Implicit differentiation

Suppose we know some relation between x and y, e.g.

$$x^2 + y^2 = 1.$$

Here, y isn't a function of x.

But if we restrict attention to $y \geq 0$, then y is a function of x; similarly for $y \leq 0$. These functions are *implicitly* defined by $x^2 + y^2 = 1$.

Restricting to a function in this way, it makes sense to differentiate with respect to x -

$$0 = \frac{d}{dx}1 = \frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}x^2 + \frac{d}{dx}y^2 = 2x + \frac{dy}{dx}2y$$

and we conclude that, whichever function we chose,

$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

for all x at which the function is differentiable.

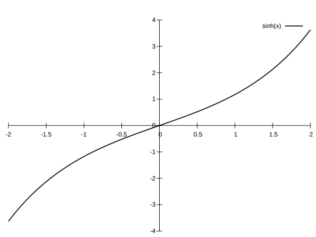
Confirm this agrees with the chain rule.

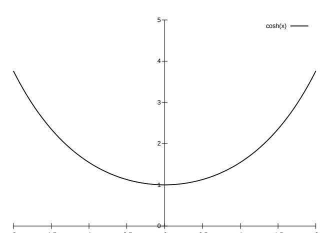
Hyperbolic functions

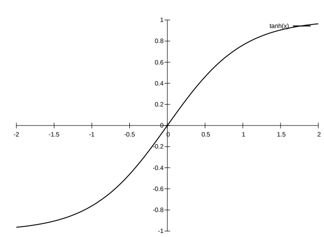
$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

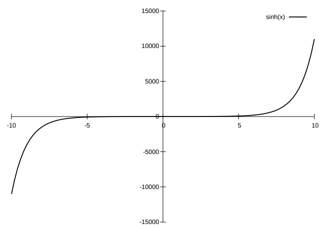
$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

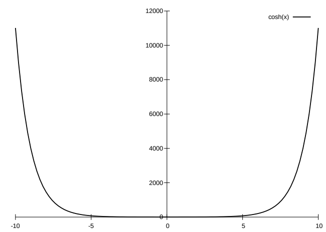
$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

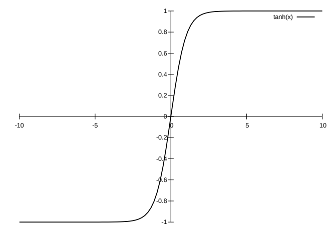




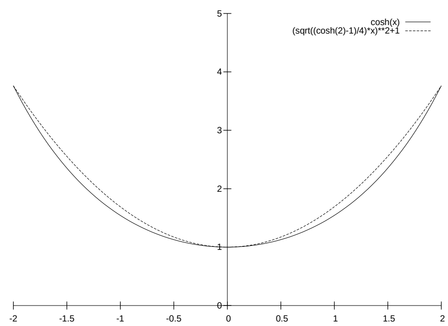






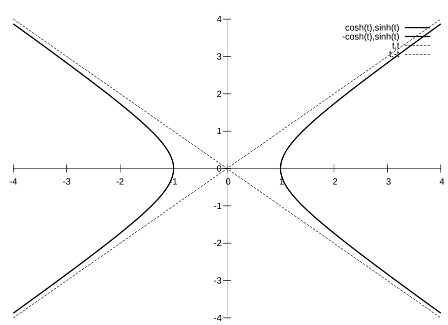


Catenary; hanging string/cable; sturdy arch.



Fundamental identity:

$$\cosh^2 - \sinh^2 = 1$$



Derivatives:

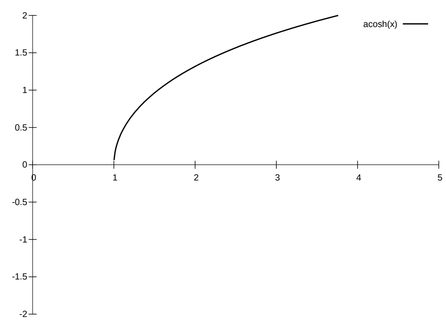
$$\frac{d}{dx}\cosh(x) = \frac{e^x - e^{-x}}{2} = \sinh(x)$$

$$\frac{d}{dx}\sinh(x) = \frac{e^x + e^{-x}}{2} = \cosh(x)$$

(So $\cosh'' = \cosh$ and $\sinh'' = \sinh$)

$$\frac{d}{dx}\tanh(x) = \frac{\sinh'(x)\cosh(x) - \sinh(x)\cosh'(x)}{\cosh^2(x)} = \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)} = \frac{1}{\cosh^2(x)}$$

Inverses: arccosh domain $[1, +\inf)$, range $[0, +\inf)$;



arcsinh domain $\mathbb{R}$, range $\mathbb{R}$;
arctanh domain $(-1, 1)$, range $\mathbb{R}$.

$$y = \cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$(e^x)^2 - 2ye^x + 1 = 0$$

$$e^x = \frac{2y \pm \sqrt{4y^2 - 4}}{2}$$

$$e^x = y \pm \sqrt{y^2 - 1}\,\mathrm{arccosh}(y) \qquad = x = \ln(y + \sqrt{y^2 - 1})$$

(taking positive square root since negative would give us negative x)

$$\mathrm{arccosh}'(\cosh(x)) = \frac{1}{\cosh'(x)} = \frac{1}{\sinh(x)}$$

but $\sinh(x) = \sqrt{\cosh^2(x) - 1}$, so

$$\mathrm{arccosh}'(x) = \frac{1}{\sqrt{x^2 - 1}}$$

Similarly,

$$\operatorname{arcsinh}'(\sinh(x)) = \frac{1}{\cosh(x)} = \frac{1}{\sqrt{1+\sinh^2(x)}}$$

$$\operatorname{arcsinh}'(x) = \frac{1}{\sqrt{1+x^2}}$$

and

$$\operatorname{arctanh}'(\tanh(x)) = \frac{1}{\tanh'(x)} = \cosh^2(x)$$

but $1 - \frac{\sinh^2}{\cosh^2} = \frac{1}{\cosh^2}$ so $\cosh^2 = \frac{1}{1-\tanh^2}$ so

$$\operatorname{arctanh}'(x) = \frac{1}{1-x^2}$$

$f(x)^{g(x)}$

$$\begin{aligned}
\frac{d}{dx}x^x &= \frac{d}{dx}\left((e^{\ln x})^x\right) \\
&= \frac{d}{dx}e^{x\ln x} \\
&= (\frac{d}{dx}(x\ln x))e^{x\ln x} \\
&= (1+\ln x)e^{x\ln x} \\
&= (1+\ln x)x^x
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}f(x)^{g(x)} &= \frac{d}{dx}\left((e^{\ln f(x)})^{g(x)}\right) \\
&= \frac{d}{dx}(e^{g(x)\ln f(x)}) \\
&= (g'(x)\ln f(x) + \frac{g(x)f'(x)}{f(x)})(e^{g(x)\ln f(x)}) \\
&= (g'(x)\ln f(x) + \frac{g(x)f'(x)}{f(x)})f(x)^{g(x)}
\end{aligned}$$

Minima and maxima

Definition: c is the minimum value of f if $f(x) = c$ for some x, and $f(x) \geq c$ for all x in $\text{dom}(f)$. We say then that f has its global / absolute minimum at x, and that f attains its minimum value at x.

Examples: 0 is minimum value of x^2, but it has no maximum value.

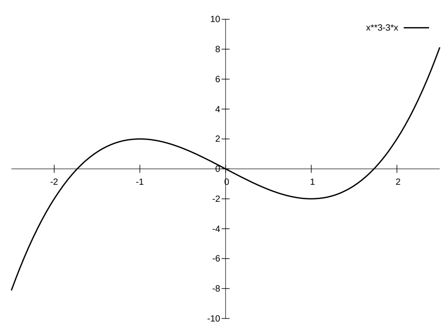
x^2 restricted to have domain $[-2, 2]$ has maximum value 4, attained at both -2 and 2.

e^x has no minimum value nor maximum value.

Definition: f has a local minimum at b in $\text{dom}(f)$ if there is an interval around b on which f is defined and $f(x) \geq f(b)$.

(Here, an "interval around b" is one which contains b and which doesn't have b as an endpoint. e.g. $(-1, 1)$ is an interval around 0, but $[0, 1)$ isn't.)

Example: $x^3 - 3x$ has a local minimum at 1 and a local maximum at -1.



$\cos(x)$ has a local maximum at $2k\pi$.

Definition: A critical point (or critical number) of f is a solution to $f'(x) = 0$, or a point in $\text{dom}(f)$ where f is not differentiable.

Fact: If f has a local min/max at b and f is differentiable at b, then $f'(b) = 0$.

So if f has a local min/max at b, then b is a critical point of f.

Remark: 0 is a critical point of x^3, but it is **not** a local min/max.

Fact: If dom(f) is a closed interval $[a, d]$ and f is continuous on $[a, d]$, then f has a minimum value and has a maximum value.

Remark: If f has a global minimum at b, then it has a local minimum at b unless b is in the boundary of dom(f).

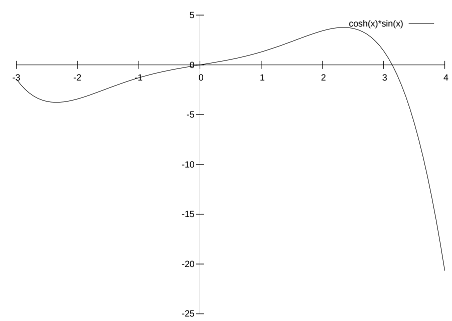
Conclusion: So if dom(f) is a closed interval $[a, d]$, and if f is continuous on $[a, d]$, then the minimum value of f is the least of the values at the endpoints and at any local minima.

So if furthermore f is differentiable on (a, d), we can find the minimum value by

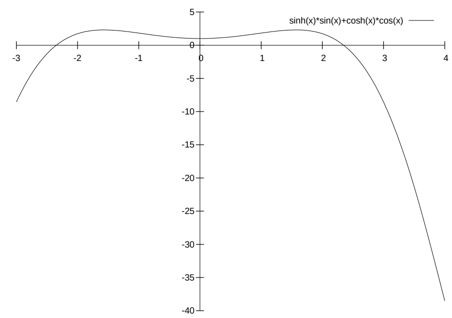
- Finding the zeros of f'
- Evaluating f at each such zero and at a and d
- The minimum value is the least of these numbers.

If f is differentiable except at a few points, we just need to also check the value of f at those points.

Example: Find the minimum value and maximum value of $f(x) = \cosh(x)\sin(x)$ as x varies between -3 and 4.



$f'(x) = \sinh(x)\sin(x) + \cosh(x)\cos(x)$. Use Newton's method to find zeroes.



$(f''(x) = \cosh(x)\sin(x) + \sinh(x)\cos(x) + \sinh(x)\cos(x) - \cosh(x)\sin(x))$
Critical points are ± 2.347
$f(2.347) = 3.764$ (this is the global max)
$f(-2.347) = -3.764$
$f(-3) = -1.421$
$f(4) = -20.67$ (this is the global min)

MVT

Theorem [Rolle's Theorem]: "What goes up and then comes down must hover instantaneously in between"

Suppose f is differentiable on (a, d) and continuous at the endpoints a and d.

Suppose $f(a) = f(d)$.

Then f has a critical point in (a, d).

Proof: f has a maximum and a minimum value.

If they are equal, then f is constant, and any point is critical.

Else, f has a global min/max in (a, d), which is a local min/max, hence a critical point.

Theorem [MVT]: "Rolle's theorem at an angle"

Suppose f is differentiable on (a, d) and continuous at the endpoints a and d.

Let $s = \frac{f(d)-f(a)}{d-a}$

Then for some b in (a,d),

$$f'(b) = s.$$

Proof: Let $g(x) = f(x) - s(x-a)$.

Then $g(a) = f(a)$ and $g(d) = f(d) - \frac{f(d)-f(a)}{d-a}(d-a) = f(d) - (f(d) - f(a)) = f(a) = g(a)$.

By Rolle's theorem, $g'(b) = 0$ for some b in (a,d).

But $g'(x) = f'(x) - s$, so

$$f'(b) = g'(b) + s = s.$$

Midterm information:

The test will be 90 minutes long and consist of have 20 multiple choice questions.

It will cover sections 1.6 (including inverse trig), 2.5, 2.7, 2.8, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.11, 4.1, 4.8. There will be a question or two explicitly addressing Maple.

There will be two seatings for the exam on the Thursday evening; 6:30–8:00 and 8:15–9:45. See course website to find out which sitting and which room you'll be in.

There will be an early seating for the exam on the Wednesday, at 2:30. If you are unable to make the Thursday seatings, you can request the early seating time on the homework website.

Midterm 1 syllabus in some detail

- Inverse functions - concept; invertible $\leftrightarrow$ 1-1.
- Limits - concept, calculation.
- Continuity.
- IVT - statement; use.
- Derivatives - limit definition; tangent lines; differentiability; diffble $\rightarrow$ conts; higher derivatives.
- Differentiation techniques: linearity; product rule; chain rule; differentiating polynomials, powers (x^b), exponentials (b^x), logarithms, $f(x)^{g(x)}$, trig/hyp, inverse trig/hyp.
- trig/hyp: definitions (triangles for trig, $\cosh(x) = \frac{e^x+e^{-x}}{2}$ etc for hyp); fundamental identities ($\cos^2 + \sin^2 = 1 = \cosh^2 - \sinh^2$), inverses (arcfoo), differentiating.
- Implicit differentiation
- Newton's method.
- Minima and maxima: finding local and global mins/maxes; critical points