Review of Handy Basic Concepts

Quadratic Equation & Factoring:

First let's remember the quadratic equation. A quadratic, or 2^{nd} order polynomial, can be written in the form: $ax^2 + bx + c$, where a, b, c are real constant coefficients. The roots, values of x such that the polynomial has an overall value of zero, are given by the quadratic equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

There of course will be either two distinct real roots (if $b^2 - 4ac > 0$), one real root (if $b^2 - 4ac = 0$), or two imaginary roots, (if $b^2 - 4ac < 0$).
In the case of two distinct roots, we can let $x = \alpha$ and $x = \beta$ represent the two roots, and our quadratic factors to:

$$ax^2 + bx + c = a(x - \alpha)(x - \beta)$$

In the case of one real root, call that $x=\alpha$, we get:

$$ax^2 + bx + c = a(x - \alpha)^2$$

And of course, if we have two imaginary roots, we **cannot factor** the quadratic in $\mathbb{R}$, the real numbers.

"Difference of" Factoring:

We know "diffference of squares" allows us to factor:

$$x^2 - b^2 = (x - b)(x + b)$$

And similarly, we can factor anything of the form:

$$(\alpha^2 - \beta^2) = (\alpha - \beta)(\alpha + \beta)$$

But of course we don't have to stop there. There are higher "difference" formulas. For instance, difference of cubes:

$$(\alpha^3 - \beta^3) = (\alpha - \beta)(\alpha^2 + ab + \beta^2)$$

And in fact, this sort of factoring can be used on a difference for any such integer powers:

$$(\alpha^n - \beta^n) = (\alpha - \beta)(\alpha^n + \alpha^{n-1}\beta + \alpha^{n-2}\beta^2 + \ldots. + \alpha^n\beta^{n-2} + \alpha\beta^{n-1} + \beta^n)$$

So for example:

$$(x^4 - 81) = (x^4 - 3^4) = (x-3)(x^3 + 3x^2 + 9x + 27)$$

Elementary Binomial Expansion:

Now say we have an expression of the form:

$$(\alpha + \beta)^n$$

Notice that the power here is of the entire bracket, not just of α and β individually as it was previously. In cases such as this for fixed finite positive integer n, we can expand the expression by multiplying it out, for example:

$$(\alpha + \beta)^2 = (\alpha + \beta)(\alpha + \beta) = (\alpha^2 + 2\alpha\beta + \beta^2)$$

$$(\alpha + \beta)^3 = (\alpha + \beta)(\alpha + \beta)^2 = (\alpha + \beta)(\alpha^2 + 2\alpha\beta + \beta^2) = (\alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3)$$

But of course, for higher integers this becomes very hard. What shall we do?
Well, it turns out, that for a n-th power, the coefficients correspond to the n-th row of Pascal's Triangle:

Pascal's Triangle	
Row 0:	1
Row 1:	1 1
Row 2:	1 2 1
Row 3:	1 3 3 1
Row 4:	1 4 6 4 1
Row 5:	1 5 10 10 5 1
⋮	⋮

Each row of Pascal's Triangle starts and ends with a "1". All of the other entries are calculated by adding the elements immediately diagonally above. For instance:

Element 2 of row 2: 1+1=2
Element 2 of row 3: 1+2=3
Element 3 of row 4: 3+3=6
Element 3 of row 5: 4+6=10 etc. etc. etc.

So if we wish to expand a binomial power, all we have to do is look at the specific row corresponding to that power:

$$(\alpha + \beta)^n = \alpha^n + c_2\alpha^{n-1}\beta + c_3\alpha^{n-2}\beta^2 + \ldots + c_{n-1}\alpha^2\beta^{n-2} + c_n\alpha\beta^{n-1} + \beta^n$$

Where c_i represents the ith element of the row in Pascal's Triangle corresponding to our n power.
For example, we can now expand the expression $(x+h)^5$ by looking at the 5th row of the triangle:

$$\textit{Row } 5: \quad 1 \;\; 5 \;\; 10 \;\; 10 \;\; 5 \;\; 1$$

to give us the expression:

$$(x+h)^5 = x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5$$

Multiplying by the Conjugate:

As we progress through this course, it will often become convenient to get rid of square root terms, especially when they are in a difference. For instance if I wanted to rationalize the denominator of:

$$\frac{1}{\sqrt{3}-\sqrt{2}}$$

What could I do to clean this up? One common trick is to look at the denominator of the above fraction and view it as one part of a difference of squares. Remember that:

$$(\alpha^2-\beta^2)=(\alpha-\beta)(\alpha+\beta)$$

So if we thought of $\sqrt{3}$ and $\sqrt{2}$ as the α and β, we would hope we could reverse the difference of squares argument and multiply the denominator by $\sqrt{3}+\sqrt{2}$. (To get $(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})=3-2=1$) But to use this on our fraction, we can only multiply by a net value of 1, or the fraction will change in value. So we multiply both numerator and denominator by the same value:

$$\frac{1}{\sqrt{3}-\sqrt{2}}=\frac{1}{\sqrt{3}-\sqrt{2}}\bullet 1=\frac{1}{\sqrt{3}-\sqrt{2}}\bullet\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}$$

$$=\frac{\sqrt{3}+\sqrt{2}}{(\sqrt{3}+\sqrt{2})(\sqrt{3}+\sqrt{2})}=\frac{\sqrt{3}+\sqrt{2}}{1}=\sqrt{3}+\sqrt{2}\quad(\textit{Much prettier})$$

Often in first year calculus we see equations of the form:

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

and we'll wish to simplify the numerator. We can do this by again viewing the numerator as part of a difference of squares problem, and multiply top and bottom of the rational expression by the correspoding "conguate" from the difference of squares form: $\sqrt{x+h}+\sqrt{x}$. That is to say:

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}\bullet\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}=\frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}=\frac{h}{h(\sqrt{x+h}+\sqrt{x})}=\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

At least if h is not zero. Notice, that this calculation is quite handy if we are taking the limit that $h\to 0$. In that case the original expression is undefined, that is $0/0$, but the aforementioned division top and bottom by h is allowed, and the simplified expression is easily evaluated as:

$$\frac{1}{\sqrt{x+0}+\sqrt{x}}=\frac{1}{2\sqrt{x}}\quad(\textit{if }\ x\neq 0)$$

Completing the Square:

Often, as when we're looking for position, direction, vertex etc. of a quadratic, it is handy to turn a quadratic of the form $x^2 + bx + c$ into the form $(x + \alpha)^2 + \beta$. We do this by completing the square.

We know that we can expand $(x + \alpha)^2 + \beta = x^2 + 2\alpha x + \alpha^2 + \beta$, and if we claim this is equivalent to $x^2 + bx + c$, then $b = 2\alpha$ and $\alpha^2 + \beta = c$. Knowing this, we can rewrite:

$$x^2 + bx + c = x^2 + 2\alpha x + c = (x^2 + 2\alpha x + \alpha^2) - \alpha^2 + c = (x + \alpha)^2 + c - \alpha^2$$
$$= (x + \tfrac{b}{2})^2 + (c - (\tfrac{b}{2})^2)$$

So for instance we can complete the square for examples such as:

$$x^2 + 3x + 2 = (x + \tfrac{3}{2})^2 + (2 - (\tfrac{3}{2})^2) = (x + \tfrac{3}{2})^2 - \tfrac{1}{4}$$

and $$x^2 - 2x + 3 = (x + (-1))^2 + (3 - (-1)^2) = (x - 1)^2 + 2$$

Simplification of Fractions:

Just as a reminder, here are some basic fraction manipulations:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \qquad \frac{ax + ay}{ab} = \frac{a(x + y)}{ab} = \frac{x + y}{b} \quad (\text{if } a \neq 0)$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{cb}{bd} = \frac{ad + cb}{bd} \qquad \frac{a}{b} + \frac{a}{d} = a\left(\frac{1}{b} + \frac{1}{d}\right) = a\left(\frac{d + b}{bd}\right) = \frac{a(d + b)}{bd} = \frac{ad + ab}{bd}$$

$$\frac{a/b}{c/d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc} \qquad \frac{a}{\left(b/c\right)} = \frac{a}{1} \cdot \frac{c}{b} = \frac{ac}{b}, \qquad \frac{\left(a/b\right)}{c} = \frac{a}{b} \cdot \frac{1}{c} = \frac{a}{bc}$$

Domain/Interval Notation:

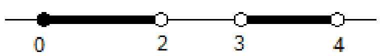
Instead of having to use bulky set construction notation for indicating intervals, we have a nice simplified notation for indicating intervals. "(" or ")" to indicate when the endpoint is not included in the interval (ie. an open end). "[" or "]" to indicate endpoints where the value is included in the interval, and "∪" to indicate that the region is composed of multiple intervals, that is to "add" intervals together. Typical examples are:

$(1,3] \Leftrightarrow 1 < x \leq 3$ Half-Open Interval

$(1,4) \Leftrightarrow 1 < x < 4$ Open Interval

$[0,3] \Leftrightarrow 0 \leq x \leq 3$ Closed Interval

$(\infty, 3.5) \Leftrightarrow x \leq 3.5$ An Infinite Interval



$[0,2) \cup (3,4) \Leftrightarrow 0 \leq x < 2$ or $3 < x < 4$

A region consisting of two intervals