One-sided limits, $\lim_{x\to a^+}$ or $\lim_{x\to a^-}$, are also allowed.

Moreover, we have the same result if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ are both $\pm\infty$ [even if one is $+\infty$ and the other $-\infty$], rather than both being 0.

[To see that it works for infinite limits: consider $\frac{1}{f(x)}$ and $\frac{1}{g(x)}$, which both tend to 0 at a]

Examples:

- $\lim_{x\to 0} \frac{\sin(x)}{x^2-4x} = \lim_{x\to 0} \frac{\cos(x)}{2x-4} = -\frac{1}{4}$.
- $\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{2x}{1} = 2$

  $\frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{(x-1)} = x+1$
- $\lim_{x\to+\infty} \frac{e^x}{x} = \lim_{x\to+\infty} \frac{e^x}{1} = +\infty$
- $\lim_{x\to+\infty} \frac{e^x}{x^2} \underset{L'H}{=} \lim_{x\to+\infty} \frac{e^x}{2x} \underset{L'H}{=} \lim_{x\to+\infty} \frac{e^x}{2} = +\infty$
-

$$\begin{aligned}
\lim_{x\to 0^+} x\ln x &= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \\
&= \lim_{x\to 0^+} \frac{\frac{d}{dx}\ln x}{\frac{d}{dx}\frac{1}{x}} \\
&= \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} \\
&= \lim_{x\to 0^+} -x \\
&= 0
\end{aligned}$$

$\lim_{x\to 0^+} \ln x = -\infty$

$\lim_{x\to 0^+} 1/x = +\infty$

so L'Hôpital applies

$\lim_{x \to 0^+} \cot(x) = +\infty = \lim_{x \to 0^+} \frac{1}{x}$

$$
\begin{aligned}
\lim_{x \to 0^+} \cot(x) - \frac{1}{x} &= \lim_{x \to 0^+} \left(\frac{\cos(x)}{\sin(x)} - \frac{1}{x} \right) \\
&= \lim_{x \to 0^+} \frac{x\cos(x) - \sin(x)}{x\sin(x)} \\
&= \lim_{x \to 0^+} \frac{\cos(x) - x\sin(x) - \cos(x)}{\sin(x) + x\cos(x)} \\
&= \lim_{x \to 0^+} \frac{-x\sin(x)}{\sin(x) + x\cos(x)} \\
&= \lim_{x \to 0^+} \frac{-(\sin(x) + x\cos(x))}{\cos(x) + \cos(x) - x\sin(x)} \\
&= \frac{0}{2} \\
&= 0
\end{aligned}
$$

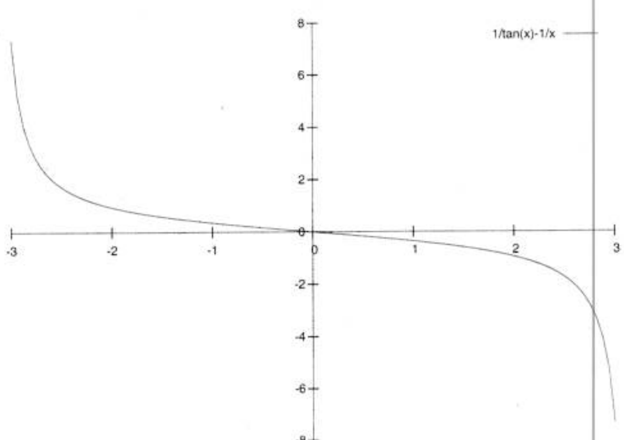
$\lim_{x \to 0^+} x\cos x - \sin x$

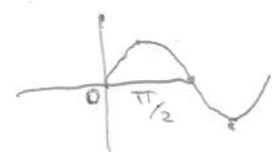
$= 0 - 0$

$= 0$

$\lim_{x \to 0^+} x \sin x$

$= 0$





Sketching curves

$$f(x) = \frac{x^2 - 1}{(x-1)(x-2)(x-3)}$$