, \ldots, b_n)$ holds
2. for any L-formula $\psi(u_1, \ldots, u_n)$, if $\psi(b_1, \ldots, b_n)$ holds, then

$$\forall u_1 \ldots u_n \, (\varphi(u_1, \ldots, u_n) \to \psi(u_1, \ldots, u_n))$$

holds.

Finally, for a language L, an L-structure U is called *atomic* iff for any tuple $(b_1, \ldots, b_n) \in U^n$ there exists an L-formula isolating its complete L-type.

Equivalent conditions for atomicity

We state three equivalent reformulations of atomicity.

Condition IV.2.0.13. *The* L_A*-structure* $U(\overline{\mathbb{Q}})$ *is atomic.*

There are definable imaginaries $U(\overline{\mathbb{Q}})/\!\sim_H$; since $U(\overline{\mathbb{Q}})/\!\sim_H$ is isomorphic to the set of algebraic points of an algebraic variety, each imaginary is algebraic, i.e. contained in a finite definable set. Therefore, the types of those imaginaries $U(\overline{\mathbb{Q}})/\!\sim_H$ are atomic. Types which are atomic over an atomic set of parameters, are atomic themselves. By Corollary III.3.2.2 any L_A-formula is equivalent to an L_A-formula with parameters in the imaginaries $U(\overline{\mathbb{Q}})/\!\sim_H$, for some finite index subgroup $H \lhd_{\mathrm{fin}} \pi(U)$.

Thus, the condition above is equivalent to the following slightly more geometric condition.

Condition IV.2.0.14. *For every tuple* $b \in U(\overline{\mathbb{Q}})^n$*, there exist an (parameter-free)* $\mathrm{L}^{\mathrm{an}}(\overline{\mathbb{Q}})$*-formula isolating the complete* L_A*-type of* $b \in U(\mathbb{C})^n$*.*

Instead of considering formulae which isolate (determine) complete L_A-types, we may require that they determine $\mathrm{Aut}_{L_A}(U(\overline{\mathbb{Q}}))$-orbits. In this situation, the two conditions are equivalent, but in general they are not. For a finite index subgroup $H \lhd_{\mathrm{fin}} \pi$ and a tuple $(b_1, \ldots, b_n) \in U(\overline{\mathbb{Q}})^n$, the set $(Hb_1, ..., Hb_n) = H^n(b_1, \ldots, b_n)$ is a closed analytic set, and therefore L^{an}-formulae are allowed to include the relation $\sim_{H^n(b_1,\ldots,b_n)}$.

This observation allows to express the condition above in the following equivalent way:

Condition IV.2.0.15. *For every tuple $b = (b_1, \ldots, b_n) \in U(\overline{\mathbb{Q}})^n$, there exist a finite index subgroup H and a parameter-free n-ary $\mathrm{L}^{\mathrm{an}}(\overline{\mathbb{Q}})$-formula φ such that all tuples*

$$\{(h_1 b_1, \ldots, h_n b_n) : h_1, \ldots, h_n \in H \ \& \ \models \varphi(h_1 b_1, \ldots, h_n b_n)\}$$

have the same L_A-type and lie in the same $\mathrm{Aut}_{L_A}(U(\overline{\mathbb{Q}}))$-orbit.

Let us denote $H_\varphi(b) = \{h_1, \ldots, h_n \in H : U(\mathbb{C}) \models \varphi(h_1 b_1, \ldots, h_n b_n)\}$ the subgroup of H determined by φ. The condition says that L^{an}-formulae can isolate subgroups of H small enough.

An element of $\mathrm{Aut}_{L_A}(U(\overline{\mathbb{Q}}))$ is completely determined by its action on factor set $U(\overline{\mathbb{Q}})/\!\sim_H$ for all $H \lhd_{\mathrm{fin}} \pi$. Each factor set $U(\overline{\mathbb{Q}})/\!\sim_H$ has a structure of an algebraic variety over k; make an identification. Under these identification, some elements of $\mathrm{Aut}_{L_A}(U(\overline{\mathbb{Q}}))$ correspond to Galois automorphisms in $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, i.e. $\mathrm{Aut}_{L_A}(U(\overline{\mathbb{Q}}))$ and $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ share a subgroup well-defined up to a conjugacy class. On the other hand, the elements of $\pi(U)$ do not correspond to Galois automorphisms. It is known that, for any $H \lhd_{\mathrm{fin}} \pi$, one may choose finitely many parameters in $U(\overline{\mathbb{Q}})/\!\sim_H$ so that any element of $\mathrm{Aut}_{L_A}(U(\overline{\mathbb{Q}}))$ fixing them induces a Galois automorphism of $U(\overline{\mathbb{Q}})/\!\sim_H$; this is implied by the fact that a non-locally modular substructure of an algebraically closed field interprets the field, with finitely many parameters. Thus, Galois orbits on $U(\overline{\mathbb{Q}})/\!\sim_H$ could form an obstruction to the action of $\mathrm{Aut}_{L_A}(U(\overline{\mathbb{Q}}))$.

An example is in order.

Example IV.2.0.16. *The group $\pi(U)$ of deck transformations acts transitively on fibres of the covering map by L_A-automorphisms; therefore the empty formula of 1 variable possesses the property of Condition IV.2.0.15 and the complete L_A-type of a single element of $U(\overline{\mathbb{Q}})$ is trivially atomic.*

Take $A = \mathbb{C}^*$ and consider the L_A-type of a pair $a, b \in \mathbb{C} = U_A$. If $p(x)$ and $p(y)$ satisfy a multiplicative relation, say $p(x)^2 = p(y)$ in $\mathbb{C}^*$, then $x \sim_Z y$ where $Z \subset \mathbb{C}^2$ is the $2\pi i\mathbb{Z}$-invariant closure of the set defined by $2x = y$, and therefore $H_{x \sim_Z y}(x, y) = \{x, y \in H : 2x = y\}$. On the other hand, if $p(x)$ and $p(y)$ do not satisfy any multiplicative dependance, it holds that the $H_\varphi(x, y)$ is always a finite index subgroup of $2\pi i\mathbb{Z} \times 2\pi i\mathbb{Z}$, which could be proved by Kummer-theoretic considerations. We prove these and other facts for $E = A$ an elliptic curve in the proof of Proposition IV.1.0.9.

IV.3 $A = E$ an elliptic curve: the equivalence of analytic language L_A and linear language L^{lin}

Let $A = E$ be an elliptic curve defined over a number field $k \subset \mathbb{C}$ embedded in $\mathbb{C}$. For convenience we also assume that E has a rational point $0 \in E(\mathbb{Q})$. We also assume that all endomorphisms of E are defined over the field k.

In this section we analyse the expressive power of language L_A on the universal covering space to show that the L_A-structure U is bi-interpretable with a 2-sorted structure $p : \mathbb{C} \to E(\mathbb{C})$ in a language L^{lin} describing End$E$-module structure on $\mathbb{C} = U$ and the algebraic variety structure on $E(\mathbb{C})$. We state the main result in Proposition IV.3.3.1. The analysis allows us to check the Condition IV.2.0.15 to establish atomicity of $U_E(\overline{\mathbb{Q}})$.

Generalities on complex elliptic curves

Let $E = A$ be an elliptic curve over a number field k embedded into $\mathbb{C}$; we identify it with a closed subset of $\mathbb{P}^2(\mathbb{C})$ defined by $y^2z = 4x^3 - g_2xz^2 - g_3z^3$. The fundamental group $\pi_1(E(\mathbb{C}), 0)$ of $E(\mathbb{C})$ is isomorphic to $\mathbb{Z}^2$. The universal covering space of $E(\mathbb{C})$ is complex-analytically isomorphic to $\mathbb{C}$: choose an identification of the universal covering space U of E with the field of complex numbers $\mathbb{C}$. Via the covering map $\rho : \mathbb{C} \to E(\mathbb{C})$, the addition $+ : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ of complex numbers makes $E(\mathbb{C})$ into an Abelian group with $0 = \rho(0)$ as its zero point, $0 + 0 = 0$. The kernel of the covering map as an Abelian group homomorphism is canonically identified with the fundamental group $\pi_1(E(\mathbb{C}), 0)$ which is a lattice $\Lambda \cong \pi_1(E(\mathbb{C})) \subset \mathbb{C}$, i.e. a discrete subgroup of $\mathbb{C}$ whose $\mathbb{R}$-linear span is $\mathbb{C}$, and so $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$ as complex analytic manifolds. Complex analytic endomorphisms of $E(\mathbb{C})$ preserving 0, lift up to complex analytic endomorphisms of $\mathbb{C}$, which are well-understood; this allows us to show a complex-analytic endomorphism of $E(\mathbb{C})$ is always induced by a multiplication by an element $z \in \mathbb{C}$ such that $z\Lambda \subset \Lambda$. The numbers z form the *dual lattice* $\Lambda^\vee = \{z \in \mathbb{C} : z\Lambda \subset \Lambda\}$. Because $E(\mathbb{C})$ is projective, complex analytic endomorphisms are also endomorphism of $E(\mathbb{C})$ as an algebraic variety over $\mathbb{C}$. In fact, those algebraic morphisms are defined over a number field extending the field of definition of E, and so they are endomorphisms of E an algebraic variety E over the extension. The ring of endomorphisms of E is denoted by $\operatorname{End} E$, and, for curves over $\mathbb{C}$, there exists a canonical embedding $\operatorname{End} E = \Lambda^\vee \subset \mathbb{C}$.

For generic E and therefore generic lattice Λ, the condition $z\Lambda \subset \Lambda$ implies z is an integer, and so $\operatorname{End} E = \mathbb{Z}$; however, if $\operatorname{End} E \neq \mathbb{Z}$, then $L = \operatorname{End} E \otimes \mathbb{Q}$ is an imaginary quadratic extension of $\mathbb{Q}$, i.e. a field $\mathbb{Q}(z)$ where $z \notin \mathbb{R}$ satisfies an quadratic equation $z^2 + q_1z + q_2 = 0$ with coefficients $q_1, q_2 \in \mathbb{Q}$.

IV.3.1 Analysis of language $L_A(E)$

An analysis of the geometry of $E(\mathbb{C})$ yields that the language L_A essentially describes the structure of the universal covering space as an $\mathrm{End}\, E$-module, and the pull-back of algebraic structure on $E(\mathbb{C})$; the analysis also implies quantifier elimination for L_A, which allows us the reformulation of Proposition IV.1.0.8 as very explicit statements (Propositions IV.1.0.9 and IV.1.0.11) readily reducing to the Kummer theory of the underlying elliptic curve, which are known (Bashmakov's theorem, results of Ribet) and Serre's results about the image of Galois action on Tate module.

The geometric facts we use in the analysis of language L_A are Poincaré's complete reducibility theorem and a basic Minkowski-sum argument using the holomorphic convexity of $\mathbb{C}$. The latter argument could have been replaced by some properties of Albanese morphisms.

Model-theoretically the important implication is that the structure on U consists of a combination of a locally modular structure of End$E$-module on U and the pull-back of the structure on $E(\mathbb{C})$ as an algebraic variety over k.

$\mathrm{End}\, E$*-module structure via* $\sim^c_B$

We follow Example II.1.4.3 to interpret $\mathrm{End}\, E$-module structure in $L_A(E)$.

Consider the morphism $+ : E(\mathbb{C}) \times E(\mathbb{C}) \to E(\mathbb{C})$ of addition and an étale morphism $f \in \mathrm{End}\, E$, and the closed diagonal subvariety $\Delta \subset E(\mathbb{C}) \times E(\mathbb{C})$. By our assumptions, they all are defined over k, and so are the graphs $\{f(z) = y\} \subset E(\mathbb{C})^2$, $\{x + y = z\} \subset E(\mathbb{C})^3$ of the morphisms $f \in \mathrm{End}\, E$ and $+ : E \times E \to E$.

$$\begin{aligned} (x_1, y_1) &\sim_\Delta (x_2, y_2) \iff x_1 - y_1 = x_2 - y_2 \\ (x_1, y_1) &\sim_{\{f(x)=y\}} (x_2, y_2) \iff f(x_1) - y_1 = f(x_2) - y_2 \\ (x_1, y_1, z_1) &\sim_{\{x+y=z\}} (x_2, y_2, z_2) \iff x_1 + y_1 - z_1 = x_2 + y_2 - z_2 \end{aligned} \tag{IV.3.1}$$

Then, for $x, y, z \in U = \mathbb{C}$, we have that

$$(x, y) \sim_{\{f(z)=y\}} (0, 0) \iff f(x) = y$$

$$(x, y, z) \sim_{\{x+y=z\}} (0, 0, 0) \iff x + y = z$$

These formulae show that $\mathrm{End}\, E$-module structure on $\mathbb{C}$ is $L_A(E)$-definable over a parameter $0 \in E(\mathbb{C})$.

The goal of the rest of this section is to prove the converse, namely that $L_A(E)$, and L^{an} in general, are interpretable in $\mathrm{End}\, E$-module $\mathbb{C}$ together with the pull-back of algebraic variety structure on $E(\mathbb{C})$.

The key geometric lemma

Let $0 \in Z \subset E(\mathbb{C})^n$ be a subvariety containing 0, and let $B = Z + Z + \ldots + Z$ be the least subgroup of $E(\mathbb{C})^n$ containing Z. Such a subgroup exists by dimensional considerations, such as Zilber's irreducibility principle.

Recall that $x \sim^c_Z y$ iff points x and y of $U^n = \mathbb{C}^n$ lie in the same connected component of the preimage of $p^{-1}(Z(\mathbb{C}))$ in the universal covering space $\mathbb{C}^n$.

Lemma IV.3.1.1. *Let B be the least Abelian subvariety containing Z. Then for a natural number N large enough, the following equivalence holds for any points $x, y \in \mathbb{C}^n$:*

$$x \sim^c_Z y \iff x \sim^c_B y \ \&\ p\left(\frac{x}{N}\right), p\left(\frac{y}{N}\right) \text{ lie in the same connected component of } \frac{1}{N}Z(\mathbb{C})$$

Proof. Let $Z' \subset B'$ be connected components of $p^{-1}(Z) \subset p^{-1}(B)$. Recall that $\pi(U)$ denotes the deck group of the covering space U, and that for a subset $Z' \subset U$, we denote $\pi(Z') = \{\gamma \in \pi : \gamma Z' = Z'\}$. We identify $\pi(U)$ with the lattice $\{\gamma \cdot 0 : \gamma \in \pi\}$, and we prove the lemma by a Minkowski-sum argument using that lattice.

We prove the lemma in three steps: first we prove that there is a finite sum $Z' + \ldots + Z' = B'$, that $\pi(Z')$ has at most finite index in $\pi(Z' + \ldots + Z')$, and that the latter condition implies the result.

The space U is isomorphic to $\mathbb{C}$ and admits a norm which we denote $|\cdot|$. There is a constant $C \in \mathbb{R}$ such that $A \subset p(\{z : |z| < C\})$. Consider sums $z'_1 + \ldots + z'_n, z'_1, \ldots, z'_n \in Z'$, and take a point $b' \in B', |b'| \leqslant C$. Represent b' as a sum $b' = z'_1 + \ldots + z'_n$. By changing $z'_1, \ldots, z'_{n-1}$ by elements of $\pi(Z')$ if necessary, we may assume $|z'_1| < C, \ldots, |z'_{n-1}| < C$; then z'_n is changed also by a sum of elements of the group $\pi(Z')$, i.e. an element of $\pi(Z')$, and so continues to lie in Z'. However, by the triangle inequality $|z'_n| < |z'_1| + \ldots + |z'_{n-1}| + |b'| \leqslant nC$. The implication of this argument is that for a sufficiently large constant $C \in R$ any element $b' \in p^{-1}(B)$ of bounded norm $|b'| < C$ is a sum of $2n$ elements of Z' of a bounded norm $|z'_i| \leqslant nC$. A finite sum of bounded analytic sets is analytic; this implies that the sum $Z' + \ldots + Z'$ is a closed analytic set.

Now consider the set of sums $\lambda \cdot 0 + z'_1 + \ldots + z'_n, |z'_1|, \ldots, |z'_n| \leqslant C, z'_1, \ldots, z'_n \in Z'$ and $\lambda \in \pi(U)$. For $|\lambda|$ large, say $|\lambda \cdot 0| \geqslant 2nC$, all these sets are uniformly bounded away from 0. For each of finitely many λ's satisfying $|\lambda| \leqslant 2nC$, since all variables vary over compact sets, this set is compact, and therefore it either contains $0 \in \mathbb{C}$ or is bounded away from $0 \in \mathbb{C}$. If $0 = \lambda \cdot 0 + z'_1 + \ldots + z'_n$, then we actually have $\lambda \cdot 0 \in Z' + \ldots + Z'$ is representable as an n-sum. Thus we get that for λ

not an n-sum, the sums $\lambda \cdot 0 + z_1' + \ldots + z_n', |z_1'|, \ldots, |z_n'| \leqslant C, z_1', \ldots, z_n' \in Z'$ are separated away from 0 uniformly in λ.

The argument above implies that each $b' \in p^{-1}(B)$ of sufficiently small norm $|b'| < \varepsilon$ is representable as a $3n$-sum of elements of Z'.

Thus, analytic sets B' and the $3n$-sum of Z' coincide in a small ball around $0 \in \mathbb{C}$. Fact III.1.2.1 implies the smooth analytic set B' is contained in the 3n-sum of Z' and therefore that they coincide.

Let us now show that $\pi(Z' + \ldots + Z')$ lies in a bounded neighbourhood of $\pi(Z')$, and contains $\pi(Z')$ as a finite index subgroup. Take a sum $z_1' + \ldots + z_n' = \gamma + z_1'' + \ldots + z_n''$; shift z_i', z_i'''s to be of norm bounded by c, i.e. pick $\lambda_i', \lambda_i'' \in \pi(Z')$ so that $|z_i' - \lambda_i'| < C, |z_i'' + \lambda_i''| < C$ for $1 \leqslant i \leqslant n$. Then

$$\gamma = z_1'' + \ldots + z_n'' - z_1' - \ldots - z_n' =$$
$$(z_1'' - \lambda_1'') + \ldots + (z_n'' - \lambda_n'') - (z_1' - \lambda_1') - \ldots$$
$$-(z_n' - \lambda_n') + \lambda_1'' + \ldots + \lambda_n'' - \lambda_1' - \ldots - \lambda_n'$$

and

$$|\gamma - (\lambda_1'' + \ldots + \lambda_n'' - \lambda_1' - \ldots - \lambda_n')| \leqslant$$
$$|(z_1'' - \lambda_1'') + \ldots + (z_n'' - \lambda_n'') - (z_1' - \lambda_1') - \ldots - (z_n' - \lambda_n')| \leqslant 2nC.$$

That is, $\pi(Z' + \ldots + Z') \subset \pi(Z') + \{z : |z| \leqslant 2nC\}$. This implies $\pi(Z' + \ldots + Z') \subset \pi(Z') + \{\lambda : |\lambda| \leqslant 2nC\}$ which completes the second step.

Choose N so that $\{\lambda : |\lambda| \leqslant 2nC\} \cap N\pi(A) = \{0\}$, and denote $H = n\pi(A)$. The subgroup H is a finite index subgroup. Now take $x, y \in B'$ satisfying the right-hand side of the equivalence. Since Z' surjects on a connected component of $\frac{1}{N}Z(\mathbb{C})$, there exist points $x', y' \in Z' \subset B'$ such that $x' \sim_H x, y' \sim Hy$. This and $x', y', x, y \in B'$ imply that there is $h_x, h_y \in \pi(B') \cap H$ such that $h_x x = x', h_y y = y'$. However, $\pi(B') \cap H = \pi(Z') \cap H$, and therefore $x, y \in Z'$ as required. □

Remark IV.3.1.2 (Maximal subvarieties with a given fundamental subgroup). The argument above depends only on the fundamental group of Z and B, or rather their images in $\pi_1(A(\mathbb{C}), 0)$. In terms of the fundamental groups, Lemma IV.5.3.2 says that for each irreducible subvariety Z there exists an Abelian subvariety $B \supset Z$ containing Z and such that the images of fundamental groups of Z and B differ by finite index at most. The Abelian subvarieties form a essentially linear structure; we use that later to analyse language L_A.

In fact, an analogue of the class of Abelian subvarieties with the property mentioned above has been considered by Kollar [Kol95]; the fibres of a *relative Shafarevich* morphism provide such a class; see the next remark for some explanation.

Remark IV.3.1.3 (Albanese and Shafarevich morphisms). A more natural way to prove this statement is via the existence of Albanese morphisms or relative Shafarevich morphisms of Kollar [Kol95]. Kollar [Kol95] considers morphisms $f : A \to C$ such that for an irreducible closed subvariety $i : Z \hookrightarrow A$ it holds $f(Z)$ is a point iff H contains a finite index subgroup of $i_*\pi_1(\hat{Z})$ where $\hat{Z}$ is the normalisation of Z; such morphisms are called *(relative) H-Shafarevich morphisms* $f : A \to C$; we may take B to be a fibre of such a morphism. The fibres of H-Shafarevich morphisms, $H < \pi_1(A, 0)$ vary, give a uniform and explicit description of relations $\sim_B$ needed to generalise Lemma IV.3.1.1. By Lemma IV.3.1.1 such "equivalence" relations $\sim_{f_H^{-1}(x)}$ are enough to interpret L_A on $U_A(\mathbb{C})$, together with the pull-back of algebraic structure on étale covers of A. One further hopes that the geometry defined by these relations is locally modular, a technical model theory term for being *essentially linear*.

Lack of explicit references in the literature and technical difficulties have prevented us from trying to prove Lemma IV.3.1.1 via the more general approach described in Remark IV.3.1.3 above.

We describe the structure induced by the relations $\sim_B$ next; it is essentially that of an End$E$-module.

Relations $\sim^c_B$, for B an Abelian subvariety, are definable in an End E*-module structure*

Recall a closed algebraic subgroup B of Abelian variety E^n is an Abelian subvariety.

An $m \times n$-matrix $M \in M(m \times n, \operatorname{End} E)$ gives rise to an endomorphism from E^m to E^n; we identify the two.

Lemma IV.3.1.4. *A closed algebraic subgroup B of E^n is a connected component of the kernel of an* End E*-morphism $M = M_B : E^n \to E^m$ represented by an $m \times n$-matrix M over* End E*,*

$$B = \ker M, M \in M(m \times n, \operatorname{End} E).$$

An Abelian subvariety B of E^n, i.e. a translation of such a subgroup, is a connected component of a set

$$M^{-1}(M(b)) = \ker M + b,$$

for a $m \times n$-matrix $M \in M(m \times n, \operatorname{End} E)$ over End E *and an element $b \in B$.*

Proof. This is a reformulation of Lemma IV.5.3.2; we prove it there. □

The endomorphisms End$E$ act on $\mathbb{C}$ by multiplication by complex numbers; thus the kernel of an endomorphism *on* $\mathbb{C}$ is a hyperplane defined by a linear equation with coefficients in $\mathbb{Z}$ if End$E = \mathbb{Z}$, or in a quadratic imaginary extension $\mathbb{Q}(\mathrm{End}E) \subset \mathbb{C}$. In particular, the kernel of an endomorphism in $M(\mathbb{Q}\mathrm{End}E)$ on $\mathbb{C}^n$ is connected.

An important corollary for us is that

Corollary IV.3.1.5. *For an Abelian subvariety $B(\mathbb{C}) \subset E^n(\mathbb{C})$, there exists a morphism $M \in M(n \times n, \mathrm{End}E) : U^n \to U^n$ such that*

$$p^{-1}(B(\mathbb{C})) = M^{-1}(M(b)),$$

for any point $b \in B(\mathbb{C})$.

The main lemma of the reduction

In notation of Lemma IV.3.1.4 above,

Lemma IV.3.1.6. *For an Abelian subvariety B of E^n and an $m \times n$-matrix $M \in M(m \times n, \mathrm{End}\, E)$ as above,*

$$x \sim^c_B y \iff M_B(x) = M_B(y)$$

Proof. A reformulation of Corollary IV.3.1.5 above. □

For subvarieties Z and B of E^n as in Lemma IV.3.1.1, we then have
(IV.3.2)

$$x \sim^c_Z y \iff M_B(x) = M_B(y) \;\&$$
$$p\left(\tfrac{x}{n}\right), p\left(\tfrac{y}{n}\right) \text{ lie in the same connected component of } \tfrac{1}{n}Z(\mathbb{C}).$$

L^{an} *via* End E*-module structure*

Formula IV.3.2 shows that the language L^{an} is interpretable in a language describing the End$E$-module structure on U together with the pull-back of the algebraic structure on $E(\mathbb{C})$. Let us define such a language.

The ring End E is an integral domain, and the End E-module U is uniquely divisible, and thus we introduce function symbols for elements of ring of fractions $\mathbb{Q}(\mathrm{End}\, E) = \mathrm{End}\, E \otimes \mathbb{Q}$ of End E.

Definition IV.3.1.7. *Let $L^{\mathrm{lin}} = L^{\mathrm{lin}}_E(k)$ be the language consisting of*

1. *(*End$E$*-module structure) binary symbol $\{+\}$ and function symbols $\{f : f \in \mathbb{Q}(\mathrm{End}\, E)\}$,*

2. *(pull-back of algebraic structure on E) a relation $\sim^o_Z$ for each closed subvariety Z of E^n defined over k*

interpreted as follows:

1. $\{+\}\cup\{f : f \in \mathbb{Q}(\mathrm{End}\, E)\}$ *are interpreted according to the* End E*-module structure on* $\mathbb{C}$*, i.e. to the embedding* $\mathrm{End}E \hookrightarrow \mathbb{C}$.
2. *the relations $\sim^o$ are interpreted as follows:*
   $x \sim^o_Z y \iff$ *points $p(x), p(y) \in E(\mathbb{C})$ lie in the same irreducible (over $\mathbb{C}$) component of $Z(\mathbb{C}) \subset E(\mathbb{C})$.*

Similarly we define $L^{\mathrm{lin}}(K)$, for arbitrary subfield $K \subset \mathbb{C}$.

Note that the relations $\sim^o_Z$ describe only the pull-back to $U = \mathbb{C}$ of the algebraic variety structure on $E(\mathbb{C})$. It might be more geometric to consider a slight modification of L^{lin}; namely, consider a two-sorted universe consisting of a sort U and a sort A for the variety A. The modified language contains an additional function symbol $p : U \to A$ and function symbols $\{f, +\}$ on U as above; however, we restrict relations $\sim^o_Z$ to the sort A only. We use this modified language to state and prove the quantifier elimination theorem.

Formula (IV.3.2) implies the goal of our analysis of L^{an} in this §:

Lemma IV.3.1.8. *The relation $\sim^c_Z$ is interpretable in L^{lin}_E. By Corollary III.3.1.2 of Decomposition Lemma III.1.3.1 of Chapter III, a relation $\sim_Z$ is expressible as a disjunction of relations $\sim^c_Y$, and thus any relation $\sim_Z$ is interpretable in $L^{\mathrm{lin}}(E)$.*

In particular, the language $\mathrm{L}^{\mathrm{an}}(E)$ is interpretable in L^{lin}_E (with parameters).

Language L^{lin}_E is interpretable in language $L_A(E)$ over the parameter $0 \in \mathbb{C}$.

Proof. The first part follows from formula (IV.3.2). The latter part follows from formulae (IV.3.1). □

Note that for subvariety Z smooth (or normal), the relations $\sim_Z$ and $\sim^c_Z$ are equivalent, and we do not need to refer to Decomposition Lemma of Chapter III.

IV.3.2 Elimination of quantifiers for $L^{\mathrm{lin}}(E)$

Lemma IV.3.2.1 (Quantifier elimination). *Every formula of the language L^{lin}_E is equivalent to a quantifier-free formula of the language L^{lin}_E.*

Proof. This follows from the next lemma. □

The following is an explicit technical statement we prove by induction.

Let $v_0, \ldots, v_n$ be variables varying in $U = \mathbb{C}$, and let $\alpha(v_0, \ldots, v_n), \beta(v_0, \ldots, v_n)$ denote linear combinations over $\mathbb{Q}(\mathrm{End}E)$.

Lemma IV.3.2.2 (Quantifier elimination). *For any formula*

$$\varphi(v_0, \dots, v_n) \in L_E^{\mathrm{lin}}$$

with free variables $v_0, \dots, v_n$ there exists integers m, N, quantifier-free formulae φ_i depending only on the variables shown explicitly, and $\mathrm{End}E$*-linear combinations α_i, β_j such that*

$$\varphi(v_0, \dots, v_n)$$

is equivalent to a quantifier-free formula of the form

$$\bigvee_{0 \leqslant k \leqslant m} \left(\bigwedge_i \alpha_i(v_0, \dots, v_n) = 0 \ \& \ \bigwedge_j \beta_j(v_0, \dots, v_n) \neq 0 \ \& \ \varphi_k(p(v_1/N), \dots, p(v_n/N) \right)$$

where $\varphi_k((p(v_1/N), \dots, p(v_n/N))$ is a formula depending only $p(v_1/N), \dots, p(v_n/N)$ and not the variables $v_1, \dots, v_n$ themselves.

Proof. We prove this by induction on the number of quantifiers in φ.

Base of induction: quantifier-free formulae

A disjunctive normal form of a quantifier-free formula can be easily made into such a form.

Moreover, a negation of a formula of such a form has also such a form; thus the case $\forall v_n \in V \varphi$ is dual to the case $\exists v_n \in V \varphi$, and it is enough to consider only the latter one.

Eliminating an existential quantifier

Assume φ is of the form above, and consider $\exists v_n \varphi(v_0, \dots, v_n, w)$. Since an existential quantifier $\exists$ commutes with disjunction $\vee$, we may further assume that $\varphi(v_0, \dots, v_n)$ is
(IV.3.3)

$$\exists v_n \left(\bigwedge_i \alpha_i(v_0, \dots, v_n) = 0 \ \& \ \bigwedge_j \beta_j(v_0, \dots, v_n) \neq 0 \ \& \ \varphi_k(p(v_1/N), \dots, p(v_n/N)) \right).$$

Assume that there is a linear combination α_i involving v_n non-trivially, i.e. for some i $\alpha_i(v_0, \dots, v_n) = cv_n + \alpha'(v_0, \dots, v_{n-1}), c \neq 0$. Since $\mathrm{End}E$ is an integral domain, the equality $\alpha_i(v_0, \dots, v_n) = cv_n + \alpha'(v_0, \dots, v_{n-1}) = 0$ implies $v_n = \frac{1}{c}\alpha_i'(v_0, \dots, v_{n-1})$. Therefore, if we replace all occurrences of v_n in formula (IV.3.3) by $\frac{1}{c}\alpha_i'(v_0, \dots, v_{n-1})$ and delete the quantifier, then we get an equivalent formula.

Now assume there is no linear combination $\alpha_i(v_0, \dots, v_n)$ depending non-trivially on v_n. Rewrite β_j's to make the dependance of β_j's on v_n explicit: $\beta_j(v_0, \dots, v_n) = b_j v_n + \beta'_j(v_0, \dots, v_{n-1})$. We claim (IV.3.3) is equivalent to

(IV.3.4)
$$\bigwedge_i \alpha_i(v_0, \dots, v_n) = 0 \;\&\; \bigwedge_{b_j=0} \beta'_j(v_0, \dots, v_n) \neq 0 \;\&\; \exists v_n \varphi_k(p(v_1/N), \dots, p(v_n/N)).$$

Evidently (IV.3.3) implies (IV.3.4). To prove the converse, take $v_0, \dots, v_n \in \mathbb{C}$ which satisfies (IV.3.4). If $v_0, \dots, v_n$ fails (IV.3.3), then $v_0, \dots, v_n$ fails some inequality $\beta_j(v_0, \dots, v_n) = b_j v_n + \beta'(v_0, \dots, v_{n-1}) \neq 0$, i.e. $v_n = \frac{1}{b_j}\beta'(v_0, \dots, v_{n-1})$. However, if we replace v_n by v'_n such that $p(v'_n/N) = p(v_n/N)$, then the validity of $\varphi_k((v_1/N), \dots, p(v_n/N))$ is obviously preserved; the validity of $\alpha_i(v_0, \dots, v_n) = 0$ is preserved since v_n does not occur in it. On the other hand, since there infinitely many such v'_n's, one can easily force the inequalities $v_n \neq \frac{1}{b_j}\beta'(v_0, \dots, v_{n-1})$, which concludes the proof of the equivalence of the formulae (IV.3.3) and (IV.3.4). □

IV.3.3 Conclusion: atomicity of U_E as an L^{lin}-structure

The analysis above shows the following.

Proposition IV.3.3.1. *The L_A-structure U_E and L^{lin}-structure $p : \mathbb{C} \to E(\mathbb{C})$ are bi-interpretable. The language L^{lin} has quantifier elimination. The substructure $U(\overline{\mathbb{Q}})$ is an elementary submodel of the universal covering space $U(\mathbb{C})$ as L_A-structures.*

Proof. The first two claims follow from Lemma IV.3.1.8 and Lemma IV.3.2.1. To prove the third claim, note that an analysis of the proof of the quantifier elimination shows that the output of the quantifier elimination procedure does not depend on the ambient model, i.e. does not depend on the field K in $U(K)$, K algebraically closed. □

We record a consequence.

Corollary IV.3.3.2. *A complete L_A-type or L^{lin}-type is isolated by the complete quantifier-free L^{lin}-type. For tuples $(v_0, \dots, v_n), (u_1, \dots, u_n)$ of elements of U_E, if infinite sequences $((p(v_1/j))_j, \dots, (p(v_n/j))_j)$ and $((p(u_1/j))_j, \dots (p(u_n/j))_j)$ are* Gal*-conjugated, then complete L_A-types of $u_1, \dots, u_n$ and $v_0, \dots, v_n$ are the same.*

Proof. Implied by Quantifier Elimination Lemma IV.3.2.1 and the definitions of basic predicates of the language L_A. □

Proposition IV.3.3.3. *The structure $U_E(\overline{\mathbb{Q}})$ is atomic, and is an elementary substructure of the universal covering space $U_E(\mathbb{C})$ as an L_A-structure.*

Proof. We apply Condition IV.2.0.15; we also refer to the arithmetic facts stated in the proof of Proposition IV.1.0.9; we refer there for a detailed exposition of the facts we refer to.

If E has complex multiplication, then by formula (IV.5.2) there are finitely many orbits of Galois action on the set of sequences $((p(\lambda/j))_j$, $p(\lambda) = 0$ and $\lambda \in \Lambda$ generates Λ as an End$E$-module. This implies for some finite index subgroup Λ' of Λ, all points $\lambda + \Lambda'$ have the same type, thereby Condition IV.2.0.15 implies the atomicity of the type of λ.

If E does not have complex multiplication, pick $\lambda_1, \lambda_2 \in \Lambda$ to be $\mathbb{Z}$-generators of Λ. We show in §IV.5.2 there are finitely many orbits on such generators, and therefore the type of the pair is atomic.

The type of a tuple in U is isolated by the type of its maximal linearly independent subtuple, thus it is sufficient to prove the atomicity of the types of linearly independent tuples over the kernel. An analysis similar to the above immediately reduces the atomicity of such types to Kummer theory Lemma IV.5.3.1. □

IV.4 Fundamental groupoid functor

In this section we derive Proposition IV.1.0.11 from Proposition IV.1.0.9 and introduce the relevant notions of category theory in detail.

IV.4.1 Definitions and Results

The example: fundamental groupoid functor $\pi_1^{\mathrm{top}}(E(\mathbb{C}))$ as a two-functor

Before defining a 2-functor formally, we give an example of Grothendieck [Gro] which have motivated the notion. A 2-functor is an algebraic structure made to capture the basic properties of paths in topological spaces.

The path 2-functor $\underline{\Omega}$ on the category $\mathfrak{Top}$ of topological spaces is a tuple $(\mathrm{Pt}, \Omega, s, t, \cdot)$ consisting of a *functor of points* $\mathrm{Pt} : \mathfrak{Top} \to \mathfrak{Sets}$ and a *paths functor* $\Omega : \mathfrak{Top} \to \mathfrak{Sets}$ subject to the following conditions:

1. $\mathrm{Pt}(T)$ is the set of points of topological space T and the morphism $\mathrm{Pt}(f) : \mathrm{Pt}(T_1) \to \mathrm{Pt}(T_2)$ is $f : T_1 \to T_2$ as a map of sets.
2. $\Omega(T)$ is the set of all paths in topological space T, i.e. continuous functions $\gamma : [a, b] \to T$, $a, b \in \mathbb{R}$; similarly $\Omega(f), f \in \mathrm{Hom}_{\mathfrak{Top}}(T_1, T_2)$ is the map taking a path $\gamma : [a, b] \to T_1$ into $f \circ \gamma : [a, b] \to T_2$.

3. $s_T, t_T : \Omega(T) \to \mathrm{Pt}(T)$ are functions from the set of paths in T to their endpoints in T; the function $s(\gamma) = \gamma(a)$ (*source*) gives the beginning point of a path γ, and the function $t(\gamma) = \gamma(b)$ (*target*) gives the ending point of path γ.
4. $\cdot_T : \Omega(T) \times \Omega(T) \to \Omega(T)$ is the partial operation of the concatenations of paths, taking $\gamma_1 : [a, b] \to T$, $\gamma_2 : [b, c] \to T$ into $\gamma = \gamma_1\gamma_2$, $\gamma_{|[a,b]} = \gamma_1, \gamma_{|[b,c]} = \gamma_2$.

Thus, a 2-functor from $\mathfrak{Top}$ to $\mathfrak{Sets}$ consists of two functors $\mathrm{Pt}, \Omega : \mathfrak{Top} \to \mathfrak{Sets}$, and two natural transformations $s, t : \Omega \to \mathrm{Pt}$ from functor Ω to Pt; s stands for source and t stands for target. For each T, there is also a functorial associative operation $\cdot_T$ defined on $\Omega_{x,y}(T) \times \Omega_{y,z}(T) \to \Omega_{x,z}(T)$, where $\Omega_{x,y}(T) = \{\gamma \in \Omega(T) : s(\gamma) = x, t(\gamma) = y\}$, etc; the operation $\cdot$ makes $\Omega_{x,x}(T)$ into a group; in the example above, $\Omega_{x,x}(T)$ is the set of all loops in T based at $x \in T$.

In particular, for each T the set $\Omega(T)$ carries the structure of a groupoid; in fact, it is conventional to consider $\underline{\Omega}$ as a functor to the category of groupoids.

If in item (2) we define $\Omega(T)$ to be the set of all paths *up to fixed point homotopy*, then we obtain the notion of the fundamental groupoid functor. The advantage of the original definition is that one may try and define n-functors describing n-dimensional homotopies on topological space T, cf. Grothendieck [Gro] for motivations; Voevodsky-Kapranov [VK91] propose an exact definition. Grothendieck [Gro] explains that it is essential not to insist on strict associativity etc, but rather to consider *all the identities to hold up to a homotopy of higher dimension*.

Abstract fundamental groupoid functors

Here we define the path-lifting property for a 2-functor, and an abstract fundamental groupoid functor as a functor preserving direct products, possessing the path-lifting property and with a particular functor of points.

The motivating example is the restriction of the 2-functor described in the previous section, to the subcategory of $\mathfrak{Top}$ consisting of complex algebraic varieties. The properties below try to capture the essential algebraic properties of this 2-functor.

Definition IV.4.1.1. *Let $\underline{\Omega}$ be a 2-functor from a subcategory $\mathfrak{C}$ of the category of varieties over an algebraically closed field K, into $\mathfrak{Sets}$. We say $\underline{\Omega}$ is an* abstract fundamental groupoid functor *if $\underline{\Omega}$ satisfies the following properties:*

1. *the functor* Pt *is the functor of K-rational points:*

$$\mathrm{Pt}(X) = X(K) = \mathcal{M}or_{\mathrm{Var}/K}(0, X)$$

(here 0 denotes a single point variety defined over $\mathbb{Q}$)

2. *the functor Ω preserves the direct product:*

$$\Omega(X \times Y) = \Omega(X) \times \Omega(Y)$$

$$\Omega(f \times g) = \Omega(f) \times \Omega(g)$$

$$s_{X \times Y} = s_X \times s_Y, t_{X \times Y} = t_X \times t_Y$$

$$(\gamma_1 \times \gamma_2) \cdot_X (\gamma_1' \times \gamma_2') = (\gamma_1 \cdot_X \gamma_1') \times (\gamma_2 \cdot_X \gamma_2')$$

(the direct product is taken in the category of $\mathfrak{Sets}$.)

3. *unique path-lifting property: if $p \in \mathrm{Hom}_{\mathfrak{E}}(X, Y)$ is an étale morphism of algebraic varieties, then for any points $x, y \in \mathrm{Pt}(X)$ the map*

$$\Omega(p) : \bigcup_{y \in \mathrm{Pt}(X)} \Omega_{x,y}(X) \to \bigcup_{z \in \mathrm{Pt}(Y)} \Omega_{p(x),z}(Y)$$

is a bijection.

Universality of the fundamental groupoid functor

Let

$$\mathfrak{E} \subset \mathbb{V},\ \mathcal{O}b\,\mathfrak{E} = \{E^n : n \geqslant 0\},\ \mathcal{M}or_{\mathfrak{E}}(X, Y) = \mathcal{M}or_{\mathrm{Var}/K}(X, Y)$$

be the full subcategory of the category of varieties whose objects are the Cartesian powers of E including $E^0 = 0$ a variety consisting of the single point 0. By the definition of a full subcategory, the morphisms of $\mathfrak{E}$ are morphisms of varieties between the objects of $\mathfrak{E}$.

The Galois group $Gal(\overline{\mathbb{Q}}/k)$ acts on the category $\mathfrak{E}$; the action leaves the objects invariant but permutes the morphisms. Recall we assume that all endomorphisms of E preserving $0 \in E(k)$ are defined over its field k of definition.

Recall a groupoid $\Omega(E)$ is *connected* iff for every $x, y \in \mathrm{Pt}(E)$ there exists $\gamma \in \Omega(E)$ "going from point x to point y", i.e. $x = s(\gamma), y = t(\gamma)$. The set of such paths satisfying $x = s(\gamma), y = t(\gamma)$ is denoted by $\Omega_{x,y}(E)$.

Using the notion of an abstract fundamental groupoid functor, we restate Proposition IV.1.0.11; then we show it is essentially equivalent to Proposition IV.1.0.9. The proof basically reconstructs the "universal covering space" V as the set of all paths $\bigcup_{* \in E(\overline{\mathbb{Q}})} \Omega_{0,*}(E)$ leaving a particular point; functoriality of $\underline{\Omega}$ allows us to define $\mathrm{End}E$-module structure on V; the unique path-lifting property of Ω ensures unique divisibility; in the beginning of §IV.4.2 we give formulae for an explicit construction of $\mathbb{C}$ as the spaces of paths on $\mathfrak{E}(\mathbb{C})$ with a particular starting point.

Proposition IV.4.1.2 (Universality of fundamental group functor). *Assume that E is an elliptic curve defined over a number field k and does not have complex multiplication. Let $\mathfrak{E}$ be the full subcategory of Cartesian powers of E as above.*

Let $\underline{\Omega}$ be an abstract fundamental groupoid 2-functor on category $\mathfrak{E}$. Assume

1. $\underline{\Omega}(E)$ *is a connected groupoid*
2. *there is an isomorphism*

$$\Omega_{0,0}(E) \cong \mathbb{Z}^2 \text{ as } \mathrm{End}E\text{-modules.}$$

Then there exists an automorphism $\sigma \in Gal(\overline{\mathbb{Q}}/k)$ such that the 2-functors $\underline{\Omega}, {\pi_1^{\mathrm{top}}}_{|\overline{\mathbb{Q}}} \circ \sigma : \mathfrak{E} \to \mathfrak{S}ets$ are naturally equivalent (as 2-functors), $\underline{\Omega} \cong {\pi_1^{\mathrm{top}}}_{|\overline{\mathbb{Q}}} \circ \sigma$:

$$\begin{array}{ccc} \mathfrak{E} & \xrightarrow[\underline{\Omega}]{} & \mathfrak{S}ets \\ \sigma\Big\downarrow & & \Big\Downarrow \text{natural transformation} \\ \mathfrak{E} & \xrightarrow{{\pi_1^{\mathrm{top}}}_{|\overline{\mathbb{Q}}}} & \mathfrak{S}ets \end{array}$$

Here $\mathrm{Pt}_{{\pi_1^{\mathrm{top}}}_{|\overline{\mathbb{Q}}}}(V) = V(\overline{\mathbb{Q}})$ and $\Omega_{{\pi_1^{\mathrm{top}}}_{|\overline{\mathbb{Q}}}}(V) = \{\gamma \in \pi_1^{\mathrm{top}}(V(\mathbb{C})) : s(\gamma), t(\gamma) \in V(\overline{\mathbb{Q}})\}$.

IV.4.2 Uniqueness of extensions implies the universality of the fundamental groupoid functor

In this § we show that Proposition IV.1.0.9 implies Proposition IV.1.0.11. We do so by explicitly constructing an extension as in Proposition IV.1.0.9 from a 2-functor as in Proposition IV.1.0.11, and then showing that the equivalence of constructed extensions implies the equivalence of 2-functors.

The construction is a formalisation of the geometric observation that the universal covering space with a basepoint can be canonically identified with the set of homotopy classes of paths leaving the basepoint of the base. The identification is via the unique path-lifting property of the covering map, and depends only on the choice of a basepoint in the universal covering space. In case of the universal covering space $\mathbb{C}$ of $E(\mathbb{C})$, the correspondence is given by the *period map*

$$\gamma \longmapsto \int_\gamma dz.$$

Recovering V as the universal covering space from the 2-functor $\underline{\Omega}$

Let $\underline{\Omega} = (\mathrm{Pt}, \Omega, s, t, \cdot)$ be a 2-functor from $\mathfrak{E}$ to $\mathfrak{S}ets$ satisfying the conditions of Definition IV.4.1.1. We want to construct an extension

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow V_{\underline{\Omega}} \xrightarrow{\varphi} E(\overline{\mathbb{Q}}) \longrightarrow 0$$

of End$E$-modules. The topological intuition above suggests that we set

$$V = V_{\underline{\Omega}} = \bigcup_{y \in \mathrm{Pt}(E)} \Omega_{0,y}(E) = \{\gamma \in \Omega(E) : s(\gamma) = 0\} \text{ (disjoint union)},$$
$$\varphi(\gamma) = t(\gamma)$$

where $0 \in E(k)$ is the zero point of the elliptic curve E.

Abelian group structure on $\Omega(E)$

The functoriality of $\underline{\Omega}$ transfers End$E$-module structure on $E(\overline{\mathbb{Q}})$ to that on V; namely, let us check that the maps $\Omega(f)$, $f \in \mathrm{End}E$, and $\Omega(m)$, where $m : E \times E \to E$ is the morphism of addition on E, define End$E$-module structure on V, or rather that their restriction to V does.

By assumption the functor $\underline{\Omega}$ preserves direct products so $\Omega(E \times E) = \Omega(E) \times \Omega(E)$, and thus there is a map

$$\Omega(m) : \Omega(E) \times \Omega(E) \to \Omega(E).$$

Maps s, t are natural transformations of Ω to Pt (as functors to $\mathfrak{S}ets$) and so $s \circ \Omega(m) = \mathrm{Pt}(m) \circ s$, $t \circ \Omega(m) = \mathrm{Pt}(m) \circ t$ is the map of addition on end-points. Therefore,

$$\Omega(m)(\Omega_{x,y}(E) \times \Omega_{v,w}(E)) \subset \Omega_{x+v,y+w}(E)),$$

and in particular

$$\Omega(m)(\Omega_{0,y}(E) \times \Omega_{0,z}(E)) \subset \Omega_{0,y+z}(E)).$$

Thus $\Omega(m) : V \times V \to V$ gives us a binary operation. It is straightforward to show that the preservation of direct product and functoriality implies that $\Omega(m)$ makes $\Omega(E)$ into an Abelian group. Let us check this.

By definition, associativity of $m : E \times E \to E$ means that $m \circ (m \times \mathrm{id}_E) = m \circ (\mathrm{id}_E \times m) : E \times E \times E \to E$; by preservation of direct product this implies $\Omega(m) \circ (\Omega(m) \times \mathrm{id}_E) = \Omega(m) \circ (\mathrm{id}_E \times \Omega(m))$ and so $\Omega(m)$ is associative. Similarly, commutativity of m means $m \circ (\mathrm{id}_1 \times \mathrm{id}_2) = m \circ (\mathrm{id}_2 \times \mathrm{id}_1) : E \times E \to E$; that similarly implies the commutativity of $\Omega(m)$. In the language of morphisms, the existence of a zero for the additive law translates to the existence of a morphism $0 : \{0\} \to E$ subject to the identifies: $m \circ (\mathrm{id} \times 0) = p_2 : \{0\} \times E \to E$ and $m \circ (0 \times \mathrm{id}) = p_1 : E \times \{0\} \to E$ corresponding to a commutative diagram:

$$\begin{tikzcd} \{0\} \times E \arrow[r] \arrow[dr] & E \times E \arrow[d] \\ & E \end{tikzcd}$$

Apply Ω to get $\Omega(m) \circ (\Omega(\mathrm{id}) \times \Omega(0)) = \Omega(p_2) : \Omega(\{0\}) \times \Omega(E) \to \Omega(E)$ and $\Omega(m) \circ (\Omega(0) \times \Omega(\mathrm{id})) = \Omega(p_1) : \Omega(E) \times \Omega(\{0\}) \to \Omega(E)$. Preservation of direct product implies $\Omega(m) \circ (\mathrm{id} \times \Omega(0)) = \Omega(p_2) : \Omega(\{0\}) \times \Omega(E) \to \Omega(E)$ and $\Omega(m) \circ (\Omega(0) \times \mathrm{id}) = \Omega(p_1) : \Omega(E) \times \Omega(\{0\}) \to \Omega(E)$. This implies that $\Omega(0)(\Omega(\{0\}))$ is a zero point in V.

Existence of (right) inverse corresponds to the existence of a morphism $i : E \to E$ subject to the following commutative diagram:

$$\begin{tikzcd} E \arrow[r] \arrow[d, "(\mathrm{id}_E, i)"'] & \{0\} \arrow[d, "0"] \\ E \times E \arrow[r, "m"'] & E \end{tikzcd}$$

Again functoriality ensures that $\Omega(i)$ satisfies a similar diagram, thus proving the existence of inverses.

The above checks that $\Omega(m)$ is an associative commutative partial operation on $\Omega(E)$ possessing a zero element and inverses; it is immediate to check V is closed under $\Omega(m)$ and inverse $\Omega(i)$, and so is a group.

Action of "fundamental group" $\Omega_{0,0}(E)$ on V via concatenation and by $\Omega(m)$-multiplication

Take a *loop* $\lambda \in \Omega_{0,0}(E)$ and $\gamma \in \Omega_{0,y}$; then both concatenation and $\Omega(m)$-product of λ and γ are well-defined; let us show that $\lambda \cdot \gamma = \Omega(m)(\lambda \times \gamma)$:

$$\Omega(m)(\lambda \times \gamma) = \Omega(m)(\lambda \cdot 0 \times 0 \cdot \gamma) = \Omega(m)(\lambda \times 0) \cdot \Omega(m)(0 \times \gamma) = \lambda \cdot \gamma.$$

The latter equality follows from the inverse element equality of morphisms $m(\mathrm{id} \times 0) = m(0 \times \mathrm{id}) = \mathrm{id}$. In the classical example, this observation corresponds to the following calculation:

$$\int_{\gamma_1 \cdot \gamma_2} dz = \int_{\gamma_1} dz + \int_{\gamma_2} dz = \int_{\gamma_1 \gamma_2} dz.$$

Here $\gamma_1 \cdot \gamma_2$ denotes the concatenation of the paths and $\gamma_1 \gamma_2$ denotes the pointwise product of the paths.

Divisibility of End$E$*-module structure and path-lifting property*

Analogously, a morphism $f \in \mathrm{End}E$, $\Omega(f) : \Omega(E) \to \Omega(E)$ defines a map $\Omega(f) : \Omega(E) \to \Omega(E)$. Arguments similar to the ones above allow us to prove that V is an End$E$-module with the operations defined above. Let us now prove that the path-lifting property implies that V is uniquely divisible. Indeed, we know that any isogeny $f \in \mathrm{End}E$ is étale ([Mum70]) and we may apply the path-lifting property to get a bijection

$$\Omega(f) : \bigcup_{0 \in \mathrm{Pt}(E)} \Omega_{0,y}(E) \longrightarrow \bigcup_{z \in \mathrm{Pt}(E)} \Omega_{0,z}(E).$$

That is, by definition of V, the map $\Omega(f)$ is a bijection on V, and V is uniquely divisible as required.

Finally, to get a short exact sequence as in Proposition IV.1.0.9, set $\varphi(w) = t(w)$. Then naturality of target map t implies $\varphi(w)$ is homomorphism of End$E$-modules; the connectivity of $\Omega(E)$ implies that φ is surjective. The kernel $\ker\varphi$ of $\varphi : V \to E(\overline{\mathbb{Q}})$ is $\Omega_{0,0}(E)$ and is isomorphic to $\mathbb{Z}^2$ by assumption.

Construction of a natural transformation $h' : \underline{\Omega} \to \pi_1^{\mathrm{top}}{}_{|\overline{\mathbb{Q}}} \circ \sigma$

For notational convenience, let $\underline{\Omega}' = (\mathrm{Pt}, \Omega', s', t', \cdot')$ denote the functor $\pi_1^{\mathrm{top}}{}_{|\overline{\mathbb{Q}}}$.

Let $W \to^{\psi} E(\overline{\mathbb{Q}})$ be the End$E$-module extension constructed from $\underline{\Omega}' = \pi_1^{\mathrm{top}}{}_{|\overline{\mathbb{Q}}}$ in the same way. Since V, W are both uniquely divisible extensions of $E(\overline{\mathbb{Q}})$ by $\mathbb{Z}^2$, we may apply Proposition IV.1.0.9 to get an End$E$-linear map $h : V \to W$ and $\sigma \in Gal(\overline{\mathbb{Q}}/k)$ such that $\sigma \circ \varphi = \psi \circ h$.

We want to get a natural transformation $h' : \underline{\Omega} \to \underline{\Omega}' \circ \sigma$. Recall a natural transformation $h' : \underline{\Omega} \to \underline{\Omega}' \circ \sigma$ is a family of maps (of sets with no further structure) $h_A^{\Omega} : \Omega(A) \to \Omega' \circ \sigma(A)$ and $h_A^{\mathrm{Pt}} : \mathrm{Pt}(A) \to \mathrm{Pt}' \circ \sigma(A)$ satisfying certain compatibly conditions expressed by commutative diagremmes (IV.4.1),(IV.4.2),(IV.4.3).

Define $h_A^{\mathrm{Pt}} : A(\overline{\mathbb{Q}}) \to A(\overline{\mathbb{Q}})$ by $h_A^{\mathrm{Pt}}(x) = \sigma(x) \in \mathrm{Pt}'(\underline{A}) = (\sigma A)(\overline{\mathbb{Q}}) = A(\overline{\mathbb{Q}})$ to be the map induced by Galois automorphism $\sigma : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$.

By assumption of connectivity of $\Omega(E)$, any $\gamma \in \Omega(E)$ can be represented as product $\gamma = \gamma_1^{-1} \cdot \gamma_2$ where $\gamma_1, \gamma_2, s(\gamma_1) = s(\gamma_2) = 0$; define a map $h'_E : \Omega(E) \to \Omega'(E)$ by setting $h'_E(\gamma) = h(\gamma_1)^{-1} \cdot h(\gamma_2)$. If $\gamma_1^{-1}\gamma_2 = {\gamma'_1}^{-1}\gamma'_2$ then $\gamma_1^{-1}\gamma'_1 = \gamma_2{\gamma'_2}^{-1} \in$

$\Omega_{0,0}(E)$, and so

$$\begin{gathered}h(\gamma_1)^{-1}\cdot h(\gamma_2) = h((\gamma_1{\gamma'_1}^{-1})\gamma'_1)^{-1}\cdot h((\gamma_2{\gamma'_2}^{-1})\gamma'_2) = \\ h((\gamma_1^{-1}\gamma'_1)+\gamma'_1)^{-1}\cdot h((\gamma_2{\gamma'_2}^{-1})+\gamma'_2) = \\ (h(\gamma_1^{-1}\gamma'_1)+h(\gamma'_1))^{-1}\cdot h(\gamma_2{\gamma'_2}^{-1})+h(\gamma'_2) = \\ (h(\gamma'_1))^{-1}\cdot h(\gamma'_2) = h({\gamma'_1}^{-1})\cdot h(\gamma'_2),\end{gathered}$$

which proves that h' is well-defined. Similar calculations check that h'_E preserves concatenation $\cdot_E$: $h'(\gamma\cdot\gamma') = h'(\gamma)\cdot h'(\gamma')$ for arbitrary $\gamma,\gamma'\in\Omega(E)$.

We define $h'_{E^n} = h^{\Omega}_{E^n} : \Omega(E^n)\to\Omega'(E^n)$ from an arbitrary Cartesian power $\Omega(E^n)=\Omega(E)^n$ by $h'_{E\times E}(\gamma\times\gamma') = h'_E(\gamma)\times h'_E(\gamma')$ etc.

Checking that h' is a natural transformation of $\underline{\Omega}$ into $\underline{\Omega}'\circ\sigma$

To check that h' is a natural transformation of $\underline{\Omega}$ to $\underline{\Omega}'\circ\sigma$, we need to check commutativity of the following diagrams (note that σ on the *left*-hand side is *not* a morphism but a functor!):

(IV.4.1)
$$\begin{tikzcd} E^n \arrow[r, "\mathrm{Pt}"'] \arrow[d, "\sigma"'] & E^n(\overline{\mathbb{Q}}) \arrow[d, "\sigma_{E^n}"] \\ E^n \arrow[r, "\mathrm{Pt}"] & E^n(\overline{\mathbb{Q}}) \end{tikzcd} \qquad \begin{tikzcd} A(\overline{\mathbb{Q}}) \arrow[r, "\mathrm{Pt}(f)"'] \arrow[d, "\sigma_A"'] & B(\overline{\mathbb{Q}}) \arrow[d, "\sigma_B"] \\ A(\overline{\mathbb{Q}}) \arrow[r, "\mathrm{Pt}(f\circ\sigma)"] & B(\overline{\mathbb{Q}}) \end{tikzcd}$$

(IV.4.2)
$$\begin{tikzcd} E^n \arrow[r, "\Omega"'] \arrow[d, "\sigma"'] & \Omega(E^n) \arrow[d, "h'_{E^n}"] \\ E^n \arrow[r, "\Omega'"] & \Omega'(E^n) \end{tikzcd} \qquad \begin{tikzcd} \Omega(A) \arrow[r, "\Omega(f)"'] \arrow[d, "h'_A"'] & \Omega(B) \arrow[d, "h'_B"] \\ \Omega'(A) \arrow[r, "\Omega'(f\circ\sigma)"] & \Omega'(B) \end{tikzcd}$$

(IV.4.3)
$$\begin{tikzcd} \Omega(A) \arrow[r, "s_A"] \arrow[d, "h'_A"'] & \mathrm{Pt}(A) \arrow[d, "\sigma_A"'] \\ \Omega'(A) \arrow[r, "s'_A"] & \mathrm{Pt}(A) \end{tikzcd} \qquad \begin{tikzcd} \Omega(A) \arrow[r, "t_A"'] \arrow[d, "h'_A"'] & \mathrm{Pt}(A) \arrow[d, "\sigma_A"'] \\ \Omega'(A) \arrow[r, "t'_A"] & \mathrm{Pt}(A) \end{tikzcd}$$

(IV.4.4)
$$\begin{tikzcd} \Omega(A)\times\Omega(A) \arrow[r, "\cdot_A", "\text{partial}"'] \arrow[d, "h'_A\times h'_A"'] & \Omega(A) \arrow[d, "h'_A"'] \\ \Omega'(A)\times\Omega'(A) \arrow[r, "\cdot'_A", "\text{partial}"'] & \Omega'(A) \end{tikzcd}$$

The first pair (IV.4.1) expresses that $h_A^{\mathrm{Pt}} = (\sigma_A)_A$ is a natural transformation of set-valued functors from Pt to Pt $\circ\, \sigma$; the diagrams are commutative just by definition of the action of $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on category $\mathfrak{C}$. Analogously the second pair (IV.4.2) expresses that h' is a natural transformation of set-valued functors Ω to $\Omega' \circ \sigma$. The first diagram in (IV.4.2) says that h'_{E^n} is a map from $\Omega(E^n)$ to $\Omega'(\sigma(E^n)) = \Omega'(\sigma(E^n))$; the second one in (IV.4.2) expresses the linearity of h' with respect to the morphisms of $E^n \to E^m$; that follows from End$E$-linearity of $h : V \to V$.

The third pair (IV.4.3) expresses compatibility of the end-point functions and h'_A; for $A = E$, this follows from the main property $\sigma \circ s = s \circ h$ when restricted to $W^n \subset \Omega(E^n)$, and that h' preserves concatenation $\cdot_A$; preservation of direct products allows us to extend this to arbitrary Cartesian power E^n. The diagram (IV.4.4) expresses the fact that h' preserves concatenation; this is by the definition of h'.

This concludes the proof that h' is a natural transformation and that of derivation of Proposition IV.1.0.11 from Proposition IV.1.0.9. □

IV.5 Extensions of $E(\overline{\mathbb{Q}})$ by Λ

In this section we state and prove Proposition IV.1.0.11; the proof is a model theoretic argument based on Kummer theory and the description of the image of Galois representation on Tate module $T(E)$. However, we tried to be very explicit and have avoided any model-theoretic terminology in the exposition of the proof. The only model theory left in the proof is in the level of motivations; as was noted earlier, those model-theoretic motivations are useful to try and generalise the formulation to other varieties and contexts in general.

IV.5.1 Transitivity of $Gal(K/\mathbb{Q})$ action on the uniquely divisible End E-module extensions of $E(\overline{\mathbb{Q}})$ by Λ

For convenience we restate Proposition IV.1.0.9. We conjecture the Proposition holds for any algebraically closed field K, char$K = 0$ instead of $\overline{\mathbb{Q}}$.

Proposition IV.5.1.1. *Let Λ be an* End$E$*-module isomorphic to either $\mathbb{Z}^2$ if* End$E = \mathbb{Z}$*, or to an order in the ring of integers in the ring* $\mathbb{Q}(\mathrm{End}\,E)$ *of fractions of the ring* End$E$ *of endomorphisms of an elliptic curve E defined over a number field. Assume further that all the endomorphisms of E are definable over k.*

1. *There exists a uniquely divisible extension in* $\mathrm{Ext}^1_{\mathrm{End}E\text{-}mod}(E(\overline{\mathbb{Q}}), \Lambda)$.
2. (a) *If* End$E = \mathbb{Z}$*, then the Galois group acts transitively on the set of uniquely divisible extensions* $\mathrm{Ext}^1_{AbGroups}(E(\overline{\mathbb{Q}}), \Lambda)$.

(b) *The action of the Galois group has only finitely many orbits on the set of uniquely divisible extensions in* $\mathrm{Ext}^1_{\mathrm{End}E\text{-}mod}(E(\overline{\mathbb{Q}}), \Lambda)$.

In other words,

1. *There exists a uniquely divisible* End$E$*-module* V *and a short exact sequence of* End$E$*-modules*

$$0 \longrightarrow \Lambda \longrightarrow V \longrightarrow E(\overline{\mathbb{Q}}) \longrightarrow 0.$$

2. (a) *If* E *has no complex multiplication and* End$E = \mathbb{Z}$*, then for any uniquely divisible* $\mathbb{Z}$*-module extensions* W, V *of* $E(\overline{\mathbb{Q}})$ *by* Λ*, fitting into the short exact sequences as above, there exist a commutative diagram:*

$$\begin{tikzcd}
0 \arrow[r] & \Lambda \arrow[r] \arrow[d, "\exists h_{|\Lambda}"'] & V \arrow[r, "\varphi"] \arrow[d, "\exists", "h\in\mathrm{Hom}\,(V,W)"'] & E(\overline{\mathbb{Q}}) \arrow[r] \arrow[d, "\exists\sigma\in Gal(\overline{\mathbb{Q}}/k)"] & 0 \\
0 \arrow[r] & \Lambda \arrow[r] & W \arrow[r, "\psi"] & E(\overline{\mathbb{Q}}) \arrow[r] & 0
\end{tikzcd}$$

(b) *If* End$E \neq \mathbb{Z}$*, then there exist finitely many uniquely divisible* End$E$*-module extensions* $W_1, ..., W_n$ *of* $E(\overline{\mathbb{Q}})$ *by* Λ*, fitting into the short exact sequences as above, such that for any uniquely divisible* End$E$*-module extension* V *there exist a commutative diagram:*

$$\begin{tikzcd}
0 \arrow[r] & \Lambda \arrow[r] \arrow[d, "\exists h_{|\Lambda}"'] & V \arrow[r, "\varphi"] \arrow[d, "\exists", "h\in\mathrm{Hom}\,(V,W)"'] & E(\overline{\mathbb{Q}}) \arrow[r] \arrow[d, "\exists\sigma\in Gal(\overline{\mathbb{Q}}/k)"] & 0 \\
0 \arrow[r] & \Lambda \arrow[r] & W_i \arrow[r, "\psi_i"] & E(\overline{\mathbb{Q}}) \arrow[r] & 0
\end{tikzcd}$$

Proof. By §IV.3 and [Sil85], there exists an embedding $k \hookrightarrow \mathbb{C}$ of the field k of definition of E such that the kernel of the universal covering map is isomorphic to any such Λ:

$$0 \longrightarrow \Lambda \longrightarrow \mathbb{C} \xrightarrow{\ p\ } E(\mathbb{C}) \longrightarrow 0$$

The endomorphisms act on the complex plane $\mathbb{C}$ by multiplication by complex numbers, and so in particular $\mathbb{C}$ is a uniquely divisible End$E$-module. The set $E(\overline{\mathbb{Q}})$ of points of E over an algebraically closed subfield is closed under addition and End$E$-multiplication i.e. is an End$E$-submodule, necessarily uniquely divisible; so is then $V = p^{-1}(E(\overline{\mathbb{Q}}))$; take that to obtain a short exact sequence as above. This proves (1).

To prove (2), we use the Kummer theory of elliptic curves and the results about the image of Galois representation on the torsion points. Essentially, what we need is that all the restrictions on the image of the Galois action on the Tate module $\prod_l T(E)$ come from the geometry of E and in this case, are linear. That is exactly what is provided by the Kummer theory and other results we use.

Pick a maximal linearly independent set $v_0, v_1, v_2, .. \in V$; let $V_n = \mathrm{End}Ev_0 + \ldots + \mathrm{End}Ev_n$ be the submodule generated by $v_0, \ldots, v_n$, and let $\mathbb{Q}V_n = (\mathrm{End}E)^{-1}V_n = \{v : \exists N \in \mathbb{N}(Nv \in V_n)\}$ be its divisible closure. We construct by induction a partial $\mathrm{End}E$-module linear map $h_n : \mathbb{Q}V_n \to W_i$ inducing a partial Galois map $\sigma_n : \varphi(\mathbb{Q}V) \to E(\overline{\mathbb{Q}})$ so that $h = \cup h_n$ is an isomorphism of V and W_i; then the construction implies $\sigma = \cup\sigma_n$ is a total Galois map on $E(\overline{\mathbb{Q}})$ to $E(\overline{\mathbb{Q}})$, and thus there is a commutative diagram as above.

For a point $a \in E(\mathbb{C})$, let us call *a compatible sequence of division points starting at a* a sequence $(a^{(i)})_{i\in\mathbb{N}}$ satisfying $a^{(i)} = ja^{(ij)}, i, j \in \mathbb{N}$. Compatible sequences of division points starting at $0 \in E(\overline{\mathbb{Q}})$ form a *Tate module* $T(E)$. A compatible sequence of *l-primary* division points is a subsequence $(a^{(l^j)})_{j\in\mathbb{N}}$ of a compatible sequence of division points; such sequences of l-primary points starting at 0 form *l-adic Tate module* $T_l(E)$.

IV.5.2 The image of Galois representations on Tate module $T_l(E)$

Choose $v_0 \in \Lambda = \ker\varphi$; then $\mathbb{Q}V_0 = \varphi^{-1}(E_{\mathrm{tors}})$, where $E_{\mathrm{tors}} = \{x \in E(\overline{\mathbb{Q}}) : nx = 0 \text{ for some } n \in \mathbb{N}\}$ is the set of *torsion* points of $E(\overline{\mathbb{Q}})$.

As the base step of the induction, in this subsection we construct a commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Lambda & \longrightarrow & V_0 & \xrightarrow{\varphi} & E_{\mathrm{tors}} & \longrightarrow & 0 \\
 & & {\scriptstyle h_0}\big\downarrow & & \big\downarrow{\scriptstyle h_0\in\mathrm{Hom}\,(V_0,W)} & & \big\downarrow{\scriptstyle \sigma\in Gal(\overline{\mathbb{Q}}/k)} & & \\
0 & \longrightarrow & \Lambda & \longrightarrow & W & \xrightarrow{\psi\circ\tau_i} & E_{\mathrm{tors}} & \longrightarrow & 0
\end{array}
\tag{IV.5.1}
$$

where $\tau_1, ..., \tau_n$ is some fixed finite collection of $\mathrm{End}E$-endomorphisms of E_{tors} independent of V, W. For curves without complex multiplication, there is no need to consider τ_i's, i.e. $n = 1$, $\tau_1 = \mathrm{id}$.

Note that when E has complex multiplication, the diagram above is somewhat reminiscent of a diagram appearing in the main theorem of complex multiplication [Shi71, Chapter 5, Theorem 5.4], also Lang [Lan83]. Global class field theory provides an explicit description on the image of Galois action in $\mathrm{Aut}(E_{\mathrm{tors}})$ in terms of the action of the ring of idéles on some lattices in the universal covering space.

E has complex multiplication

Pick an arbitrary isomorphism $h_0 : \Lambda \to \Lambda$ and extend it uniquely to $h_0 : V_0 \to W$; we may do so by unique divisibility of V and W. Define an End$E$-morphism $\tau : E_{\text{tors}} \to E_{\text{tors}}$ by $\tau(x) = \psi \circ h_0 \circ \varphi^{-1}(x)$. The calculation $\psi \circ h_0(y+\Lambda) = \psi \circ h_0(y)$ shows it is well-defined; linearity of τ follows from that of φ, h_0 and φ.

Ideally we would like to be able to choose $h_0 : \Lambda \to \Lambda$ so that $\tau = \tau_h : E_{\text{tors}} \to E_{\text{tors}}$ is induced by a Galois automorphism. Here we prove a weaker statement below.

Denote $\mathcal{O} = \text{End}E$ and $E[n] = \{x \in E(\overline{\mathbb{Q}}) : nx = 0\}$ the n-torsion of E. The set $E[n]$ is a free 1-dimensional $\mathcal{O}/_{n}\mathcal{O}$-module ([Lan78, Ch.8§15,Fact 1]), and $\text{Aut}_{\mathcal{O}}(E[n]) = \text{Aut}_{\mathcal{O}/n\mathcal{O}-\text{mod}}(E[n]) \cong (\mathcal{O}/_{n}\mathcal{O})^*$, where $(\mathcal{O}/_{n}\mathcal{O})^*$ denotes the group of invertible elements in $(\mathcal{O}/_{n}\mathcal{O})^*$.

An $\mathcal{O}$-automorphism of E_{tors} is given by a compatible system of $\mathcal{O}$-automorphisms of $E[n]$, $n > 0$; thus we see that there is an action of $\hat{\mathcal{O}} = \lim\limits_{n} \mathcal{O}/_{n}\mathcal{O}$ on E_{tors} as an End$E$-module; the fact that $\text{Aut}_{\mathcal{O}}(E[n]) = \text{Aut}_{\mathcal{O}/n\mathcal{O}-\text{mod}}(E[n]) \cong (\mathcal{O}/_{n}\mathcal{O})^*$ implies that $\text{Aut}_{\mathcal{O}}(E_{\text{tors}}) \cong \hat{\mathcal{O}}$.

Now we refer to a consequence of the main theorem of complex multiplication, namely that, in notation of [Lan78, Ch.8§15,Fact 2], the image of Galois group

$$G_K = Gal(K(E_{\text{tors}}) : K) \to \prod((\text{End}E)_l)^*$$

is open of finite index, i.e. $Im\, G_K$ is a finite index subgroup of $\hat{\mathcal{O}}^* = \prod_l \mathcal{O}_l^*$. Choose $\tau_1, ..., \tau_n$ to be representatives of conjugacy classes $\mathcal{O}^*/Im\, G_K$; we then have that for some i $\tau_i \tau = \sigma \in Im\, G_K$; this choice of $h_0, \sigma = \tau_i \tau$ makes the diagram $(IV.5.1)$ commutative, as required.

E does not have complex multiplication

Assume that E does not have complex multiplication, i.e. End $E \cong \mathbb{Z}$; identify End$E = \mathbb{Z}$, $T(E) = \hat{\mathbb{Z}}^2$, and $\text{Aut}(T(E)) = \text{SL}_2(\hat{\mathbb{Z}})$. The maps $\varphi : V \to E(\overline{\mathbb{Q}})$, $\psi : W \to E(\overline{\mathbb{Q}})$ define embeddings $\iota_\varphi : \Lambda \to T(E)$, $\iota_\psi : \Lambda \to T(E)$ by

$$\iota_\varphi : \lambda \mapsto (\varphi(\lambda/j))_{j \in \mathbb{N}},$$

$$\iota_\psi : \lambda \mapsto (\psi(\lambda/j))_{j \in \mathbb{N}}.$$

The images $\iota_\varphi(\Lambda), \iota_\psi(\Lambda)$ of the both maps are dense in $T(E)$ due to the surjectivity of $\varphi, \psi : \mathbb{Q}\Lambda \to E_{\text{tors}}$.

Take a pair of elements $\lambda_0, \lambda_1 \in \ker\varphi \cong \Lambda$ generating Λ as an Abelian group; we want to find $\lambda_0', \lambda_1' \in \ker\psi \cong \Lambda$ and $\sigma = \sigma_0 \in Gal(\overline{\mathbb{Q}}/k)$ such that

$$\sigma\iota_\varphi(\lambda_0) = \iota_\psi(\lambda_0'),$$

$$\sigma\iota_\varphi(\lambda_1) = \iota_\psi(\lambda_1').$$

Under identification $\ker\varphi = \mathbb{Z}^2$, since vectors $\lambda_0, \lambda_1 \in \mathbb{Z}^2$ generate lattice $\mathbb{Z}^2$, it holds that $\det(\lambda_0, \lambda_1) = 1$. That implies that $\det(\iota_\varphi(\lambda_0), \iota_\varphi(\lambda_1))$ is a unit in $\hat{\mathbb{Z}} = \prod_l \mathbb{Z}_l$. Similarly $\det(\iota_\psi(\lambda_0'), \iota_\psi(\lambda_1'))$ has to be a unit in $\hat{\mathbb{Z}}$.

By [Ber88, Theorem 3], the image of the Galois group $Gal(\overline{\mathbb{Q}}/k)$ in $\mathrm{Aut}(T(E)) = \mathrm{SL}_2(\hat{\mathbb{Z}})$ contains an open subgroup $\mathrm{SL}_2(N\hat{\mathbb{Z}}) = \ker\Big(\mathrm{SL}_2(\hat{\mathbb{Z}}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})\Big)$, for some $N \in \mathbb{N}$ large enough. Now, there is $L \in \mathrm{SL}_2(N\hat{\mathbb{Z}})$ such that $L(u_0) = u_0'$ and $L(u_1) = u_1'$ iff $\det(u_0, u_1) = \det(u_0', u_1')$ and $u_0 \in u_0' + N\hat{\mathbb{Z}}$, $u_1 = u_1' + N\hat{\mathbb{Z}}$. Thus it is enough to take $\mathbb{Z}$-linearly independent λ_0', λ_1' such that $\psi(\lambda_0'/N) = \varphi(\lambda_0/N)$ and $\psi(\lambda_1'/N) = \varphi(\lambda_1/N)$. Necessarily then $\det(\iota_\psi(\lambda_0'), \iota_\psi(\lambda_1'))$ is a unit. This implies that $\det(\lambda_0', \lambda_1') = 1$ and thus, λ_0', λ_1' generate $\mathbb{Z}^2$. The latter statement is independent of the identification $\ker\psi = \mathbb{Z}^2$, and this concludes the proof in the case of no complex multiplication.

IV.5.3 Kummer theory

The main statement of the Kummer theory for an elliptic curve

Let us state the main lemma in a form convenient to us to make an inductive process; that is a form natural from model-theoretic point of view and corresponding to the property of *atomicity* over the kernel. In the model-theory language, it says that the type of a linearly independent tuple in U is isolated by a quantifier-free formula of language L^{lin}.

Lemma IV.5.3.1 (Kummer theory for an elliptic curve). *Let E be an elliptic curve defined over a number field k. Let $a_1, \ldots, a_n \in E(\overline{\mathbb{Q}})$ be a sequence of points linearly independent over* $\mathrm{End}E$. *Then there exists $N \in \mathbb{Z}$ such that any two compatible sequences $(a_1^{(i)}, \ldots, a_n^{(i)})_{i\in\mathbb{N}}$, $(b_1^{(i)}, \ldots, b_n^{(i)})_{i\in\mathbb{N}}$ of division points in $E(\overline{\mathbb{Q}})$ starting at $a_1, \ldots, a_n$ and such that $a_1^{(N)} = b_1^{(N)}, \ldots, a_n^{(N)} = b_n^{(N)}$, are $Gal(\overline{\mathbb{Q}}/k)$-conjugated by $\sigma \in Gal(\overline{\mathbb{Q}}/k)$,*

$$\sigma(a_i^{(j)}) = b_i^{(j)},\ \textit{for all } 0 \leqslant i \leqslant n, j \in \mathbb{N}.$$

Proof of Lemma. See Bashmakov [Bas72] for original results for elliptic curves; see [Rib79, JR87, Rib87] for Kummer theory of Abelian varieties, and see [Ber88] for a summary of results of Kummer theory of Abelian varieties; we quote [Ber88, Theorem 2] from that paper.

We now introduce the notations of Bertrand [Ber88, Theorem 2]. Let $G = A \times L$ be a product of an Abelian variety by a torus L so that after a finite extension of k it satisfies Poincaré's complete reducibility theorem (as a variety *over* k). Let l denote a prime. Let $G_{l^\infty} = \{x \in G(\bar{k}) : \exists n\, l^n x = 0\}$ be the l^∞-torsion of G. For a point $P \in G(k)$, let G_P be *the smallest algebraic subgroup of* G *containing* P, i.e. the Zariski closure of subgroup $\mathbb{Z}P$ of G, and *let* G_P° *be its connected component through the origin*, and finally let

$$\xi_{l^\infty}(P) : \mathrm{Gal}(\bar{k}/k(G_{l^\infty}, P)) \longrightarrow T_{l^\infty}(A \times L)$$

$$\sigma \longmapsto \sigma(P_{l^\infty}) - P_{l^\infty}, \text{ for some } P_{l^\infty} \in T_{l^\infty}(A \times L).$$

It can be checked by direct computation checks the map $\xi_{l^\infty}(P) : \mathrm{Gal}(\bar{k}/k(G_{l^\infty}, P) \to T_{l^\infty}(A \times L)$ does not depend on the choice of $P_{l^\infty} \in T_{l^\infty}(A \times L)$.

Let $T(G_P^\circ)$ be the sequences of $T(E)$ consisting of elements of G_P°; then, according to [Ber88, Theorem 2], the image of $\xi_\infty(P) = \prod_l \xi_{l^\infty}$ has finite index in $T(G_P^\circ) = \prod_l T_{l^\infty}(A \times L)$, i.e. the image contains $NT(G_P^\circ)$, for some natural number $N \in \mathbb{N}$ large enough.

We claim that if we take $G = E^n$, $P = (a_1, \ldots, a_n) \in E^n(\overline{\mathbb{Q}})$, then N above is N required in Lemma. By the result cited above, it is enough to prove that if $a_1, \ldots, a_n \in E(\overline{\mathbb{Q}})$ are End$E$-linearly independent, then $G_P^\circ = E^n$.

An algebraic subgroup of E^n is necessarily an Abelian subvariety; by the Corollary of Poincaré's complete reducibility theorem (Lemma IV.5.3.2 below) if $\dim G_P^\circ \neq E^n$, then for some $m \in \mathbb{N}$, the connected component of G_P°/m lies in the kernel of some non-trivial morphism $f : E^n \to E^m$. The kernel of f satisfies the End$E$-linear relation corresponding to $f : E^n(\bar{k}) \to E^n(\bar{k})$ as a morphism of End$E$-modules. Since by assumption P satisfies no End$E$-linear relation, this is a contradiction, and $G_P^\circ = E^n$.

Any Abelian subvariety is a connected component of the kernel of a morphism

The following Lemma has been just used to relate the more geometric formulation of Bertrand [Ber88] and the more explicit statement of Lemma IV.5.3.1 following Bashmakov [Bas72] and Ribet [Rib79] specific to Abelian varieties.

Lemma IV.5.3.2 (Corollary of Poincaré's complete reducibility theorem). *Let $B \subset E^n$ be an irreducible Abelian subvariety. Then there is a morphism $f : E^n \to E^m$ such that B is a connected component of* $\ker f$.

In particular, this implies that an Abelian subvariety $B \neq E^n$ satisfies a non-trivial End *-linear relation.*

Proof. By Poincaré's reducibility theorem, there exist an Abelian subvariety $B' \subset E^n$ such that $B(\bar{k})+B'(\bar{k}) = E^n(\bar{k})$ and the intersection $B(\bar{k})\cap B'(\bar{k})$ is finite. This implies there is an isogeny $f : B \times B' \to E^n, (x,y) \to (x+y)$, and $f(B) = B$. There exists another isogeny $f' : E^n \to B \times B'$ such that $f' \circ f = [m]$ is a multiplication by a natural number $m \in \mathbb{N}$; let B_m and B'_m be the connected components of $B/m \subset E^n$ and $B'/m \subset E^n$, respectively, passing through the origin. Then we know that $mB_m = ff'(B_m) = B$, $mB'_m = ff'(B'_m) = B'$ but also $f'(B_m)$ and $f'(B'_m)$ do not intersect within $B \times B'$; to conclude, $B_m, B'_m \subset E^n$ and $B_m \cap B'_m = \emptyset$ and $B_m + B'_m = E^n$. Let $p_2 : B \times B' \to B'$ be the projection on the second coordinate; since 0 is a connected component of $\ker f$, it implies that B_m is a connected component of kernel of $f \circ p_2 \circ f' : E^n \to E^n$ and B is a connected component of $f \circ p_2 \circ f' \circ [m] : E^n \to E^n$. □

This concludes the proof of the lemma IV.5.3.1 stating the Kummer theory for an elliptic curve. □

An inductive argument based on Kummer theory

Assume now that we are on inductive step $n-1$, i.e. we have defined an End$E$-linear map $h_{n-1} : \mathbb{Q}V_{n-1} \to W$ and $\sigma_{n-1} \in Gal(\overline{\mathbb{Q}}/k)$, $h_{n-1}(v_i) = w_i, 0 < i < n$ such that $\psi(h_{n-1}(v)) = \sigma_{n-1}\varphi(v)$ for every $v \in \mathbb{Q}V_{n-1}$. Consider a compatible system $(\varphi(v_1/j))_j, \ldots, (\varphi(v_n/j))_j, j \in \mathbb{N}$ of division points in $E(\overline{\mathbb{Q}})$, and take N as in Kummer theory Lemma IV.5.3.1. By the induction hypothesis we have $\sigma_{n-1}\varphi(v_1/j) = \psi(v_1/j), \ldots, \sigma_{n-1}\varphi(v_{n-1}/j) = \psi(v_{n-1}/j)$ for any j. Choose $w_n \in W$ such that $\sigma_{n-1}\varphi(v_n/N) = \psi(w_n/N)$; that is possible by surjectivity of $\psi : W \to E(\overline{\mathbb{Q}})$. By Kummer theory lemma, for N large enough, there exists $\sigma' \in Gal(\overline{\mathbb{Q}}/k)$ such that $\sigma'\sigma_{n-1}\varphi(v_1/j) = \psi(v_1/j), \ldots, \sigma'\sigma_{n-1}\varphi(v_{n-1}/j) = \psi(v_{n-1}/j)$, and $\sigma'\sigma_{n-1}\varphi(v_n/j) = \psi(w_n/j)$; let $\sigma_n = \sigma'\sigma_{n-1}$. By construction we have that $\sigma_n|_{\varphi(\mathbb{Q}V_{n-1})} = \sigma_{n-1}|_{\varphi(\mathbb{Q}V_{n-1})}$ and $\sigma_n\varphi(v_i/j) = \psi(v_i/j), 0 < i < n+1$. This implies $\sigma_n\varphi(v) = \psi(v)$ for arbitrary $v \in \mathbb{Q}V_n$, thereby completing the induction step.

After countably many steps we construct a total End$E$-linear map $h = \cup h_n : V \to W$ and $\sigma : \varphi(V) \to E(\overline{\mathbb{Q}})$. Since $\varphi(V) = E(\overline{\mathbb{Q}})$, the Galois map σ is defined on the whole of $E(\overline{\mathbb{Q}})$. Since Galois map σ is surjective, this implies $h : V \to W$ is surjective, too. This completes the proof of Proposition IV.1.0.9.

The last argument could in fact have been avoided by a little more careful inductive construction of h: instead of always choosing w_n to match $\varphi(v_n)$ we could have on odd steps pick an arbitrary w_n and then choosen v_n so that $\sigma_n(\varphi(v_n)) = \psi(w_n)$

while on even steps preserving the old behaviour. It is very easy to force surjectivity of the constructed map $h : V \to W$ this way; it is a very common argument in model theory called “a back-and-forth argument”. □

IV.5.4 Concluding Remarks

Image of Galois representation

Remark IV.5.4.1. Note that for the arguments in §IV.7.4 it is essential that the image of the Galois action is as large as possible subject to linear dependencies. However, this is something specific to elliptic curves and false for higher dimensional Abelian varieties. This observation shows that the straightforward generalisation to higher dimensional Abelian varieties is false. We discuss this more in Lemma IV.6.3.1.

Kummer theory

Remark IV.5.4.1 says that in higher dimensions, the base of the induction breaks down due to additional restrictions on the image of the Galois action. This does not happen in the the later steps of the induction process based on Kummer theory.

Remark IV.5.4.2 (Generalisations of Kummer theory argument). Since Kummer theory is known in much larger generality, say for a product of arbitrary Abelian varieties, complex tori $\mathbb{C}^*$ and complex lines $\mathbb{C}$ ([Ber88]), it seems straightforward to generalise the Kummer theory argument above to such a product A. Thus, one would prove that if there exists $h_0 : \ker \varphi \to \ker \psi$ and a Galois map $\sigma_0 : E(k(T(E)) \to E(k(T(E)))$ in $\mathrm{Gal}(k(T(E))/k)$ such that $\varphi \circ \sigma_0 = h_0 \circ \psi$, then there exists $h : V \to W$ and $\sigma \in Gal(\overline{\mathbb{Q}}/k)$ making the diagram (2a) commute. That is, a morphism between kernel from $\Lambda \subset V$ to $\Lambda \subset W$ extends to a morphism on the whole of V. Model-theoretically, this means that the types of the points of universal covering space lying over algebraic points $A(\overline{\mathbb{Q}})$ is *atomic over the kernel* in the *linear* language.

Remark IV.5.4.3 (A geometric formulations of Kummer theory). Note that the formulation of Kummer theory in Bertrand [Ber88] is more geometric than the way we state it in Lemma IV.5.3.1; instead of “linearly independence of coordinates of a point $P \in E^n$” he speaks of a “connected component of Zariski closure of group $\mathbb{Z}P$ in semi-Abelian variety E^n”, which is much more invariant and robust notion. The way we state Proposition IV.1.0.8 and discussions in Chapter III agree with his formulation well: we also prefer to speak only about the properties of *connectivity* and *analytic irreducibility*, albeit that of subsets of the universal covering space. It is also more close to the functorial formulation of Proposition IV.1.0.11.

Remark IV.5.4.4 (Failure of Kummer theory for extensions of Abelian varieties by tori). According to Ribet [Rib87, JR87], Kummer theory may fail for non-trivial extensions of Abelian varieties by $(\mathbb{C}^*)^n$ due to "existence of an additional morphism"; he gives a motivic interpretation in [Rib87]. It is natural to ask if an analogous argument could still be carried despite the failure of Kummer theory. To state a correct conjecture, we may want to consider the functorial formulation of Proposition IV.1.0.11 or rather that of Proposition IV.1.0.8 in a different language, perhaps with respect to *another* variety A.

IV.6 Abelian varieties and Chern classes

In this section we analyse the language $L_A(A)$ for $A = L^*$ an ample homogeneous $\mathbb{C}^*$-bundle over an Abelian variety. We show that this language can definably speak of the first Chern class associated to line bundle L and the Riemannian hermitian form on the universal covering space of Abelian variety $A(\mathbb{C})$; an implication important for us is that it is also able to speak about the Weil pairing on $T(A)$ fixed by Galois action. Another important implication is that the first-order $L_A(L^*)$-theory of the universal covering of L^* is unstable.

This shows that the language L_A is different from one considered in Zilber [Zild], and the corresponding model $U_{L^*_A}(\overline{\mathbb{Q}})$ could be atomic, cf. Lemma IV.6.3.1.

IV.6.1 A construction of a 2-form on $\pi_1(A(\mathbb{C}), 0)$

Let A be a an Abelian variety over $\mathbb{C}$ and let $L = L_A$ be a line bundle over A. Given these data, we want to construct the Riemannian form of variety A (Mumford [Mum70, Theorem of Appel-Humpert and Lemmas on p.20])

$$E_L : \pi_1(A, 0) \times \pi_1(A, 0) \to \mathbb{Z},$$

which we call the *1-st Chern class of line bundle L over the Abelian variety A*; we justify the name later in §IV.6.2 by showing that our construction is just another view on the construction given in Mumford. The form E_L is an element of $\bigwedge^2 H_1(\Lambda, \mathbb{Z}) = H^2(\pi_1(A), \mathbb{Z})$ associated to line bundle L. The construction works for any topological space with a commutative fundamental group, and in fact depends only on the fundamental group of L^*.

Let L^*_A be the corresponding homogeneous $\mathbb{C}^*$-bundle on A obtained by deleting the zero section.

The form E_L is defined as follows:

$$E_L : \pi_1(A,0) \times \pi_1(A,0) \to \pi_1(\mathbb{C}^*,1)$$

$$E_L : \Lambda \times \Lambda \to \mathbb{Z} \tag{IV.6.1}$$

$$E_L : (\gamma_1,\gamma_2) \to [\widetilde{\gamma_1},\widetilde{\gamma_2}]$$

where $\widetilde{\gamma_1},\widetilde{\gamma_2} \in \pi_1(L_A^*,(0,1))$ are arbitrary loops in L_A^* such that $p(\widetilde{\gamma_1}) = \gamma_1, p(\widetilde{\gamma_2}) = \gamma_2$. The orientation on $\mathbb{C}^*$ allows to canonically identify group $\pi_1(\mathbb{C}^*,1)$ and $\mathbb{Z}$.

Let us check $E_L(\gamma_1,\gamma_2) = [\widetilde{\gamma_1},\widetilde{\gamma_2}]$ is homotopic to an element of $\pi_1(\mathbb{C}^*,1)$.

First of all, $E_L(\gamma_1,\gamma_2)$ is a loop by construction. The loop $[\gamma_1,\gamma_2] = p([\widetilde{\gamma_1},\widetilde{\gamma_2}])$ is contractible by assumption; apply the lifting property of $\mathbb{C}^*$-fibration $p : L_A^* \to A$ and the homotopy contracting $[\gamma_1,\gamma_2]$ to the trivial loop at 0. The homotopy will lift to a homotopy with a loop in the fibre. The loop $E_L(\gamma_1,\gamma_2)$ is well-defined because $\pi_1(\mathbb{C}^*,1)$ lies in the centre of $\pi_1(L_A^*)$ (cf. Example V.1.2.3 in Appendix).

Let us do this construction in terms of the language L_A on the universal covering space $\widetilde{L_A^*}$ of L_A^*. We know that L_A is able to define $\pi_1(L_A^*,y_0)$, for a point $y_0 \in L_A^*$; that group is enough to reconstruct the form.

There is a natural map $p_* : \pi_1(L_A^*,y_0) \to \pi_1(A,0)$ if we choose $y_0 \in L_A^*$ lying above $0 \in A$. For elements $\gamma_1,\gamma_2 \in \pi_1(A,0)$ let $c_L(\gamma_1,\gamma_2) = [\widetilde{\gamma_1},\widetilde{\gamma_2}]$ where $\widetilde{\gamma_1},\widetilde{\gamma_2}$ are arbitrary elements such that $p_*(\widetilde{\gamma_1}) = \gamma_1, p_*(\widetilde{\gamma_2}) = \gamma_2$. The element $c_L(\gamma_1,\gamma_2) \in \ker(p_* : \pi_1(L_A^*,y_0) \to \pi_1(A,0))$ is well-defined because the extension $\pi_1(L_A^*,0)$ is a central extension of $\pi_1(A,0)$. Note all these arguments can be done in L_A-structure $\widetilde{L_A^*}$.

Lemma IV.6.1.1. *The L_A-structure $\widetilde{L_A^*}$ associated to a homogeneous $\mathbb{C}^*$-bundle L_A^* over an Abelian variety is able to define a bilinear form*

$$E_L : \Lambda \times \Lambda \to \mathbb{Z},$$

where $\Lambda = \pi_1(A(\mathbb{C}),0)$ is the lattice of the fundamental group of $A(\mathbb{C})$.

The form E_L is the Riemannian form associated to the Abelian variety $A(\mathbb{C})$ as a variety over $\mathbb{C}$.

Proof. The arguments preceeding the lemma present a construction of such a form in $\widetilde{L_A^*}$ in the language L_A, where Λ is interpreted as the set of equivalence classes of $\pi_1(L_A^*,y_0)/\pi_1(\mathbb{C}^*,1)$.

We show that it coincides with a construction in Mumford [Mum70] below. □

In fact, this is a construction of an element of $H^2(\pi_1(A,0),\mathbb{Z})$ corresponding to the group extension $\pi_1(L_A^*,y_0)$ of $\pi_1(A(\mathbb{C}),0)$ by Λ.

Corollary IV.6.1.2. *The first-order L_A-theory of $\widetilde{L_A^*}$ is unstable.*

Proof. The definable set $\pi_1(\mathbb{C}^*, 1) \subset \pi_1(L_A^*, y_0)$ carries an Abelian group structure; let us show it carries a ring structure isomorphic to $\mathbb{Z}$.

Choose $\lambda_1, \lambda_2 \in \Lambda$ to be $\mathbb{Z}$-generators of Λ; then it may be shown that $E_L(\lambda_1, \lambda_2) = \mu_0$ is a generator of Abelian group $\pi_1(\mathbb{C}^*, 1)$. Define the product μ of $\mu_1, \mu_2 \in \pi_1(\mathbb{C}^*, 1)$ by the following formula:

$$\exists \nu_1 (E_L(\lambda_1, \nu_1) = \mu_1 \ \& \ E_L(\lambda_2, \nu_1) = 0) \ \&$$
$$\exists \nu_2 (E_L(\lambda_1, \nu_2) = 0 \ \& \ E_L(\lambda_2, \nu_2) = \mu_2) \ \&$$
$$\mu = E_L(\lambda_1, \lambda_2)$$

Take $\mu_1 = a\mu_0, \mu_2 = b\mu_0$. Non-degeneracy of E_L implies that $E_L(\lambda_1, \nu_1) = a\mu_0 \ \& \ E_L(\lambda_2, \nu_1) = 0$ implies $\nu_1 = a\lambda_2$ and similarly $\nu_2 = b\mu_0$. Then bilinearity of form E_L implies $\mu = ab\mu_0$. □

IV.6.2 A classical construction of the Chern class of a line bundle

In this subsection we present a definition of a Chern class of a line bundle over an Abelian variety by Mumford [Mum70], and show it is equivalent to ours.

Recall $L = L_A$ is a line bundle over A, and L_A^* is the corresponding $\mathbb{C}^*$-bundle on A; following Mumford, consider their universal covering spaces; let $\widetilde{L_A}$ and $\widetilde{L_A^*}$ be the universal covering spaces of the line bundle L_A and the $\mathbb{C}^*$-bundle L_A^*, respectively. The spaces $\widetilde{L_A}$ and $\widetilde{L_A^*}$ are in fact line bundles over $V = \widetilde{A}$, and, moreover, the line bundle $\widetilde{L_A}$ is the pullback of L_A along the map p, i.e. $\widetilde{L_A} = p^* L_A$. The inclusion map $i : L_A^* \hookrightarrow L_A$ induces a map $i_* : \widetilde{L_A^*} \to \widetilde{L_A}$. The complex structure on A and the bundles involved induces complex structures on $V, \widetilde{L_A}, \widetilde{L_A^*}$.

With this choice of complex structure, the induced universal covering map $\widetilde{\mathbb{C}} \to \mathbb{C}^*$ becomes the exponential map; and the map i_* is the identity on the base $V = \widetilde{A}$, and on the fibre it is the exponential map $\exp : \mathbb{C} \to \mathbb{C}^* \hookrightarrow \mathbb{C}$. Thus, we may choose trivialisations $\widetilde{L_A^*} \cong V \times \mathbb{C}$ and $\widetilde{L_A} = V \times \mathbb{C}$, so that the map i_* looks as follows:

$$i_* : \widetilde{L_A^*} \to \widetilde{L_A}$$
$$(v, z) \to (v, e^{2\pi i z}).$$

The trivialisation also allows us to consider the action of $\pi_1(A, 0) = \Lambda \subset V$ on $\widetilde{L_A^*}$ by automorphisms φ_γ

$$\varphi_\gamma(v, z) = (v + \gamma, f_\gamma(v) + z),$$

where f_γ is a function depending on $\gamma \in \Lambda$. Then

$$e_\gamma(z) = e^{2\pi i f_\gamma(z)}$$

describes the action of $\pi(A, 0)$ on the line bundle L_A.

According to Mumford [Mum70, Proposition, p.18] the element E of $H^2(\pi_1(A(\mathbb{C})), \mathbb{Z})$ associated to L_A can be defined as

$$E : \pi_1(A(\mathbb{C}), 0) \times \pi_1(A(\mathbb{C}), 0) \to \pi_1(\mathbb{C}^*, 1)$$

$$E : \Lambda \times \Lambda \to \mathbb{Z}$$
$$E : (\gamma_1, \gamma_2) = f_{\gamma_1}(v + \gamma_2) + f_{\gamma_2}(v + \gamma_1 + \gamma_2) - f_{\gamma_1}(v + \gamma_1) - f_{\gamma_2}(v)$$

Identifying $\pi_1(\mathbb{C}^*, 1) = \mathbb{Z}, \pi_1(A(\mathbb{C}), 0) = \Lambda$, a direct computation then shows $[\widetilde{\gamma_1}, \widetilde{\gamma_2}] = f_{\gamma_1}(v+\gamma_2) + f_{\gamma_2}(v+\gamma_1+\gamma_2) - f_{\gamma_1}(v+\gamma_1) - f_{\gamma_2}(v)$ where $\widetilde{\gamma_1}$ and $\widetilde{\gamma_2}$ are arbitrary loops in L_A^* such that $p(\widetilde{\gamma_1}) = \gamma_1, p(\widetilde{\gamma_2}) = \gamma_2$.

IV.6.3 A Galois invariant Weil pairing on Tate module $T_l(A)$

For an Abelian variety defined over field k, Mumford [Mum70, §20, p.186] constructs a $Gal(\overline{\mathbb{Q}}/k)$-invariant non-degenerate canonical $\mathbb{Z}_l$-bilinear pairing

$$e_l : T_l(A) \times T_l(A) \to T_l(\mathbb{C}^*)$$

between the Tate module of an Abelian variety A and its dual $\hat{A}$ which takes values in $T_l(K^*) = \lim \mu_n$ the inverse limit of groups of roots of unity in the ground field. More generally, a line bundle L on A defines a pairing (*Riemann form*) $E^L : T_l(A) \times T_l(A) \to T_l(\mathbb{C}^*)$. The pairing is non-degenerate if the line bundle L is ample.

Variety $A \times \hat{A}$ possesses a canonical ample *Poincaré* line bundle L which defines a non-degenerate pairing $E^L : \Lambda \times \Lambda \to \mathbb{Z}$; it is a construction given at Mumford [Mum70, Theorem of Appel-Humpert and Lemmas on p.20].

Mumford [Mum70, §23, p.238, also §9, p.82] shows that the Weil pairing $e_l : T_l(A) \times T_l(A) \longrightarrow \mathbb{Z}_l$ corresponds to the analytic pairing $E^L : \Lambda \times \Lambda \longrightarrow \mathbb{Z}$ via the natural map $\lambda \longmapsto (p(\lambda/n))_n$.

We have remarked earlier that the definability of a bilinear form corresponding to the Weil pairing is an essential difference of our approach to that of Zilber [Zild]. We state a result towards this observation:

Lemma IV.6.3.1. *Let A be an Abelian variety defined over a number field $k \subset \mathbb{C}$, and let $p : U \to A(\mathbb{C})$ be the universal covering space of $A(\mathbb{C})$. If $\dim A > 1$ and $\operatorname{End} A = \mathbb{Z}$, then there exists three elements $\lambda_1, \lambda_2, \lambda_3 \in \Lambda = \ker p$ such that their complete L_A-type $\operatorname{tp}_{L_A}(\lambda_1, \lambda_2, \lambda_3)$ is not isolated. Therefore, the model $U_A(\overline{\mathbb{Q}})$ is not atomic in the language L_A.*

Proof. First we remark that the reduction of $L_A(A)$ to $L^{\text{lin}}(A)$ is applicable to all Abelian varieties, so consider the structure $U_A(\mathbb{C})$ in the language $L^{\text{lin}}(A)$; thus we have chosen a point $O \in U$ above a point $0 \in A(k)$ and consider the linear language L^{lin}. Since $\dim T(A) > 2$, there exists two linearly independent elements of Λ such that $e_l((p(\lambda_1)/j)_j, (p(\lambda_2/j)_j)) = 1 \in T(\mathbb{C}^*)$; however, quantifier elimination implies that an L_A-formula may express only linear dependencies as well as Galois dependencies between $p(\lambda_1/N)$ and $p(\lambda_2/N)$, for some bounded N; this implies it may not isolate the type of this tuple (over O). The $L^{\text{lin}}(A)$-type of the pair (λ_1, λ_2) corresponds to the L_A-type of the triple $(\lambda_1, \lambda_2, \lambda_3)$ where $\lambda_3 = O$. □

IV.7 Conjectures

Let us now make several conjectures.

IV.7.1 General conjectures

The motivations of Chapter I and Proposition IV.1.0.9 tempt us to make the following conjectures; however, we do not believe they hold in this generality. Rather, we hope that the conjectures hold in some particular situations and thereby provide a link between the geometry and arithmetics of some particular classes of algebraic varieties. Neither could we make the conjectures more precise in this general form. We devote the rest of the section to stating some precise variants of these conjectures.

Conjecture IV.7.1.1. *Let V be the category of varieties over an algebraically closed field K. Then there is a collection of algebraic properties of abstract fundamental functors such that any two functors satisfying all those properties are naturally equivalent up an automorphism of V.*

The following should hold only for certain classes of varieties A.

Conjecture IV.7.1.2. *Let A be a smooth variety defined over $\mathbb{Q}$. Then the universal covering space U_A of $A(\mathbb{C})$ belongs to a naturally axiomatisable uncountably categorical excellent $L_{\omega_1\omega}(L_A)$-class possessing arbitrary large models.*

Recall that in Chapter III we show that U_A belongs to a naturally axiomatisable model stable $L_{\omega_1\omega}$-class; I was not able to restate those results of Chapter III as a universality property of a fundamental groupoid functor in a reasonably clear and non-technical form.

Remark IV.7.1.3. The proofs given in this thesis dealt only with the case of *countable* field K. As was explained in the introduction, the cardinality of K presents an essential obstruction to the proofs; however, the theory of excellent classes shows we only need more *arithmetic* information about the points of A over *countable* fields to prove the statements for fields of *large* cardinalities.

IV.7.2 A conjecture for elliptic curves

Let E be an elliptic curve defined over a Galois number field k, and let $A = \bigsqcup_{\sigma\in \mathrm{Gal}(k/\mathbb{Q})} E^\sigma$ be the disjoint union of Galois conjugates of E. Then A is a variety defined over $\mathbb{Q}$. It is convenient to let the set $A(K)$ carry an $L_A(\mathbb{Q})$-structure where we interpret $x \sim_H y$ as equality $x = y$, and $x \sim_Z y$ as that $x, y \in A(K)$ lie in the same K-irreducible component of variety $Z(K) \subset A(K)$.

Let $\mathfrak{E}$ be the full subcategory of the category V of varieties over an algebraically closed field K consisting of all Cartesian powers of $E^\sigma, \sigma \in \mathrm{Gal}(k/\mathbb{Q})$:

$$\mathcal{O}b\,\mathfrak{E} = \{\sigma E^n : n \geqslant 0, \sigma \in \mathrm{Gal}(k/\mathbb{Q})\}, \quad \mathcal{M}or_{\mathfrak{E}}(X,Y) = \mathcal{M}or_{\mathrm{Var}/K}(X,Y)$$

Then, $\mathrm{Aut}_{L_A}(A(K))$ acts on the category $\mathfrak{E}$ by automorphisms.

We express the following conjecture in terms of model theory and the fundamental groupoid functors; there is no convenient way to express the conjecture in terms of the Galois action on $\mathrm{Ext}^1_{\mathrm{End}E\text{-mod}}(E(K), \mathbb{Z}^2)$. The model-theory conjecture forces us to consider the category *up to an L_A-automorphism on $A(K)$, and not* $\mathrm{Aut}(K/\mathbb{Q})$, although neither of these groups is intrinsic to $\mathfrak{E}$.

Conjecture IV.7.2.1. *1. The first-order L_A-theory of U_A and an $L_{\omega_1\omega}(L_A)$-axiom "for any $a_0 \in U_A$ the group $x \sim_\pi a_0$ is isomorphic to $\mathbb{Z}^2$" define an uncountably categorical class, and has arbitrary large models.*

2. if $\underline{\Omega}, \underline{\Omega}' : \mathfrak{E} \to \mathfrak{Sets}$ are abstract fundamental groupoid functors and for arbitrary $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$

(a) $\Omega_{x,x}(\sigma E) \cong \Omega'_{x,x}(\sigma E) \cong \mathbb{Z}^2$ for any $x \in \sigma E(K)$,

(b) $\Omega(\sigma E)$ is connected

then there exists an automorphism $\tau : \mathfrak{E} \to \mathfrak{E}$ induced by an L_A-automorphism of $A(K) = \bigsqcup_{\sigma\in \mathrm{Gal}(k/\mathbb{Q})} E^\sigma(K)$ such that $\underline{\Omega}$ and $\underline{\Omega}' \circ \tau$ are equivalent (as 2-functors).

IV.7.3 A fundamental groupoid functor over $\mathbb{F}_p$.

The characteristic of the field is not essential for the formulation of Proposition IV.1.0.11: the notion of an étale morphism is well-defined for an arbitrary scheme. Nor does it seem to be essential for the proof of Proposition IV.1.0.9: some results towards describing the image of Galois and Kummer theory (Artin-Schreier theory) are known in positive characteristic also [Ser00].

Considerations above allow us to make a conjecture on the universality and existence of an abstract fundamental groupoid of an elliptic curve and the multiplicative group of $\bar{\mathbb{F}}_p$; we state it precisely only for $\bar{\mathbb{F}}_p$.

The group $\bar{\mathbb{F}}_p^*$ carries a structure of $\mathbb{Z}[\frac{1}{p}]$-module; $n \in \mathbb{Z}$ acts by $x^n : \bar{\mathbb{F}}_p^* \to \bar{\mathbb{F}}_p^*$ and $x^{1/p} : \bar{\mathbb{F}}_p^* \to \bar{\mathbb{F}}_p^*$ is the inverse of the bijective map provided by the Frobenious morphism $x^p : \bar{\mathbb{F}}_p^* \to \bar{\mathbb{F}}_p^*$.

A finite étale morphism $\varphi : \bar{\mathbb{F}}_p^* \to \bar{\mathbb{F}}_p^*$ has form of a polynomial $ax^n, a \in \bar{\mathbb{F}}_p$ where n is prime to p.

Conjecture IV.7.3.1 (The universal covering space of $\bar{\mathbb{F}}_p^*$). *There exists a notion of a path on the variety $\bar{\mathbb{F}}_p^*$, and it is unique. By this we mean the following.*

1. *Let $\mathfrak{C}$ be the full subcategory of Cartesian powers of the field $\bar{\mathbb{F}}_p$ in the category of varieties defined over $\bar{\mathbb{F}}_p$. There exists an abstract fundamental groupoid functor $\underline{\Omega} : \mathfrak{C} \to \mathfrak{S}ets$ such that*
   (a) *$\Omega(\bar{\mathbb{F}}_p^*)$ is connected*
   (b) *$\Omega_{1,1}(\bar{\mathbb{F}}_p^*) \cong \mathbb{Z}[\frac{1}{p}]$*
   Moreover, any two such functors are equivalent up to the Galois group action on $\mathfrak{C}$.
2. *There exists an extension*

$$0 \longrightarrow \mathbb{Z}[\tfrac{1}{p}] \longrightarrow V \longrightarrow \bar{\mathbb{F}}_p^* \longrightarrow 1$$

of $\mathbb{Z}[\frac{1}{p}]$-modules such that V is uniquely divisible. Moreover, any two such extensions are conjugated by action of $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ on $\bar{\mathbb{F}}_p^*$.

The algebraic fundamental group $\hat{\pi}_1^{\mathrm{alg}}(E(\bar{\mathbb{F}}_p), 0)$ is isomorphic to either of $\prod_{l\neq p} \mathbb{Z}_l \times \mathbb{Z}_l$ or $\mathbb{Z}_p \times \prod_{l\neq p} \mathbb{Z}_l \times \mathbb{Z}_l$ is known ([SGA1, Expose XI, Théorème 2.1, p.288], cf. also [Sil85, Ch.V§3,Th.3.1] for elliptic curves). This allows to make a conjecture about elliptic curves by requiring $\Omega_{0,0}(E) \cong \mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}[\frac{1}{p}]$ instead of 2.

A more interesting question is whether there may exist a notion of a path *irrespective of the characteristic of the ground field.*

IV.7.4 A conjecture for Abelian varieties

For Abelian varieties we make the following conjecture. We explain below why it does not hold for the universal cover of an Abelian variety itself.

Conjecture IV.7.4.1. *Let L^* be a homogeneous ample line bundle over a simple Abelian variety X of odd dimension or dimension 2 or 6, and assume* $\operatorname{End} X = \mathbb{Z}$. *Then the universal covering space U_{L^*} as an L_A-structure is a member of a naturally axiomatisable categorical $L_{\omega_1\omega}(L_A)$-class possessing arbitrary large models.*

The reason for choosing the universal covering space of L^*_A can be seen on two levels. General model-theoretic consideration of Zilber's programme [Zilc] on "logically perfect structures" say that in such a *categorical* situation, in a structure, here related to the Abelian variety, one should *be able to describe in a straightforward way the notions essential for the development of the theory of Abelian variety*, and indeed, any book on Abelian varieties devotes the first half to the line bundles on them; from that point of view, it is not surprising for us that, in particular, one needs to consider line bundles to define the Galois-invariant *Weil pairing* which is essential for us. Galois-invariant Weil pairing provides an explicit technical reason: the proof of Proposition IV.1.0.9 for A an Abelian variety would break in §IV.7.4 exactly due to the fact that the Galois action on Tate module has to stabilise the Weil pairing; on the other hand we have shown in §IV.6.1 that Weil pairing is definable in $L_A(L^*)$.

On a more technical level, the above could be clarified as follows. Proposition IV.1.0.9 is proved by induction. the base is that, up to Galois action, there is a unique extension of E_{tors}, the group of torsion points of an elliptic curve, of a particular kind. In §, the proof requires a result showing that, up to finite index, several elements of Tate module are conjugated iff they satisfy the same linear equations. For Abelian varieties of dimension 2 and higher, this is no longer true: the Galois group preserves a quadratic form on the Tate module, called the Weil pairing. Therefore, in an analogue of Proposition IV.1.0.9 one needs to consider a small class of extensions. This is done by enriching the language to be able to express the restrictions on the orbits (up to finite index, so to say).

The conjecture shows that there is no a priori reason to believe that the notion of universal covering space of a particular variety is uncountably categorical in the natural language; in the language of functors, this says that there is no a priori reason the fundamental groupoid functor restricted to the category of Cartesian powers of a particular variety has the universality property. Instead, we have to consider a suitably "self-sufficient" variety, geometry of which is rich enough to see explicitly the theory of that variety; on more technical level, all the restrictions on the Galois action should be seen *explicitly* via the geometry of the underlying variety.

The reason behind the restriction $\mathrm{End}\, A = \mathbb{Z}$ and the restrictions on dimension is that there is a theorem in Serre [Ser00] which says that for such varieties, the image of Galois action is of finite index in the group stabilising the Weil pairing on Tate module. Kummer theory is also known for all Abelian varieties [Ber88].

On the other hand, conjecturally the image of the Galois representation is equal to Mumford-Tate group; it is natural to ask whether that conjecture fits in the context here; perhaps one has to change the definition of language L_A to allow for more expressive power.

IV.7.5 A conjecture for Shimura curves

Arithmetic of Shimura curves is well-studied; in particular, for Shimura curves, there is a quite explicit description of Galois action analogous to the results on the Galois action on the Tate module of an elliptic curve: for curves without complex multiplication it is a result of Ohta ([Oht74], also [Cla03, Theorem 123, p. 90]); for curves with complex multiplication this is implied by the explicit description of Galois group provided by the theory of complex multiplication. We thank A.Yafaev for pointing and explaining us those results. Those results justify us to make a conjecture that

Conjecture IV.7.5.1. *For a Shimura curve S defined over $\mathbb{Q}$, the first-order L_A-theory of U_S and an $L_{\omega_1\omega}(L_A)$-axiom "for any $a_0 \in U_S$ the group $x \sim_\pi a_0$ is isomorphic to $\pi_1(S, p(a_0))$" define an uncountably categorical class, and have arbitrary large models.*

Assuming that the quantifier elimination holds for Shimura curves, we may use Condition IV.2.0.15 to state a conjecture on the arithmetic of Shimura curves in a very explicit manner.

Chapter V

Appendices

V.1 Appendix A: Basic notions of homotopy theory

Here we introduce basic notions of homotopy theory—that of a homotopy, of a fibration, a covering, a path, a covering homotopy property and a path-lifting property, and the notion of a fundamental group and a universal covering space.

V.1.1 Homotopy and Coverings

We introduce basic notions of homotopy theory, and some analogous notions of complex algebraic geometry. Exposition follows [Nov86, Ch.4,§§2-4].

In this § we assume that all topological spaces are sufficiently nice, i.e. Hausdorff, locally connected and locally linearly connected.

Homotopies and paths

Let X, Y be topological spaces. A *continuous homotopy of a map* $f : X \to Y$, or simply a *homotopy*, is a continuous map

$$F(x,t) : X \times I \to Y, x \in X, a \leqslant t \leqslant b$$

of a *cylinder* $X \times I$, where $I = [a, b] = \{t : a \leqslant t \leqslant b\}$ is the closed interval of real line from $a \in \mathbb{R}$ to $b \in \mathbb{R}$, and which coincides with f on *boundary* $X \times \{a\}$

$$F(x, a) = f(x), x \in X.$$

Two maps $f, g : X \to Y$ are *homotopic* if there is *a continuous homotopy F of f to g* such that

$$F(x, a) = f(x),$$
$$F(x, b) = g(x).$$

Being homotopic is obviously an equivalence relation; a *homotopy class* is a class of maps $f : X \to Y$ homotopic to each other.

If maps f and g coincide on a point, $f(x_0) = g(x_0)$, then one often requires *connecting homotopy to fix* $f(x_0)$, that is, $F(x_0, t) = f(x_0) = g(x_0)$; this is called *a homotopy fixing* $f(x_0)$.

A homotopy between two points is called a *path*; thus, a *path* γ *in* Y is just a continuous map $\gamma : [0, 1] \to Y$; a path γ is *trivial* iff $\gamma(t) = y_0, 0 \leqslant t \leqslant 1$ for all t and some point $y_0 \in A$. Thus, two points are homotopic iff they can be joined by a path. A path is always homotopic to its endpoint, the connecting homotopy just contracts the path by itself. To get a non-trivial notion of homotopy of paths, one usually considers only homotopies of paths fixing the ends. Thus, we say *two paths are fixed point homotopic* iff there is a connecting homotopy fixing their endpoints.

Covering homotopies and fibrations

Consider continuous maps $p : X \to Y$ and an arbitrary map $f : Z \to Y$. The map f is *covered by a map* $g : Z \to X$ iff $p \circ g = f$.

Definition V.1.1.1. *A map $p : X \to Y$ is called a* fibration *iff for any space Z any homotopy $F : Z \times I \to Y$ covered at the initial time $t = a$, can be covered at all times $a \leqslant t \leqslant b$ by some homotopy $G : Z \times I \to X$ so that $p \circ G(z, t) = F(z, t), G(z, a) = g(z)$. That is, if map $f(z) = F(z, a) : Z \to Y$ is covered by a map $g : Z \to X$, $f(z) = F(z, a) = p \circ g(z), z \in Z$, then there exist a homotopy $G : Z \times I \to X$ covering $F : Z \times I \to X$,*

$$G(z, a) = g(z)$$

$$F(z, t) = p \circ G(z, t).$$

Homotopy G is called a covering homotopy with initial condition *g. We also say that homotopy F* lifts *to homotopy G, and that fibration $p : X \to Y$ has* lifting property.

Quite often one weakens the definition by restricting Z to a subclass of spaces; an example of an important notion of this type is when $Z = I^n$ is required to be a direct product of intervals.

In all most important cases a covering homotopy can be constructed with the help of a *homotopical connection*, i.e. a unique recipe to cover an arbitrary homotopy of a point $y \in Y$ (i.e. a path γ in Y, $\gamma(a) = y$) by a path in X starting from an arbitrary point $x_0 = \gamma(a) \in X, y = p(x_0)$. The recipe should continuously depend on the path in Y and on the starting point of the covering path in X. Continuity on these variables ensures that the covering homotopy property for paths can be extended to arbitrary (in some reasonable sense) spaces Z.

If $p : X \to Y$ is a fibration, then the map p is called a *projection*, X the *total space*, Y *the base*, and $F_y = p^{-1}(y), y \in Y$ a *fibre* of the fibration p. Using the existence of a connection, one can prove that all the fibres F_y of a fibration of a base space Y are homotopically equivalent provided any two points of Y are homotopic, i.e. if space Y is linearly connected.

Given a connection and a path γ in the base of a fibration $p : X \to Y$, we get a map from fibre F_y to $F_{y'}$ above the ends of the path γ: a point $x \in F_y$ goes to the endpoint of the unique lifting $\tilde{\gamma}_x$ of path γ to X starting at x:

$$x \in F_y \mapsto \tilde{x}' = \gamma_x(1) \in F_{y'}, \text{ where } \tilde{\gamma}_x(0) = x.$$

The point x' varies continuously with x, and in fact, properties of a connection ensure that $x \mapsto x'$ is a homotopy equivalence between fibres F_y and $F_{y'}$. The map from F_y to $F_{y'}$ may depend on the path γ and not only its homotopy class; if the correspondence $x \to x'$ depends only on homotopy class of the path γ between points x and x', the fibration is called *flat*. An important case of a flat fibration we define next.

Definition V.1.1.2. *A* covering $p : X \to Y$ *is a fibration with a discrete fibre* F*, i.e.* F *is a space such that all its subsets are open and any point* $y \in Y$ *has a open neighbourhood* $U, y \in U \subset Y$ *such that the full preimage of* U *is homeomorphic to a direct product of* $U \times F$*, where* $F = \bigcup_\alpha \{x_\alpha\}$*:*

$$p^{-1}(U) = \bigcup_\alpha U_\alpha \cong U \times F$$

and each U_α *is a homeomorphic copy of* U*.*

Subsets $U_\alpha \subset X$ are open and pairwise non-intersecting; on each of them p is a homeomorphism with U. The homotopic connection is given by the following recipe. Given a point $x_0 \in X$, to lift a path $\gamma : [0,1] \to Y$ in Y which is small enough to fit in a neighbourhood U, we just take path $\tilde{\gamma}(t) = (\gamma(t), x_\alpha) \subset U_\alpha$, where $x_0 = (y_0, x_\alpha) \in U_\alpha \subset U \times F$. If γ is not small enough, we split it in pieces which are small enough, and lift it piece by piece.

If one drops the requirement that fibre F is discrete, then one get the notion of a *locally trivial fibration*; the proof of the covering homotopy property requires an argument, and is not a priori clear; such a fibration has homeomorphic fibres. In more detail, a map $p : X \to Y$ is called *locally trivial fibration* iff any point $y \in Y$ of the base is contained in a neighbourhood U_α such that $p^{-1}(U_\alpha) \subset X$ is homeomorphic to a direct product $U_\alpha \times F$ via a homeomorphism $\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F$ compatible with the projection $\mathrm{pr} \circ \varphi_\alpha = p$. There is also a compatibility condition on the behaviour of φ_α's on the intersection $V_{\alpha\beta} = U_\alpha \cap U_\beta$; there are two homeomorphisms corresponding to the intersection $V_{\alpha\beta} = U_\alpha \cap U_\beta$:

$$\varphi_\alpha : p^{-1}(V_{\alpha\beta}) \to V_{\alpha\beta} \times F,$$

$$\varphi_\beta : p^{-1}(V_{\alpha\beta}) \to V_{\alpha\beta} \times F.$$

The map $\lambda_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1} : V_{\alpha\beta} \times F \to V_{\alpha\beta} \times F$ leaves the fibres invariant setwise, and thus has form

$$\lambda_{\alpha\beta}(\omega, f) = (\omega, \hat{\lambda}_{\alpha\beta}(\omega)(f)), f \in F, w \in V_{\alpha\beta},$$

where $\hat{\lambda}_{\alpha\beta}(\omega) : F \to F$ is a homeomorphism of the fibre continuously depending on point ω. In a neighbourhood $W = U_1 \cap U_2 \cap U_3$ we have

$$\lambda_1 \circ \lambda_2 \circ \lambda_3 \equiv 1.$$

The maps λ_{ij} are called *gluing functions*. If we know the gluing functions $\lambda_{\alpha\beta}$ for some covering of a space Y by neighbourhoods U_α's, and the gluing functions satisfy the condition above, then we can uniquely reconstruct the fibration $p : X \to Y$. Obviously, we require that above each neighbourhood U_α the fibration decomposes into the direct product.

The notions of a locally trivial fibration and the general notion of a fibration are fundamental for the theory of manifolds, differential topology, geometry and their applications.

V.1.2 Fundamental Group, Functoriality and Long Exact Sequence of a Fibration

Definition of homotopy groups $\pi_n(X, x_0), n \geqslant 0$

Let S^n be the sphere in $\mathbb{R}^n$ defined by

$$x_1^2 + x_2^2 + \ldots + x_n^2 = 1,$$

with a distinguished point $s_0 = (1, 0, \ldots, 0)$. Then, S^1 is a circle with a basepoint, and S^0 is just a point.

Definition V.1.2.1. *For $n > 0$, the set of homotopy classes of maps of basepoint spaces $(S^n, s_0) \to (X, x_0)$ is called n-th* homotopy group *and is denoted $\pi_n(X, x_0)$. We describe group operation only for $n = 1$. For $n = 1$, the group operation is* *given by concatenation of paths, i.e. by the path which first follows the first path, and then goes along the second path:*

$$\gamma(e^{\pi it}) = \gamma_1(e^{2\pi it}), 0 \leqslant t \leqslant 1,$$

$$\gamma(e^{\pi it}) = \gamma_1(e^{2\pi i(t-1)}), 1 \leqslant t \leqslant 2.$$

For $n = 0$, the "0-th homotopy group" $\pi_0(X, x_0)$ is not a group, but is just a set with a distinguished element; it is the set of all the connected components of X. It is customary to call it a group, although it is an abuse of language.

Thus, the fundamental group $\pi_1(X, x_0)$ consists of fixed homotopy classes of loops based at the point x_0; and the group operation is just concatenation of paths; it is well-defined on the homotopy classes.

The fundamental group is *a covariant functor on the category of basepoint topological spaces*. That means that each map of basepoint spaces

$$f : X \to Y, x_0 \to y_0$$

gives rise to a map of fundamental groups

$$f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0).$$

The homomorphisms are defined sending each map $\gamma : (S^n, s_0) \to (X, x_0)$ into the composition $f \circ \gamma : (S^n, s_0) \to (Y, y_0)$.

The correspondence $f \mapsto f_*$ is *natural*, which means that a composition of maps of basepoint spaces gives rise to the composition of corresponding maps; that is, for maps $f : X \to Y, g : Y \to Z, f(x_0) = y_0, g(y_0) = z_0$ of basepoint spaces we have

$$(f \circ g)_* = f_* \circ g_*.$$

This correspondence is well-defined for any n, including $n = 0$.

The presence of a basepoint is essential for naturality.

The short exact sequence of fibration

The following observation is very important as for development of the theory of homotopy groups. It allows one to calculate the fundamental group of a fibration; the fact that this is possible is very important for the methods of the paper.

Property V.1.2.2. *To each fibration*

$$p : X \to Y, p^{-1}(y_0) = F_0, x_0 \in F_0$$

there corresponds a long exact sequence of homotopy groups

$$\to \pi_n(F_0, x_0) \longrightarrow \pi_n(X, x_0) \xrightarrow{p_*} \pi_n(Y, y_0) \xrightarrow{\partial} \pi_{n-1}(F_0, x_0) \longrightarrow \pi_{n-1}(X, x_0) \to .$$

In particular, for $n = 0, 1$ the end of the sequence looks like

$$\longrightarrow \pi_1(F_0, x_0) \longrightarrow \pi_1(X, x_0) \xrightarrow[p_*]{} \pi_1(Y, y_0) \xrightarrow[\partial]{} \pi_0(F_0, x_0).$$

For a fibration with a connected fibre we have $\pi_0(F_0, x_0) = 0$, and thus we get an exact sequence

$$\pi_1(F_0, x_0) \longrightarrow \pi_1(X, x_0) \xrightarrow{p_*} \pi_1(Y, y_0) \longrightarrow 0$$

Example V.1.2.3. *If $F_0 = \mathbb{C}^*$, then X is called a $\mathbb{C}^*$-*bundle over Y*. In this case $\pi_1(\mathbb{C}^*, 1)$ lies in the centre of $\pi_1(Y, y_0)$ and we have an exact sequence*

$$\mathbb{Z} \longrightarrow \pi_1(X, x_0) \xrightarrow{p_*} \pi_1(Y, y_0) \longrightarrow 0.$$

If further $\pi_2(Y, y_0) = 0$, we have a central extension of groups

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(X, x_0) \xrightarrow{p_*} \pi_1(Y, y_0) \longrightarrow 0.$$

Such extensions are classified by $H^2(\pi_1(Y, y_0), \mathbb{Z})$. The extension corresponding to L is sometimes referred to as the Chern class of L.

This fibration sequence is essential for us to prove the geometric properties of a Zariski-type étale topology on the universal covering space U of an algebraic variety $A(\mathbb{C})$. We also use the sequence to abstractly reconstruct Weil pairing.

V.1.3 Regular Coverings and Universal Covering Spaces

Let $p : X \to Y$ be a covering. A *regular covering* is a covering such that $p_*\pi(X) \subset \pi_1(Y)$ is a normal subgroup.

The biggest regular covering is called *universal covering space* X of base Y; it is a covering such that $\pi_1(X)$ is a trivial group. It can be shown that the universal covering space exists for many spaces, in particular it exists for all complex algebraic varieties (in complex topology). The group $\Gamma = \pi_1(X, x_0)$ acts freely and discretely on X, and its orbits coincide with the fibres $p^{-1}(y)$.

Deck Transformations of a covering

Let $p : X \to Y$ be a covering. We say that a continuous map $g : X \to Y$ is a *deck transformation* iff $p \circ g(x) = p(x), x \in X$. If X is connected, then a deck transformation g is determined by the image of a point; indeed, if two continuous deck transformations $g_1, g_2 : X \to Y$ coincide on a point x, $g_1(x) = g_2(x)$, then they coincide in a admissible open neighbourhood $U_\alpha \subset X$ of x (notation of Definition V.1.1.2). The space X can be covered by such neighbourhoods, and this proves g_1 and g_2 coincide on a connected component of X containing x.

On the other hand, the path-lifting property allows one to define a natural action of the fundamental group $\pi_1(Y, y_0)$ on the fibre $p^{-1}(y_0)$: for a loop $\lambda \in \pi_1(Y, y_0), \lambda(0) = \lambda(1) = y_0$, we set

$$\lambda \cdot x = \tilde{\lambda}(1), \tilde{\lambda}(0) = x, p(\tilde{\lambda}) = \lambda.$$

where $\tilde{\lambda}$ is the lifting of λ starting at point $x, p(x) = y_0$.

Therefore we get an action of the fundamental group $\pi_1(Y, y_0)$ on a covering space of a basepoint base space Y.

In fact, the covering is completely characterized by the fundamental group, as the following important theorem shows:

Fact V.1.3.1 (Galois correspondence between coverings and subgroups). *There is a bijective correspondence between subgroups of $\pi_1(Y, y_0)$ and basepoint covering spaces of (Y, y_0).*

For a subgroup $H < \pi_1(Y, y_0)$, the corresponding covering is denoted $p^H : \tilde{Y}^H \to Y$ with a basepoint $y^H \in \tilde{Y}^H$. The correspondence is natural, i.e. if $H_1 < H_2$, then there is a well-defined covering

$$p^{H_1,H_2} : Y^{H_1} \to Y^{H_2}, y^{H_1} \to y^{H_2}.$$

The choice of basepoint is important for the functoriality of the interdependence; otherwise there is no unique way to choose a covering corresponding to embedding $H_1 < H_2$.

In particular, there is a covering corresponding to the trivial subgroup $H = 0$; it is called *the universal covering space* of X, and in next § we give an explicit construction for the universal covering space in terms of paths. We also give an explicit construction for deck transformation.

A covering $\tilde{Y}^H$ corresponding to a normal subgroup $H \lhd \pi_1(Y, y_0)$ is called *regular*. The regular covering have the property that any two points of a fibre are conjugated by a deck transformation. For H normal, the group of deck transformations in this case is the group $\pi(Y, y_0)/H$, and it acts transitively on the fibres of covering $p : \tilde{Y}^H \to Y$.

Universal Covering Spaces and Deck Transformation

Definition V.1.3.2. *A covering $p : X \to Y$ is called the* universal covering *iff the space X is* simply connected*, i.e. its fundamental group $\pi_1(X, x) = 0$ is trivial.*

The universal covering space is usually denoted by $\tilde{Y}$.

Given a basepoint $y_0 \in Y$, we can construct the universal covering space as the set of homotopy classes of paths leaving the basepoint:

$$\tilde{Y} = \{\gamma : [0,1] \to Y : \gamma(0) = y_0\}/\{\text{homotopy fixing } \gamma(0)\}$$

with a basepoint $\tilde{y}_0$ being the trivial path in Y

$$\tilde{y}_0(t) = y_0, \text{ for all } t.$$

Each continuous transformation of basepoint spaces $f : (Y_1, y_1) \to (Y_2, y_2)$ induces a transformation on the covering spaces $\widetilde{f} : (\tilde{Y}_1, \tilde{y}_1) \to (\tilde{Y}_2, \tilde{y}_2)$, $\widetilde{f}(\tilde{y}_1) = \tilde{y}_2$.

The dependance is natural, and thus we get a functor from the category of basepoint topological spaces to itself

$$(Y, y_0) \to (\tilde{Y}, \tilde{y}_0)$$

which sends a space with a basepoint to its universal covering space with a basepoint.

The fundamental group $\pi_1(Y, y_0)$ acts naturally on the space $\tilde{Y}, \tilde{y}_0$ by prefixing a path in $\tilde{Y}$ with a loop from $\pi_1(Y, y_0)$

$$(\lambda \in \pi_1(Y, y_0), \gamma \in \tilde{Y}) \mapsto \lambda \circ \gamma$$

where $\circ$ denotes the concatenation of paths. The concatenation is well-defined as $\lambda(0) = y_0 = \lambda(1) = \gamma(0)$.

V.2 Algebraic geometry: normal varieties and étale morphisms

The word "variety" is overused similarly to the word "ring", and so in this section we first define "variety" as a certain kind of a scheme with finiteness restrictions, and introduce some notions of algebraic geometry which we use in the main body of the text; those properties are algebraic generalisations of some well-known analytic and topological properties, and so we define them via those properties, in an attempt to provide most geometric definitions.

V.2.1 Varieties, subvarieties and their fields of definitions

We restrict ourselves to considering quasi-projective varieties, and we assume all our varieties embedded into the projective space $\mathbb{P}^n(K)$ for some algebraically closed field. We say that a variety, or a point thereof, are *defined over* $k \subset K$ if it is fixed by the action of $\mathrm{Aut}(K/k)$. We do not assume a variety to be irreducible.

V.2.2 Etale morphisms.

In our context the following definition of an *étale morphism* is most useful; however, it applies only in characteristic 0. The equivalence of this definition to the usual one is given in [DS98].

Definition V.2.2.1. *A morphism $f : Y \to X$ defined over a characteristic 0 field k is* étale *iff, for an embedding $k \hookrightarrow \mathbb{C}$, the induced map $f : Y(\mathbb{C}) \to X(\mathbb{C})$ is a topological covering map, with respect to the complex topology on $Y(\mathbb{C})$ and $X(\mathbb{C})$, i.e. f induces an isomorphism of topological covering spaces of $Y(\mathbb{C})$ and $X(\mathbb{C})$. The morphism $f : Y \to X$ is called étale at a point $y \in Y$ if it is an isomorphism of an neighborhood of y in Y open in the complex topology onto an open neighbourhood of x in X open in the complex topology.*

The definition above does not depend on the embedding of k to $\mathbb{C}$; this fact and the equivalence of this definition to the usual one is via an invariant local characterisation of an étale morphism [Mil80, Ch.1,Th.3.14,p.26].

Definition V.2.2.2. *A morphism $f : Y \to X$ of affine varieties defined over an algebraically closed field is called* standard etale *morphism at a point $x \in X(k)$ iff there exist functions $a_1, \dots, a_r : X \to k$ such that Y is locally described by the equation $P(x,t) = t^r + a_1(x)t^{r-1} + .. + a_{r-1}(x)t + a_r(x) = 0$ i.e.*

$$Y(k) \cong \{(x,t); P(t) = 0, x \in X(k)\}$$

and all the roots of the polynomials $P_x(t) = P(x,t)$ are simple at any geometric point $x \in X(k)$. A morphism $f : Y \to X$ of affine varieties defined over an algebraically closed field is called standard étale morphism if it is standard étale at any geometric point $x \in X(k)$.

It is evident that a standard étale morphism induces a covering map in the complex topology.

By [Mil80, Ch.1,Th.3.14,p.26], an étale morphism is locally standard etale;

Fact V.2.2.3. *Assume $f : Y \to X$ is étale in some (Zariski) open neighbourhood of y in Y. Then there are Zariski open affine neighbourhoods V and U of y and $x = f(y)$, respectively, such that $f|V : V \to U$ is a standard étale morphism.*

We remark that the notion of an étale morphism is in fact defined over arbitrary rings, and is defined via the exactness properties of the functor $h_Y = \mathrm{Hom}_X(-, Y)$ from the category of X-schemes to Y-schemes induced by morphism $f : Y \to X$.

V.2.3 Normal closed analytic sets

Definition V.2.3.1. *A closed analytic subset X of a Stein space is* normal *if any bounded meromorphic function on X is holomorphic.*

A normalisation morphism $\mathbf{n}$ of variety Y *is a morphism $\mathbf{n} : X \to Y$ from a normal variety X such that any dominant (surjective on an open subset) morphism $f : Z \to Y$ lifts up to a unique morphism $\tilde{f} : Z \to X$ such that $f = \tilde{f} \circ \mathbf{n}$.*
Any smooth closed analytic set is normal.

We only the following two properties of a normal variety:

Fact V.2.3.2. *A normalisation morphism exists for any variety, and is functorial.*

The following express that normality is a local notion:

Fact V.2.3.3 (normality is a local notion). *1. if $p : X(\mathbb{C}) \to Y(\mathbb{C})$ is a local isomorphism in complex topology, and $Y(\mathbb{C})$ is normal, so is $X(\mathbb{C})$.*
In particular,
2. if $p : X \to Y$ is étale, then X is normal iff Y is normal.

Fact V.2.3.4. *If X is smooth, then X is normal.*

Fact V.2.3.5. *Let X be a closed analytic subset of a Stein manifold, or let X be an algebraic variety. If X is connected and normal, then X is irreducible.*

V.3 Appendix: Geometric Conjectures on the convexity of the universal covering space of a complex algebraic variety

V.3.1 Shafarevich conjecture on holomorphic convexity of universal covering spaces

In [Sha94, IX§4.3] Shafarevich proposed a conjecture that the universal covering space of an algebraic variety is holomorphically convex; thus "it has many holomorphic functions".

Recall the definition of a holomorphically convexity and separability:

Definition V.3.1.1. *A complex space U is called* holomorphically separable *if for every $x_0 \in U$ there are holomorphic functions $f_1, \ldots, f_l$ on U such that x_0 is isolated in the set $\{u \in U : f_1(u) = \ldots = f_n(u) = 0\}$.*

A complex space U is called holomorphically convex *iff either of the two equivalent conditions holds:*

(i) *for any compact subset $K \subset U$ the set*

$$\hat{K} = \{u \in U : |f(u)| \leqslant \sup |f(K)| \text{ for every holomorphic function } f \text{ on } U\}$$

is compact.

(ii) *for any infinite discrete subset S of U there exist a holomorphic function $f : U \to \mathbb{C}$ unbounded on S.*

A complex space U is called Stein *iff it is both holomorphically separable and holomorphically convex.*

Another characterisation of a Stein manifold is that

Lemma V.3.1.2. *A manifold which is biholomorphic to a closed analytic set in Euclidian space $\mathbb{C}^n$; in particular the Euclidean space $\mathbb{C}^n$ itself is Stein.*

Proof. [Č85] □

Conjecture V.3.1.3 (Shafarevich[Sha94, IX§4.3]). *The universal cover of a projective variety is holomorphically convex, or even Stein.*

Another characterisation of a Stein manifold is that a manifold which is biholomorphic to a closed analytic set in Euclidian space $\mathbb{C}^n$; in particular the Euclidean space $\mathbb{C}^n$ itself is Stein.

V.3.2 Equivalence of isomorphisms of topological and analytic vector bundles over a Stein manifold

On Stein manifolds, analytical properties are often determined by pure topology, for example analytic vector bundles over a Stein manifold are isomorphic analytically iff they are so topologically. For the sake of completeness we present several properties of this kind.

Theorem V.3.2.1 (Oka's principle). *Let X be a Stein manifold. Then*

1. *Every topological fibre bundle over X has an analytic structure*
2. *If two analytic fibre bundles over X are topologically equivalent, then they are also analytically equivalent.*

Theorem V.3.2.2 (Theorem A.). *Let $V \to X$ be an analytic vector bundle over a Stein manifold X. Then for any $x_0 \in X$ there are global holomorphic sections $s_1, \ldots, s_n \in \Gamma(X, V)$ of V such that:*

If $U = U(x_0) \subset X$ is an open neighbourhood of x_0 and $s \in \Gamma(U, V)$ is a local section of V on $U(x_0)$, then there exist an open neighbourhood $V = V(x_0) \subset U$ of x_0 and holomorphic functions $f_1, \ldots, f_N$ on V such that

$$s_{|V} = f_1 s_1 + .. + f_N s_N.$$

Theorem V.3.2.3 (Theorem B.). *Let $\pi : V \to X$ be an analytic vector bundle over a Stein manifold X. Then $H^1(X, V) = 0$.*

Moreover, if $A \subset X$ is an analytic set, U is an open covering of X and a cochain $\xi \in Z^1(U, V)$ such $\xi_{\nu\mu}|_{U_{\nu\mu} \cap A} = 0$, then we can find a cochain $\eta \in C^0(U, V)$ such that $d\eta = \xi$ and $\eta|_{U_\nu} \cap A = 0$.

V.3.3 Stein factorisation and Lefschetz-type properties of algebraic varieties

In this section we use state several somewhat unexpected results about fundamental groups of algebraic varieties; arguably one may call such properties *rigidity properties*, or *Lefschetz-type properties*, or *positivity properties*.

A Zariski open subset has real codimension at least two; the following fact should not seem surprising:

Fact V.3.3.1. *Let Y be a connected* normal *complex space and $Z \subset Y$ a Zariski closed subspace. Then $\pi_1(Y - Z) \to \pi_1(Y)$ is surjective.*

Proof. Kollar, Prop.2.10.1 □

Let us state first two facts which say that morphisms of complex algebraic *normal* varieties have rather easy and well-understood *topological* structure almost everywhere, i.e. on a dense Zariski open set. It is critical for us that these topological structure allow us to understand the corresponding morphism of covering spaces.

Fact V.3.3.2. *Let $f : X \to Y$ be a morphism of irreducible* normal *algebraic complex varieties.*

Then there exist an open subset $Y^0 \subset Y$ and $X^0 = f^{-1}(Y^0)$, and a variety Z^0 such that f factorises as follows:

$$X^0 \to^{f^0} Z^0 \to^{f^{et}} Y^0$$

where

1. $Z^0 \to Y^0$ *is a finite étale morphism*
2. $X^0 \to Z^0$ *is a topological fibre bundle (in complex topology) with connected fibres*

Moreover, if $f : X \to Y$ is dominant, then $Z^0 \to Y^0$ is surjective.

Proof. Kollar, Proposition 2.8.1. □

Note that while $f^0 : X^0 \to Y^0$ is interpretable in the theory of algebraic varieties and in L_A, as indeed any morphism of algebraic varieties is, the theory may not say anything about the induced morphism $(f^0)_* : U(X^0) \to U(Y^0)$ of the universal covering spaces of $X^0(\mathbb{C})$ and $Y^0(\mathbb{C})$.

Indeed, the language allows us to speak about the *liftings*, or *induced morphisms* only for morphisms between closed subvarieties; and even then, we lift those only to the cover U_A which generally speaking is much smaller then the universal covering space of subvarieties concerned.

We find the following useful.

Corollary V.3.3.3. *Let $f : X \to Y$ be a projective morphism of irreducible algebraic varieties over Θ, and let $g \in Y(\mathbb{C})$ be a Θ-generic point. Then any open Θ-definable set intersects all connected components of generic fibre $W_g = f^{-1}(g)$.*

Proof. Such a set intersects all generic fibres of a morphism. Factoring projection through a morphism with connected fibres, which is possible by Stein factorisation, the result follows. □

Fact V.3.3.4. *Any projective morphism $f : Y \to X$ of algebraic varieties admits a factorisation $f = f_0 \circ f_1$ as a product of a finite morphism $f_0 : Y \to Y'$ and a morphism f_1 with connected fibres.*

Proof. [Har77, Ch. III, Corollary 11.5] □

Corollary V.3.3.5. *If $f : X \to Y$ is a Θ-definable morphism of Θ-definable irreducible algebraic varieties, then there exist an open Θ-definable set $Y^0 \subset Y$ and $X^0 = f^{-1}(Y^0)$ such that the relation*

> *$x, y \in X^0$ lie in the same connected component of $X_g = f^{-1}(g)$, for some $g \in Y^0$*

is Θ-definable.

Proof. Indeed, the relation is defined by $f(x) = f(y)$ and $f^0(x) = f^0(y)$, in notation of previous lemma. □

The Fact V.3.3.2 above leads to

Fact V.3.3.6. *Let $f : X \to Y$ be a morphism of normal algebraic connected complex varieties; let $X_g = f^{-1}(g), g \in X$ be a generic fibre of f over a generic point $g \in Y(\mathbb{C})$.*

Then sequence

$$f_* : \pi_1(X_g(\mathbb{C})) \to \pi_1(X(\mathbb{C})) \to \pi_1(Y(\mathbb{C}))$$

is exact up to finite index.

Moreover, if $f : X \to Y$ is dominant, then

$$f_* : \pi_1(X(\mathbb{C})) \to \pi_1(Y(\mathbb{C}))$$

is surjective up to finite index.

If X, Y and morphism $f : X \to Y$ are defined over a field Θ, then there exists an open subset $Y_0 \subset Y$ defined over Θ such that the above conclusions hold for $g \in V_0$ not necessarily Θ-generic.

Proof. Follows from Facts V.3.3.1 and V.3.3.2 and the exact sequence of the fundamental groups of a fibration, from Kollar, Proposition 2.8.1 and Kollar, Proposition 2.10.1. □

Recall $p : U \to A(\mathbb{C})$ is the universal covering of an algebraic variety A.

Recall for a subset W' of U^n, we denote $\pi(W') = \{\gamma \in \pi^n : \gamma W' \subset W'\}$. In next lemma we will drop the assumption on normality using the assumption that the universal covering space U is holomorphically convex.

Corollary V.3.3.7. *Let W' be an étale irreducible closed set, and let $V' = \operatorname{Clpr} W'$. Assume that $p(W')$ and $p(V')$ are both normal. Then $\pi(\operatorname{pr} W')$ is a finite index subgroup of $\pi(V')$, i.e. $\pi(V') \lesssim \operatorname{pr} \pi(W')$.*

Proof. By Decomposition Lemma III.1.4.1 we may assume that W' and V' are connected components of $p_H^{-1}(W(\mathbb{C}))$, $p_H^{-1}(V(\mathbb{C}))$ for some normal algebraic varieties W and V, respectively; then by properties in §V.1.1 $\pi(W') = \pi_1(W(\mathbb{C})), w)$, $\pi(V') = \pi_1(V(\mathbb{C}), v)$, for some points $w \in W(\mathbb{C}), v \in V(\mathbb{C})$.

Furthermore, we may still assume normality because it is preserved under taking preimage under an étale map. By the previous lemma $\operatorname{pr} \pi_1(W(\mathbb{C})) \lesssim \pi_1(V(\mathbb{C}))$, as required. □

V.3.4 Extending to the case of a non-normal subvariety

The above provides an explicit description of morphisms topologically, between normal algebraic varieties.

However, it is very important for us to deal with an *arbitrary* subvarieties, not necessarily normal. We do that by considering the image of the fundamental groups in the big ambient variety which is normal.

Fact V.3.4.1. *Assume A is Shafarevich.*

Let $p : U \to A(\mathbb{C})$ be the universal covering space, let $\iota : X \to A \times A$ be a closed subvariety, and let $Y = \operatorname{Clpr} X$.

Assume that

connected components of $p^{-1}(X(\mathbb{C}))$ and $p^{-1}(Y(\mathbb{C}))$ are irreducible
Then there is a sequence of subgroups of $\pi_1(A(\mathbb{C}))^2$

$$\iota_*\pi_1(X_g(\mathbb{C})) \to \iota_*\pi_1(X(\mathbb{C})) \to \iota_*\pi_1(Y(\mathbb{C})) \to 0$$

which is exact up to finite index, and the homomorphisms are those of subgroups of $\pi_1(A(\mathbb{C}))^2$.

Proof. We prove this by passing to the normalisation of varieties W and $Z = \operatorname{Clpr} W$. The assumption about the irreducibility of connected components implies that the composite maps of fundamental groups $\pi_1(\hat{W}) \to \pi(W) \to \iota_*\pi_1(W)$ and $\pi_1(\hat{Z}) \to \pi_1(Z) \to \iota_*\pi_1(Z)$ are surjective.

To show this, first note that the universal covering spaces $\tilde{\hat{W}}(\mathbb{C})$ and $\tilde{\hat{Z}}(\mathbb{C})$ are irreducible as analytic spaces; indeed, normality is a local property, and so they are normal as analytic spaces; they are obviously connected, and for normal analytic spaces connectivity implies irreducibility.

By properties of covering maps, a morphism between analytic spaces lifts up to a morphism between their universal covering spaces (as analytic spaces); thus the normalisation map $\mathbf{n}_W : \hat{W} \to W$ lifts up to a morphism $\tilde{\mathbf{n}}_W : \tilde{\hat{W}} \to U$. The normalisation morphism $\mathbf{n}_W$ is finite and closed by Hartshorne [Har77, Ch.II,§3,Ex.3.5,3.8]; therefore $\tilde{\mathbf{n}}_W$ is also, and the image of an irreducible set is irreducible. Therefore $\tilde{\mathbf{n}}_W(\tilde{\hat{W}})$ is an irreducible subset of a connected component of $p^{-1}(W(\mathbb{C}))$. Moreover, if we choose different liftings $\tilde{\mathbf{n}}_W$, we may cover $p^{-1}(W(\mathbb{C}))$ by a countable number of such sets. Now, we use the assumption that a connected component of $p^{-1}(W(\mathbb{C}))$ is irreducible to conclude that the image $\tilde{\mathbf{n}}_W(\tilde{\hat{W}})$ coincides with a connected component of $p^{-1}(W(\mathbb{C}))$. This implies that the map of fundamental groups is surjective; this may be easily seen if one thinks of a fundamental group as the group of deck transformations.

Let $\mathbf{n}_W : \hat{W} \to W$, $\mathbf{n}_{W_g} : \hat{W}_g \to W_g$ and $\mathbf{n}_Z : \hat{Z} \to Z$ be the normalisation of varieties W,W_g and Z.

By the universality property of normalisation in §V.2.3 we may lift the normalisation morphism $\mathbf{n}_{W_g} : \hat{W}_g \to W_g$ to construct a commutative diagram:

$$\begin{array}{ccccc} \hat{W}_g & \to & \hat{W} & \to & \hat{Z} \\ \downarrow & & \downarrow & & \downarrow \\ W_g & \to & W & \to & Z \end{array}$$

By functoriality of π_1, this diagram and embedding $\iota : W \to A \times A$ gives us

$$\begin{array}{ccccc}
\pi_1(\hat{W}_g) & \to & \pi_1(\hat{W}) & \to & \pi_1(\hat{Z}) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_1(W_g) & \to & \pi_1(W) & \to & \pi_1(Z) \\
\downarrow & & \downarrow & & \downarrow \\
\iota_*\pi_1(W_g) & \to & \iota_*\pi_1(W) & \to & \iota_*\pi_1(Z)
\end{array}$$

Now, g' is Θ-generic in $\hat{W}'_{g'}$; We are almost finished now. By Fact V.3.3.6 the upper row of the diagram is exact up to finite index, and $\pi_1(\hat{W}) \to \pi_1(\hat{Z})$ are surjective, up to finite index; by assumptions on W and Z, the composite morphisms $\pi_1(\hat{Z}) \to \iota_*\pi_1(Z)$ and $\pi_1(\hat{W}) \to \iota_*\pi_1(W)$ are surjective. Diagram chasing now proves that the bottom row is also exact up to finite index, and the map $\iota_*\pi_1(\hat{W}) \to \iota_*\pi_1(\hat{Z})$ is surjective up to finite index. □

Corollary V.3.4.2. *Let W' be an étale irreducible closed set, and let $V' = \mathrm{Clpr}\, W'$. Then $\pi(\mathrm{pr}\, W')$ is a finite index subgroup of $\pi(V')$, i.e. $\pi(V') \lesssim \mathrm{pr}\, \pi(W')$.*

Proof. The proof of the analogous corollary V.3.3.7 carries on verbatim, except that now we do not need the assumption of normality. □

Recall we use Lemma III.2.2.2.

V.4 Appendix C. Number theory facts

In this § we list the facts in the proof of ω-homogeneity of the standard model $0 \to \Lambda \to \mathbb{C}^{2g} \to A(\mathbb{C}) \to 0$.

V.4.1 Galois action on the torsion points of Abelian varieties.

We translate from Serre, [Resume de course de 1984-1985 et 1985-1986].

Notations and definitions

Let K be a finite extension of $\mathbb{Q}$, let $\bar{K}$ be the algebraic closure of K, let G_K be the Galois group $\mathrm{Gal}(\bar{K}/K)$.

Let A be an Abelian variety defined over K of dimension $n \geqslant 1$.

Remark V.4.1.1. For each prime number l, let T_l be the Tate module of relative to l; that is a free $\mathbb{Z}_l$-module of rang $2n$. The group G_K operates on T_l via representation

$$\rho_l : G_K \longrightarrow \mathrm{Aut}(T_l) = \mathrm{GL}_{2n}(\mathbb{Z}_l).$$

Denote $G_{K,l}$ the image of this representation; the group $G_{K,l}$ is the Galois group of "points of l^∞-division" de A.

The family of ρ_l, for l prime, defines a homomorphism

$$\rho : G_K \longrightarrow \prod_l G_{K,l} = \prod_l \mathrm{Aut}(T_l).$$

The group $\rho(G_K)$ is the Galois group of division points of A.

Theorem V.4.1.2. *(Bogomolov) There exists a finite extension K' of K such that*

$$\rho : G_K \longrightarrow \prod_l G_{K,l}$$

is surjective.

Such $\rho_l's$ are called *independent*. The field K' depends on the Abelian varieties A.

Comparison with the group of symplectic semilitudes.

A choice of a polarisation e on A equips the Tate model T_l with an alternating form of non-zero discriminant (and which is invertible for l sufficiently large); The Galois group $G_{K,l}$ is contained in the group $\mathrm{GSp}(T_l, e_l)$ of symplectic semilitudes of T_l relative to e_l.

Theorem V.4.1.3. *Assume the following holds:*

(i) The ring $\mathrm{End}\, A$ *of $\bar{K}$-automorphisms of A is $\mathbb{Z}$*

(ii) The dimension of A is odd, or equal to either 2 or 6.

Then $G_{K,l}$ is an open subgroup of $\mathrm{GSp}(T_l, e_l)$ *for each l, and is equal to* $\mathrm{GSp}(T_l, e_l)$ *for l large enough.*

Combining with theorem above, we get the property used in the proof of ω-homogeneity.

Theorem V.4.1.4. *Assuming (i) and (ii) above, $\rho(G_K)$ is an open subgroup of* $\prod_l \mathrm{GSp}(T_l, e_l)$.

For $n = 1$, this is an old result of Serre [Ser98]; a similar result holds when $\mathrm{End}\, A \neq \mathbb{Z}$.

Galois action on torsion points of elliptic curves with complex multiplication

The theorem V.4.1.3 describes the Galois action on the torsion points of an elliptic curve without complex multiplication. The lemma below does so for elliptic curves with complex multiplication.

Let $((\operatorname{End} E)_l)^*$ denote the group of unities of the l-adic completion of the ring $\operatorname{End} E$.

Lemma V.4.1.5. *Assume than elliptic curve E has complex multiplication. Then the image*

$$G_K = Gal(K(E_{tors}) : K) \to \prod ((\operatorname{End} E)_l)^*$$

is open of finite index.

Proof. [Lan78, Ch. V, §5, Fact 1] □

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