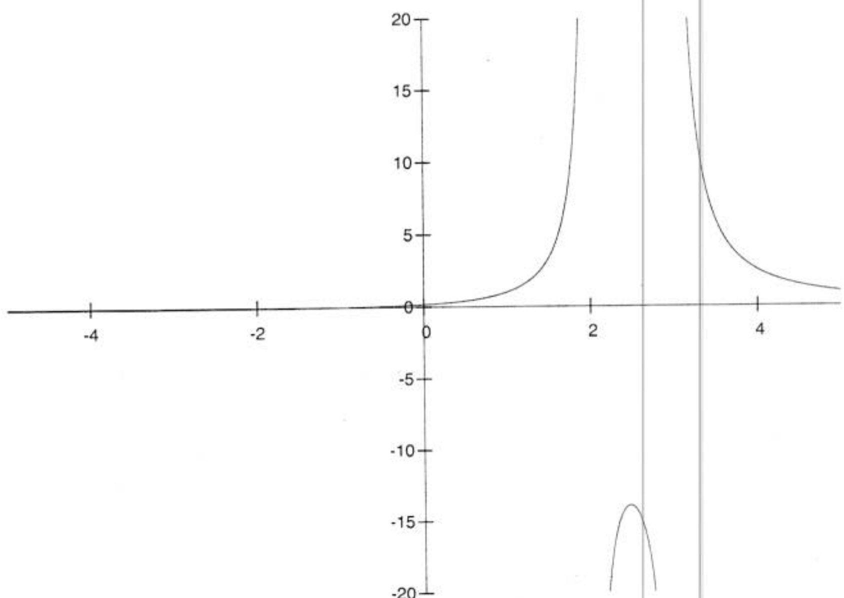


•

$$f(x) = e^{\frac{1}{x}} > 0$$

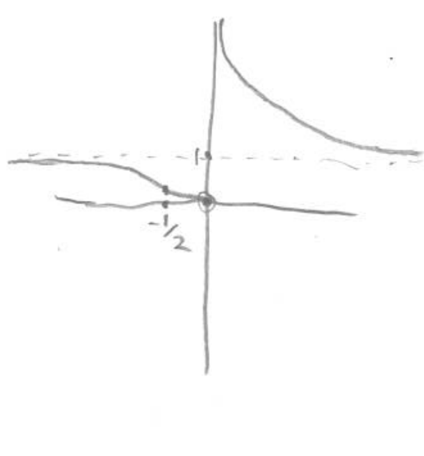
$\lim_{x \to 0^-} f(x) = \lim_{y \to -\infty} e^y = 0$

$f(x) = 0$ never!

$\lim_{x \to 0^+} f(x) = +\infty$

$$f'(x) = \frac{-e^{1/x}}{x^2} = -x^{-2}e^{1/x}$$

$\lim_{x \to +\infty} f(x) = e^0 = 1 = \lim_{x \to -\infty} f(x)$



never 0

top is always negative

bottom is always +ve

so $f'(x)$ is always -ve

so $e^{1/x}$ is decreasing everywhere

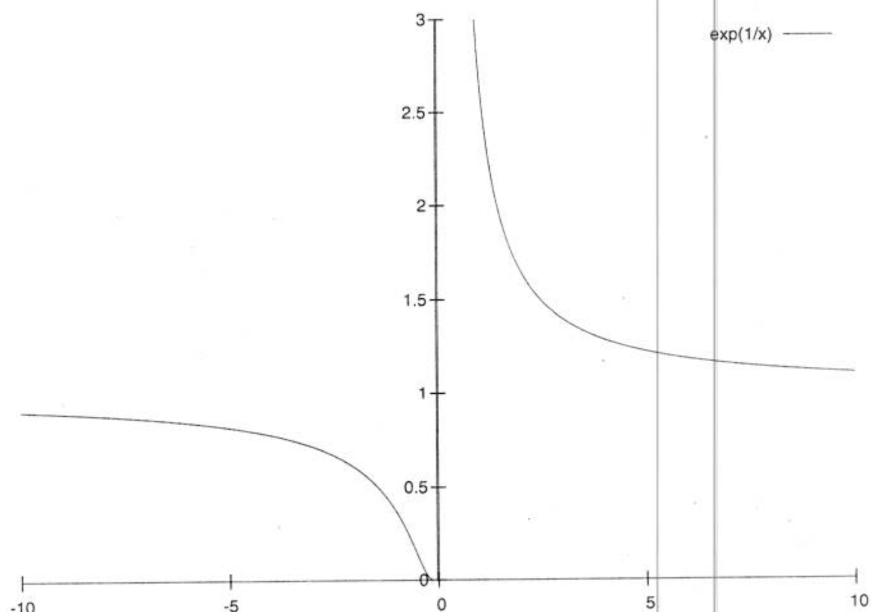
$$f''(x) = -\left(-2x^{-3}e^{1/x} + \left(-x^{-2}e^{1/x}x^{-2}\right)\right)$$
$$= e^{1/x}\left(2x^{-3} + x^{-4}\right)$$

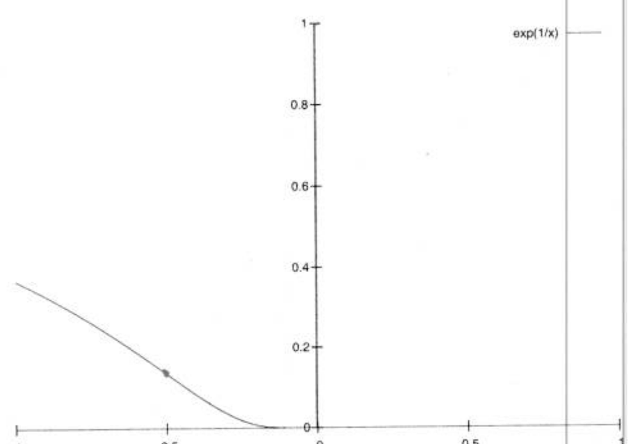
sign of $f''(x)$ = sign of $\left(2x^{-3} + x^{-4}\right) = x^{-3}\left(2 + x^{-1}\right)$

x^{-3} : $(-\infty, 0)$: $-$ve
$(0, +\infty)$: $+$ve
0 : undefined

$2 + 1/x$: $-1/2$: 0
$(-\infty, -1/2)$: $+$ve
$(-1/2, 0)$: $-$ve
$(0, \infty)$: $+$ve
0 : undefined

$f''(x)$: $(-\infty, -1/2)$: $-$ve ← concave down
$(-1/2, 0)$: $+$ve ← concave up
$(0, \infty)$: $+$ve ←
$-1/2$: 0 ← inflection





$$\lim_{x\to 0}\left(\frac{d}{dx}e^{\frac{1}{x}}\right)=0$$

•

$$f(x) = \frac{x^2}{1+x} = (1+x)^{-1}x^2$$

$$\lim_{x\to\infty} f(x) = +\infty$$

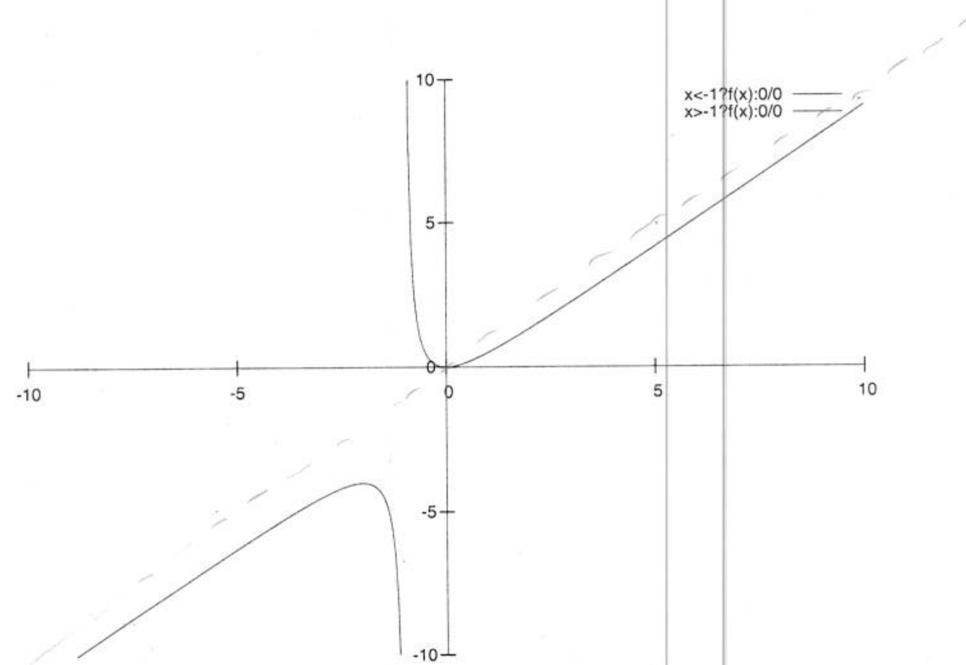
$$\lim_{x\to-\infty} f(x) = -\infty$$

$$f'(x) = -(1+x)^{-2}x^2 + 2x(1+x)^{-1}$$

$$= \frac{2x(1+x) - x^2}{(1+x)^2}$$

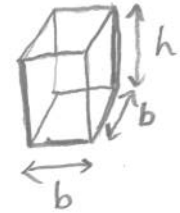
$$= \frac{x^2+2x}{x^2+2x+1}$$

so $\lim_{x\to\infty} f'(x) = 1 = \lim_{x\to-\infty} f'(x)$



Optimisation

Example: You want to construct a box with a square base and an open top. The volume should be $1m^3$, and you want to minimise the material used. What should be the dimensions of the box?



$b^2 h = 1$ $\quad b > 0$

minimise $b^2 + 4bh$ (sum of areas)

$h = 1/b^2$

minimise $b^2 + 4\frac{1}{b} = M(b)$

$M'(b) = 2b - \frac{4}{b^2}$

critical points: $2b - 4/b^2 = 0$

$2b = 4/b^2$

$b^3 = 2$

$b = \sqrt[3]{2}$

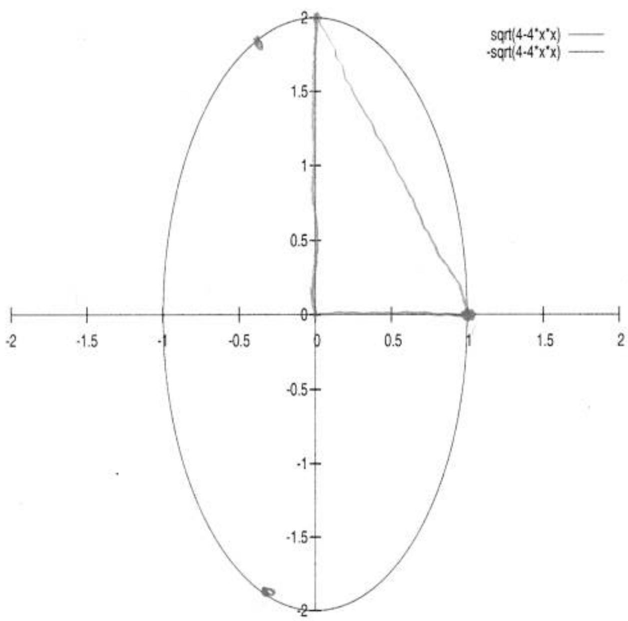
since $\lim_{b \to 0^+} M(b) = +\infty = \lim_{b \to +\infty} M(b)$

and $M(b)$ is continuous and diff^{ble} on $(0, +\infty)$

the min must be at the critical point $\sqrt[3]{2}$

Example: What are the points on the ellipse furthest away from $(1, 0)$?

$$4x^2 + y^2 = 4?$$



maximise $r(x,y)$

$= \sqrt{(1-x)^2 + y^2}$

i.e. maximise $r^2(x,y)$

$= (1-x)^2 + y^2$

$4x^2 + y^2 = 4$

so $r^2(x,y) = r^2(x) = (1-x)^2 + 4 - 4x^2$

$\frac{d}{dx} r^2(x) = -2(1-x) - 8x = 0$

$-2 = 6x$

$x = -\frac{1}{3}$

$y = \pm\sqrt{4 - 4\left(\frac{-1}{3}\right)^2}$

$= \pm\sqrt{4 - 4/9}$

$= 2 \pm \sqrt{8/9}$

$= \frac{4}{3} \pm \sqrt{2}$

$(-\frac{1}{3}, \frac{4}{3} \pm \sqrt{2}$