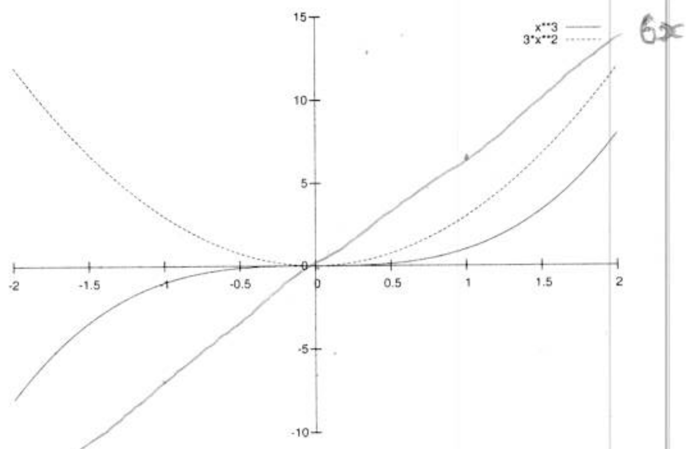
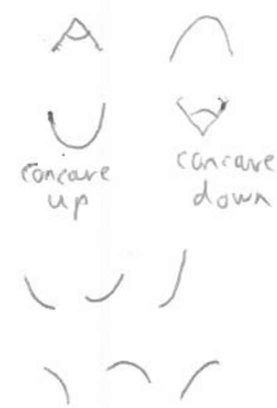




Second derivatives and "concave up/down": If $f''(x) > 0$ on an interval, then $f'(x)$ is increasing. So the slope of $f(x)$ is increasing.

We say a graph is concave upward on an interval where its slope is increasing

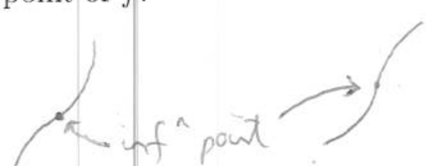
We say a graph is concave downward on an interval where its slope is decreasing.

So:

- If $f''(x) > 0$ on an interval, then $f(x)$ is concave upward on that interval.
- If $f''(x) < 0$ on an interval, then $f(x)$ is concave downward on that interval.

An inflection point is a point where $f(x)$ switches from being concave upward to being concave downward, or vice versa.

So if f'' changes sign at b, then b is an inflection point of f.

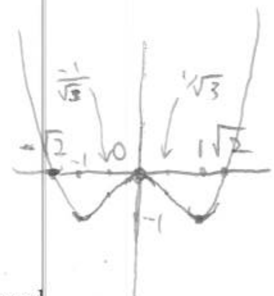


Sketching graphs, part I

Example: Let's sketch the graph of $x^2(x^2-2)$ $=$

$$f(x) = x^4 - 2x^2.$$

$f(-1) = 1 - 2 = -1$



Note $f(0) = 0$.
$f'(x) = 4x^3 - 4x = 4x(x^2 - 1)$, so $f'(x) = 0$ at -1,0,1, and

- On $(-\infty, -1)$, $f'(x) < 0$ so f is decreasing;
- On $(-1, 0)$, $f'(x) > 0$ so f is increasing;
- On $(0, 1)$, $f'(x) < 0$ so f is decreasing;
- On $(1, +\infty)$, $f'(x) > 0$ so f is increasing.

$-$ $+$ $-$ $+$ f'
-1 0 1

So -1 and 1 are local minima, and 0 is a local maximum.
$f''(x) = 12x^2 - 4$, so $f''(x) = 0$ at $-\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$, and

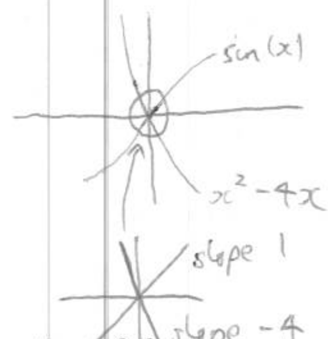
- On $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$, $f''(x) > 0$ so f is concave upward;
- On $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $f''(x) < 0$ so f is concave downward;
- On $\left(\frac{1}{\sqrt{3}}, \infty\right)$, $f''(x) > 0$ so f is concave upward.

$+$ $\frac{-1}{\sqrt{3}}$ $-$ $\frac{1}{\sqrt{3}}$ $+$ f''

So $-\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$ are inflection points of f.

Example of graph we can't yet sketch:

$$f(x) = \frac{\sin(x)}{x^2 - 4x}$$



What happens near 0?

i.e., what is $\lim_{x\to 0} f(x)$?

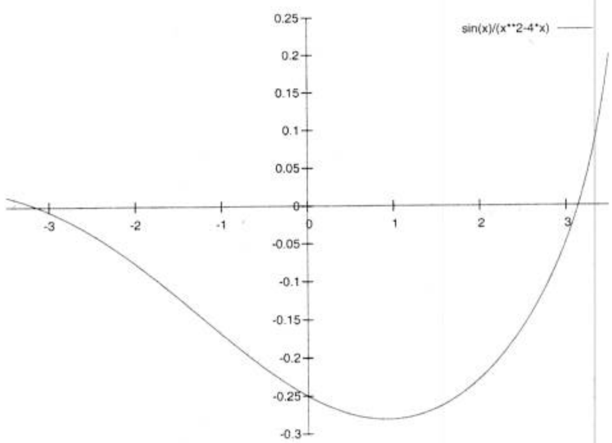
L'Hôpital

Near 0, $\sin(x)$ is well approximated by x [this is the equation of the tangent line at 0, since $\sin'(0) = \cos(0) = 1$], and $x^2 - 4x$ is well approximated by $-4x$ [since $\frac{d}{dx}x^2 - 4x \restriction_0 = 2x - 4 \restriction_0 = -4$]

So near 0, $\frac{\sin(x)}{x^2-4x} \approx \frac{x}{-4x} = -\frac{1}{4}$.

$\restriction_0$

So we expect $\lim_{x\to 0} \frac{\sin(x)}{x^2-4x} = -\frac{1}{4}$. Indeed:



This kind of reasoning yields: Theorem: [L'Hôpital's Rule] Suppose $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$.

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$, assuming the right hand limit exists.

Here, a is allowed to be $+\infty$ or $-\infty$, and so is the right hand limit.