Mean Value Theorem

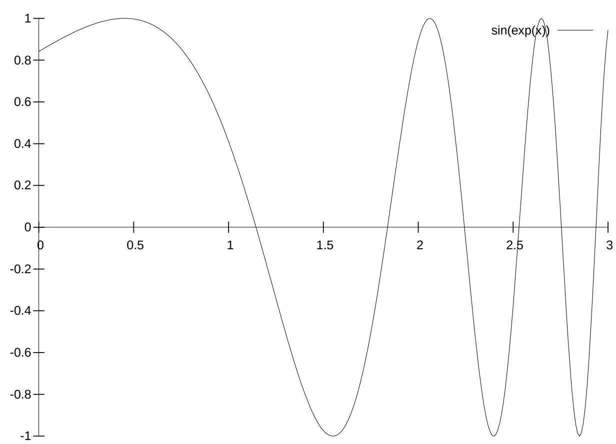
Theorem [Rolle's Theorem]: "What goes up and then comes down must hover instantaneously in between"

Suppose f is differentiable on (a, d) and continuous at the endpoints a and d.

Suppose $f(a) = f(d)$.

Then f has a critical point in (a, d).

Example: $f(x) = \sin(e^x)$: $f(\ln(\pi)) = f(\ln(2\pi))$, so $f'(x) = 0$ for some x in $(\pi, 2\pi)$.



Proof: f has a maximum and a minimum value.

If they are equal, then f is constant, and any point is critical.

Else, f has a global min/max in (a, d), which is a local min/max, hence a critical point.

Theorem [MVT]: "slanted Rolle"

Suppose f is differentiable on (a, d) and continuous at the endpoints a and d.

Let $s = \frac{f(d)-f(a)}{d-a}$.

Then for some b in (a, d),

$$f'(b) = s.$$

In other words: for some b in (a, d), the tangent line at b is parallel to the straight line between the points on the graph $(a, f(a))$ and $(d, f(d))$.

Example: $f(x) = \sin(e^x)$: $f\left(\ln\left(\frac{\pi}{2}\right)\right) = 1$, $f(\ln(\pi)) = 0$. So for some x in $\left(\ln\left(\frac{\pi}{2}\right), \ln(\pi)\right)$, $f'(x) = \frac{0-1}{\ln(\pi)-\ln\left(\frac{\pi}{2}\right)} = -1.44$.

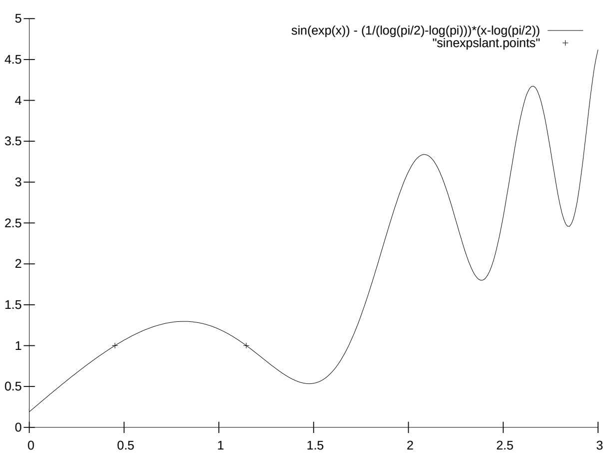
Proof: Let $g(x) = f(x) - s(x - a)$.

Then $g(a) = f(a)$ and $g(d) = f(d) - \frac{f(d)-f(a)}{d-a}(d - a) = f(d) - (f(d) - f(a)) = f(a) = g(a)$.

By Rolle's theorem, $g'(b) = 0$ for some b in (a, d).

But $g'(x) = f'(x) - s$, so

$$f'(b) = g'(b) + s = s.$$



We can use the MVT "backwards" to deduce information about $f(x)$ from information about $f'(x)$.

Example: Suppose a train travels along a straight track, and $s(t)$ is its distance in metres from its starting station t seconds after it leaves.

If $s(600) = 10000$ and between $t = 600$ and $t = 1200$ the train's speed never drops below $30ms^{-1}$, what is the least $s(1200)$ could be?

Answer: $s(1200)$ can't be less than $s(600)+30*(1200-600) = 28000$, since otherwise by the MVT $s'(t)$ would, for some t between 600 and 1200, be less than $\frac{28000-10000}{1200-600} = 30$.

Increasing/Constant/Decreasing: Let $f(x)$ be a function.

(i) If $f'(x) > 0$ for all x in an interval, then $f(x)$ is increasing on the interval.

(ii) If $f'(x) = 0$ for all x in an interval, then $f(x)$ is constant on the interval.

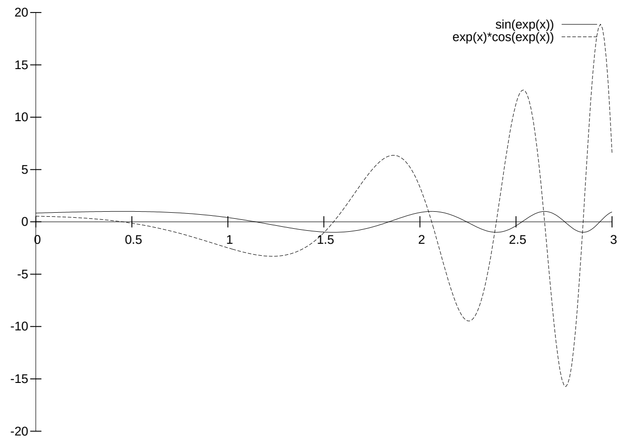
(iii) If $f'(x) < 0$ for all x in an interval, then $f(x)$ is decreasing on the interval.

Proof:

(i) If b and c are in the interval with $b < c$ and $f(c) \leq f(b)$, then by MVT $f'(x) = \frac{f(c)-f(b)}{c-b} \leq 0$ for some x in $[b, c]$.

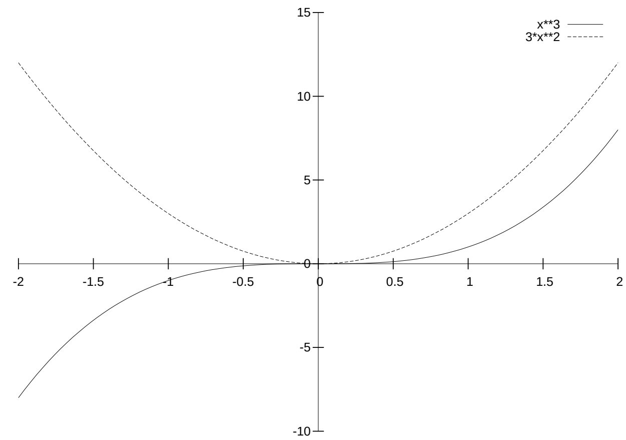
(ii) If b and c are in the interval with $b < c$ and $f(c) \neq f(b)$, then by MVT $f'(x) = \frac{f(c)-f(b)}{c-b} \neq 0$ for some x in $[b, c]$.

(iii) If b and c are in the interval with $b < c$ and $f(c) \geq f(b)$, then by MVT $f'(x) = \frac{f(c)-f(b)}{c-b} \geq 0$ for some x in $[b, c]$.



Classifying critical points: If c is a critical point of $f(x)$, then

- if $f'(x) < 0$ just to the left of c and $f'(x) > 0$ just to the right of c, then c is a local minimum;
- if $f'(x) > 0$ just to the left of c and $f'(x) < 0$ just to the right of c, then c is a local maximum;
- if $f'(x)$ is positive on both sides of c, or if it is negative on both sides of c, then c is not a local min/max.



Second derivatives and "concave up/down": If $f''(x) > 0$ on an interval, then $f'(x)$ is increasing. So the slope of $f(x)$ is increasing.

We say a graph is concave upward on an interval where its slope is increasing

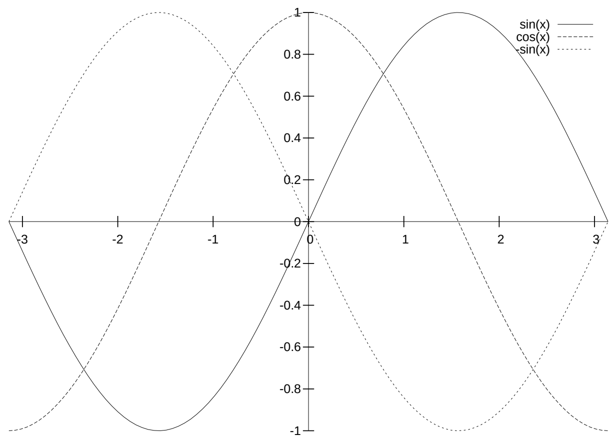
We say a graph is concave downward on an interval where its slope is decreasing.

So:

- If $f''(x) > 0$ on an interval, then $f(x)$ is concave upward on that interval.
- If $f''(x) < 0$ on an interval, then $f(x)$ is concave downward on that interval.

An inflection point is a point where $f(x)$ switches from being concave upward to being concave downward, or vice versa.

So if f'' changes sign at b, then b is an inflection point of f.


sin(x)
cos(x)
-sin(x)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-3
-2
-1
0
1
2
3

Sketching graphs, part I

Example: Let's sketch the graph of

$$f(x) = x^4 - 2x^2.$$

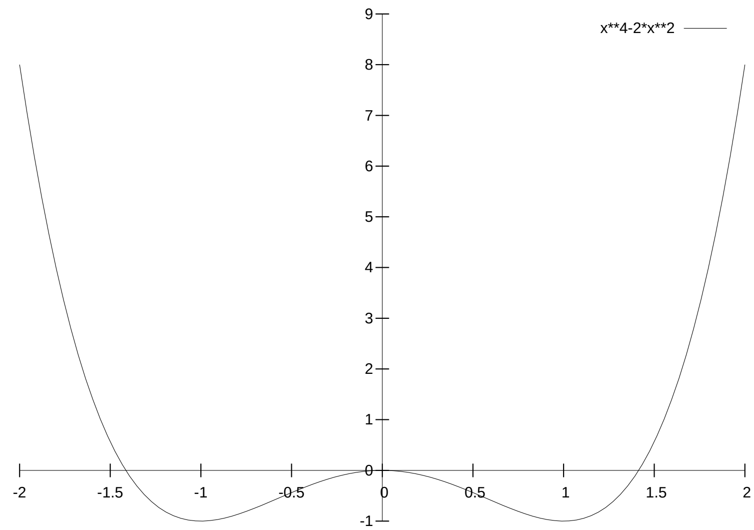
Note $f(0) = 0$.
$f'(x) = 4x^3 - 4x = 4x(x^2 - 1)$, so $f'(x) = 0$ at -1,0,1, and

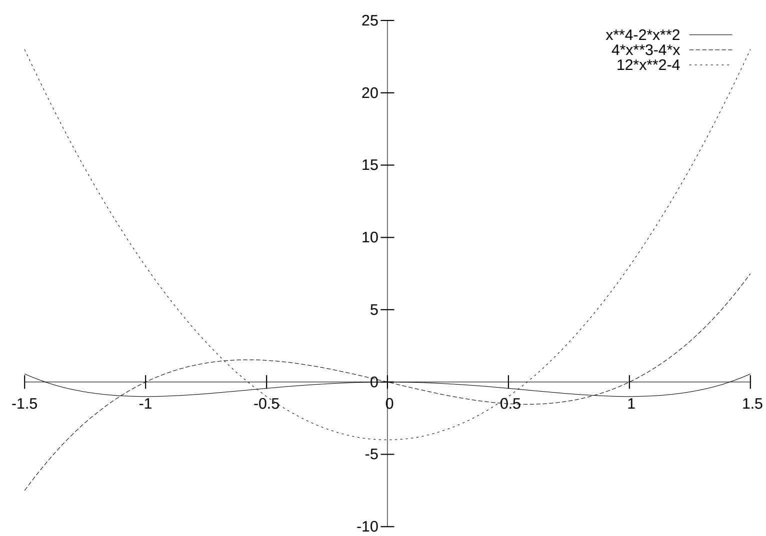
- On $(-\infty, -1)$, $f'(x) < 0$ so f is decreasing;
- On $(-1, 0)$, $f'(x) > 0$ so f is increasing;
- On $(0, 1)$, $f'(x) < 0$ so f is decreasing;
- On $(1, +\infty)$, $f'(x) > 0$ so f is increasing.

So -1 and 1 are local minima, and 0 is a local maximum.
$f''(x) = 12x^2 - 4$, so $f''(x) = 0$ at $-\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$, and

- On $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$, $f''(x) > 0$ so f is concave upward;
- On $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $f''(x) < 0$ so f is concave downward;
- On $\left(\frac{1}{\sqrt{3}}, \infty\right)$, $f''(x) > 0$ so f is concave upward.

So $-\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$ are inflection points of f.


x**4-2*x**2
9
8
7
6
5
4
3
2
1
0
-1
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2


x**4-2*x**2
4*x**3-4*x
12*x**2-4
25
20
15
10
5
0
-5
-10
-1.5
-1
-0.5
0
0.5
1
1.5

Example of graph we can't yet sketch:

$$f(x) = \frac{\sin(x)}{x^2 - 4x}$$

What happens near 0?

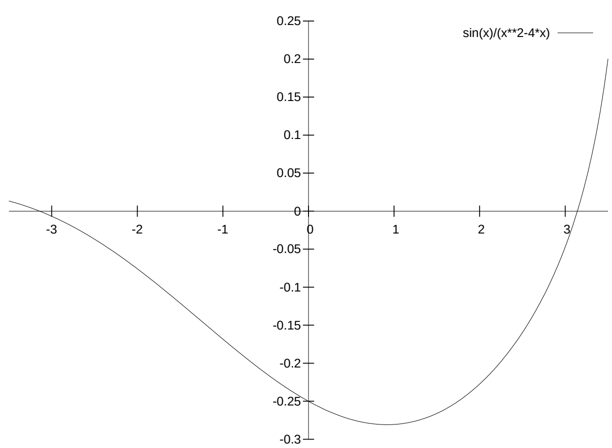
i.e., what is $\lim_{x\to 0} f(x)$?

L'Hôpital

Near 0, $\sin(x)$ is well approximated by x [this is the equation of the tangent line at 0, since $\sin'(0) = \cos(0) = 1$], and $x^2 - 4x$ is well approximated by $-4x$ [since $\frac{d}{dx}x^2 - 4x \restriction_0 = 2x - 4 \restriction_0 = -4$]

So near 0, $\frac{\sin(x)}{x^2-4x} \approx \frac{x}{-4x} = -\frac{1}{4}$.

So we expect $\lim_{x\to 0} \frac{\sin(x)}{x^2-4x} = -\frac{1}{4}$**. Indeed:**



This kind of reasoning yields: Theorem: [L'Hôpital's Rule] Suppose $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$.

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$, assuming the right hand limit exists.

Here, a is allowed to be $+\infty$ or $-\infty$, and so is the right hand limit.

One-sided limits, $\lim_{x\to a^+}$ or $\lim_{x\to a^-}$, are also allowed.

Moreover, we have the same result if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ are both $\pm\infty$ [even if one is $+\infty$ and the other $-\infty$], rather than both being 0.

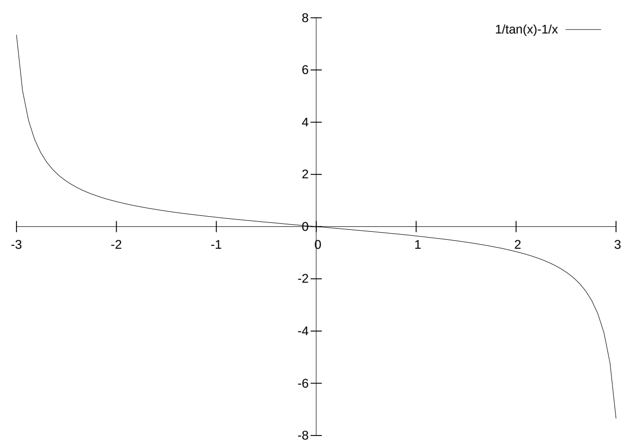
[To see that it works for infinite limits: consider $\frac{1}{f(x)}$ and $\frac{1}{g(x)}$, which both tend to 0 at a]

Examples:

- $\lim_{x\to 0} \frac{\sin(x)}{x^2-4x} = \lim_{x\to 0} \frac{\cos(x)}{2x-4} = -\frac{1}{4}$.
- $\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{2x}{1} = 2$
- $\lim_{x\to +\infty} \frac{e^x}{x} = \lim_{x\to +\infty} \frac{e^x}{1} = +\infty$
- $\lim_{x\to +\infty} \frac{e^x}{x^2} = \lim_{x\to +\infty} \frac{e^x}{2x} = \lim_{x\to +\infty} \frac{e^x}{2} = +\infty$
-

$$\begin{aligned}
\lim_{x\to 0^+} x\ln x &= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \\
&= \lim_{x\to 0^+} \frac{\frac{d}{dx}\ln x}{\frac{d}{dx}\frac{1}{x}} \\
&= \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} \\
&= \lim_{x\to 0^+} -x \\
&= 0
\end{aligned}$$

- $$\begin{aligned}\lim_{x\to 0^+} \cot(x) - \frac{1}{x} &= \lim_{x\to 0^+} \frac{\cos(x)}{\sin(x)} - \frac{1}{x} \\ &= \lim_{x\to 0^+} \frac{x\cos(x) - \sin(x)}{x\sin(x)} \\ &= \lim_{x\to 0^+} \frac{\cos(x) - x\sin(x) - \cos(x)}{\sin(x) + x\cos(x)} \\ &= \lim_{x\to 0^+} \frac{x\sin(x)}{\sin(x) + x\cos(x)} \\ &= \lim_{x\to 0^+} \frac{\sin(x) + x\cos(x)}{\cos(x) + \cos(x) - x\sin(x)} \\ &= \frac{0}{2} \\ &= 0\end{aligned}$$



-

$$\begin{aligned}
\lim_{x\to 0^+} (1+x)^{\frac{1}{x}} &= \lim_{x\to 0^+} \left(e^{\ln(1+x)}\right)^{\frac{1}{x}} \\
&= \lim_{x\to 0^+} e^{\frac{1}{x}\ln(1+x)} \\
&= e^{\lim_{x\to 0^+} \frac{\ln(1+x)}{x}} \\
&= e^{\lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1}} \\
&= e^1 = e
\end{aligned}$$

$$1.00001^{100000} = 2.7182682371922975$$