Fixed Point Techniques in Analog Systems

Diogo Poças and Jeffery Zucker

Abstract Analog computation is concerned with continuous rather than discrete spaces. Most of the physical processes arising in nature are modeled by differential equations, either ordinary (example: spring/mass/damper system) or partial (example: heat diffusion). In analog computability, the existence of an effective way to obtain solutions (either exact or approximate) of these systems is essential.

We develop a framework in which the solutions can be seen as fixed points of certain operators on continuous data streams, using the framework of Fréchet spaces. We apply a fixed point construction to retrieve these solutions and present sufficient conditions on the operators and initial inputs to ensure existence and uniqueness of these corresponding fixed points.

1 Introduction

Analog computation, as conceived by Kelvin [10], Bush [1], and Hartree [4], is a form of experimental computation with physical systems called analog devices or analog computers. Historically, data are represented by measurable physical quantities, including lengths, shaft rotation, voltage, current, resistance, etc., and the analog devices that process these representations are made from mechanical or electromechanical or electronic components [5, 7, 9].

The main objects of our study are *analog networks* or *analog systems*, [6, 11–13], whose main components are described as follows:

$$\textit{Analog network} = \textit{data} + \textit{time} + \textit{channels} + \textit{modules}.$$

We will model *data* as elements from a topological vector space $\mathscr{A}$ that is actually a Fréchet space. We will use the nonnegative real numbers as a continuous model of *time* $\mathbb{T} = [0, \infty)$. Each *channel* carries a continuous stream, represented as a function $u : \mathbb{T} \to \mathscr{A}$ (this space is denoted by $C(\mathbb{T}, \mathscr{A})$). Each *module* M is

D. Poças (✉) • J. Zucker
Department of Mathematics and Statistics, McMaster University, Hamilton, ON, Canada
e-mail: pocasd@math.mcmaster.ca; zucker@cas.mcmaster.ca

© Springer International Publishing Switzerland 2016
J. Bélair et al. (eds.), *Mathematical and Computational Approaches in Advancing Modern Science and Engineering*, DOI 10.1007/978-3-319-30379-6_63

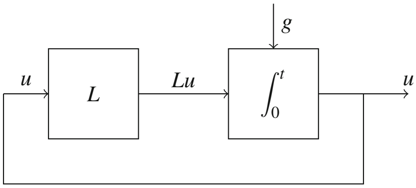


Fig. 1 Analog network (with two modules and three channels) for the time evolution problem

specified by a *stream function*

$$F_M : C(\mathbb{T}, \mathscr{A})^k \times \mathscr{A}^\ell \to C(\mathbb{T}, \mathscr{A}).$$

In this way, we can think of analog networks as directed graphs where modules are nodes and channels are edges (see Fig. 1). We will use analog systems to study the time evolution problem (also called the Cauchy problem).

Definition 1 (Time evolution problem, [3]) For a given initial condition $g \in \mathscr{A}$ and an operator $L : \mathscr{A} \to \mathscr{A}$, the *time evolution problem* is given by the system

$$\begin{cases} \frac{du}{dt} = Lu, \ t \in \mathbb{T}; \\ u(0) = g. \end{cases} \tag{1}$$

We look for a solution $u \in C(\mathbb{T}, \mathscr{A})$.

To construct an analog system that represents the time evolution problem, we can simply integrate the differential equation (1) to obtain

$$u(t) = g + \int_0^t Lu(s)ds =: F_g(u), \tag{2}$$

where we use the right hand side to define an operator $F_g : C(\mathbb{T}, \mathscr{A}) \to C(\mathbb{T}, \mathscr{A})$, which can be computed using an analog network with two modules. Introducing a feedback to implement the equality, we obtain the analog system of Fig. 1.

We can informally define a specification of the analog network as a tuple of streams describing the data on all channels which satisfy the equations given by the modules. We can then observe the equivalence between the notions of (a) solutions to the time evolution problem of Definition 1; (b) specifications of the analog system of Fig. 1; and (c) fixed points of the operator F_g of Equation (2). Henceforth we will focus on the last notion. Our goal is to provide sufficient conditions on F_g that ensure existence and uniqueness of fixed points, as well as the existence of a constructive method to obtain fixed points when they exist.

In Sect. 2 we introduce Fréchet spaces, which form the framework for our problem at hand. In Sect. 3 we assume analyticity of g to prove local existence and convergence of fixed points (Theorem 2). In Sect. 4 we first extend our results to global existence and convergence (Theorem 3) and then extend our constructive method for different choices of initial input (Theorem 5). Finally, we turn to uniqueness of fixed points and prove it for certain operators (Theorem 6).

2 Fréchet Spaces

For the remainder of this paper, we use the following notation:

Notation 1 *$\mathscr{A}$ is the space of infinitely differentiable functions, $\mathscr{A} = C^\infty(\mathbb{R})$.*

Notation 2 *The operator $L : \mathscr{A} \to \mathscr{A}$ is given by $Lu = \alpha\partial_x u$, for some $\alpha \in \mathbb{R}$.*

Thus, the operator F_g becomes

$$F_g(u)(t,x) = g(x) + \alpha \int_0^t \partial_x u(s,x)ds. \tag{3}$$

As we know, $\mathscr{A} = C^\infty(\mathbb{R})$ is not complete under the supremum norm; however, for each $x_0 \in \mathbb{R}$, $X \in \mathbb{R}^+$ and $k \in \mathbb{N}$, we can define a pseudonorm by

$$\|f\|_{X,x_0,k} = \sup_{|x-x_0|\leq X} \left| \frac{\partial^k f}{\partial x^k}(x) \right|.$$

It turns out that the notion of interest is that of Fréchet space, which we now briefly review (a detailed exposition can be found in [8, Ch. V]).

Definition 2 (Fréchet space [8]) A Fréchet space is a topological vector space X whose topology is induced by a countable family of pseudonorms $\{\|\cdot\|_\alpha\}_{\alpha\in A}$. Moreover,

- the family $\{\|\cdot\|_\alpha\}_{\alpha\in A}$ separates points, that is,

$$u = 0 \text{ if and only if } \|u\|_\alpha = 0 \text{ for all } \alpha;$$

- X is complete with respect to $\{\|\cdot\|_\alpha\}_{\alpha\in A}$, that is, for every sequence (x_n) which is Cauchy with respect to each pseudonorm $\|\cdot\|_\alpha$, there exists $x \in X$ such that (x_n) converges to x with respect to each pseudonorm $\|\cdot\|_\alpha$.

Example 1 The space $\mathscr{A} = C^\infty(\mathbb{R})$ of infinitely differentiable functions in $\mathbb{R}$ is a Fréchet space with the countable family of pseudonorms given by

$$\|f\|_{N,k} = \sup_{-N\leq x\leq N} \left| \frac{\partial^k f}{\partial x^k}(x) \right|, \tag{4}$$

for $N, k \in \mathbb{N}$.

Example 2 The space $C(\mathbb{T}, \mathscr{A})$ of continuous streams is also a Fréchet space, with the countable family of pseudonorms given by

$$\|u\|_{M,N,k} = \sup_{0 \le t \le M} \sup_{-N \le x \le N} \left| \frac{\partial^k u}{\partial x^k}(t,x) \right|, \tag{5}$$

for $M, N, k \in \mathbb{N}$.

We can see that the family of pseudonorms in Example 2 is closely related to the family in Example 1. In fact, this illustrates a useful property of Fréchet spaces; in general, the space of continuous streams over a Fréchet space is itself a Fréchet space. In other words, Fréchet spaces work well with the operation of taking continuous streams.

Proposition 1 (New Fréchet spaces from old) *If $\mathscr{A}$ is a Fréchet space with a countable family of pseudonorms $\{\|\cdot\|_\alpha\}_{\alpha \in A}$, then so is $C(\mathbb{T}, \mathscr{A})$ with the countable family of pseudonorms $\{\|\cdot\|_{M,\alpha}\}_{M \in \mathbb{N}, \alpha \in A}$, where*

$$\|u\|_{M,\alpha} = \sup_{0 \le t \le M} \|u(t)\|_\alpha.$$

Even though Fréchet spaces, as they stand, are not necessarily normed spaces, we can define a metric from the pseudonorms, under which these spaces are complete.

Proposition 2 *Given a Fréchet space, we can define a complete metric from the family of pseudonorms which induces the same topology.*

For a proof, see [8, Ch. V], in particular Theorem V.5.

The usefulness of complete metric spaces is evident due to the following.

Theorem 1 (Banach fixed point theorem) *Given a complete metric space (X, d), suppose that $T : X \to X$ is a contracting operator in the sense that there exists $0 \le \lambda < 1$ with*

$$d(T(x), T(y)) \le \lambda d(x, y) \text{ for all } x, y \in X.$$

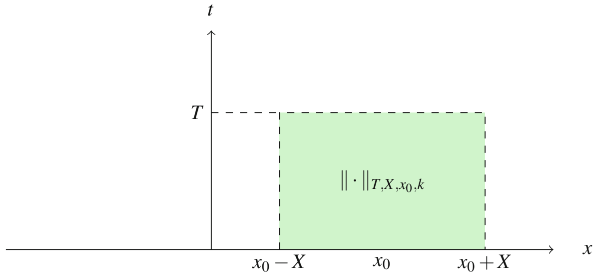
Then T has a unique fixed point x^*. Moreover, for all $x_0 \in X$ the sequence of iterations $x_n := T^n(x_0)$ converges to x^*.

3 Local Convergence Theorem

Consider the space $C(\mathbb{T}, \mathscr{A})$. Take any (arbitrary but fixed) $x_0 \in \mathbb{R}$, $X \in \mathbb{R}^+$, $T \in \mathbb{T}$. Then, for any $k \in \mathbb{N}$, we have a pseudonorm

$$\|u\|_{T,X,x_0,k} = \sup_{\substack{0 \le t \le T \\ |x - x_0| \le X}} \left| \frac{\partial^k u}{\partial x^k}(t,x) \right|. \tag{6}$$

Fig. 2 Compact rectangles



Observe that we are taking suprema on compact rectangles of the form $[0, T] \times [x_0 - X, x_0 + X]$ (see Fig. 2). The reason for taking suprema on compact rectangles will be made clear shortly with Theorem 2 (local convergence theorem). We also observe that, for each compact rectangle $\mathbb{X}' = [0, T] \times [x_0 - X, x_0 + X]$, we can define the space of *compact continuous streams* $C([0, T], C^\infty(x_0 - X, x_0 + X))$. Clearly, any function in $C(\mathbb{T}, \mathscr{A})$ can be mapped to a function in $C([0, T], C^\infty(x_0 - X, x_0 + X))$ via the restriction $u \mapsto u \restriction_{\mathbb{X}'}$. Moreover, $C([0, T], C^\infty(x_0 - X, x_0 + X))$ can be seen to be a Fréchet space with the family of pseudonorms $\| \cdot \|_{T,X,x_0,k}$ given by (6). Note that x_0, X and T are *fixed* and the indexing is on $k \in \mathbb{N}$.

Finally, observe that the operator $F_g : C(\mathbb{T}, \mathscr{A}) \to C(\mathbb{T}, \mathscr{A})$ has a restriction $F_g \restriction_{\mathbb{X}'}$ to the space $C([0, T], C^\infty(x_0 - X, x_0 + X))$.

Our next step is to prove contraction inequalities, which play an important role in fixed point techniques.

Lemma 1 (Contraction inequalities) *Consider the Fréchet space $C(\mathbb{T}, \mathscr{A})$ with pseudonorms $\| \cdot \|_{T,X,x_0,k}$ given by (6). Let $g \in C^\infty(\mathbb{R})$ and $F_g : C(\mathbb{T}, \mathscr{A}) \to C(\mathbb{T}, \mathscr{A})$ be given by (3). Then, for any $u, v \in C(\mathbb{T}, \mathscr{A})$, any pseudonorm $\| \cdot \|_{T,X,x_0,k}$ and any $m \in \mathbb{N}$, we have the following bound:*

$$\|F_g^m(u) - F_g^m(v)\|_{T,X,x_0,k} \leq \frac{(|\alpha|T)^m}{m!} \|u - v\|_{T,X,x_0,k+m}. \tag{7}$$

Proof By induction on m. □

Let us see how we can use these bounds in a proof.

Theorem 2 (Local Fréchet space convergence theorem) *Consider the Fréchet space $C(\mathbb{T}, \mathscr{A})$ with pseudonorms $\| \cdot \|_{T,X,x_0,k}$ given by (6). Take an initial input $u_0 \in C(\mathbb{T}, \mathscr{A})$ and initial condition $g \in C^\infty(\mathbb{R})$. Assume also that $u_0 = 0$ and g is analytic at x_0 with some radius of convergence*[1] *R. Let $F_g : C(\mathbb{T}, \mathscr{A}) \to C(\mathbb{T}, \mathscr{A})$ be given by (3). Then, for any $T, X \in \mathbb{R}^+$ such that $|\alpha|T + X < R$, the sequence (u_m)*

[1] Or equivalently, that g has a holomorphic extension on a disk of the complex plane with center x_0 and radius R; see Remark 1.

given by $u_m = F_g^m(u_0)$ *converges in the rectangle* $\mathbb{X}' = [0, T] \times [x_0 - X, x_0 + X]$ *to a fixed point of* $F_g \restriction_{\mathbb{X}'}$.

Proof To ease the exposition we introduce the pseudonorms on g given by

$$\|g\|_{X,x_0,k} = \sup_{|x-x_0|\le X} \left| \frac{\partial^k g}{\partial x^k}(x) \right|, \quad \text{for} \quad x_0 \in \mathbb{R},\ X \in \mathbb{R}^+,\ k \in \mathbb{N}. \tag{8}$$

Since g is analytic at x_0 with radius of convergence R, there is a sequence of real coefficients (a_j) such that, for all $x \in (x_0 - R, x_0 + R)$,

$$g(x) = \sum_{j=0}^{\infty} a_j (x - x_0)^j.$$

It also follows that $\limsup_{n\to\infty} \sqrt[n]{|a_n|} \le \frac{1}{R}$ (see Footnote 1). Moreover, we have the following bound, for any $X < R$:

$$\|g\|_{X,x_0,k} = \left| \sup_{|x-x_0|<X} \sum_{j=0}^{\infty} \frac{(j+k)!}{j!} a_{j+k} (x - x_0)^j \right| \le \sum_{j=0}^{\infty} \frac{(j+k)!}{j!} |a_{j+k}| X^j. \tag{9}$$

Let $T, X \in \mathbb{R}^+$ such that $|\alpha|T + X < R$. We show that (u_m) is a Cauchy sequence with respect to the pseudonorm $\|\cdot\|_{T,X,x_0,k}$. First observe that

$$\begin{aligned}
\sum_{m=0}^{\infty} \|u_{m+1} - u_m\|_{T,X,x_0,k} &= \sum_{m=0}^{\infty} \|F_g^m(g) - F_g^m(0)\|_{T,X,x_0,k} \\
&\overset{1}{\le} \sum_{m=0}^{\infty} |\alpha|^m \frac{T^m}{m!} \|g\|_{X,x_0,k+m} \\
&\overset{2}{\le} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} |\alpha|^m T^m X^j |a_{k+m+j}| \frac{(k+m+j)!}{m!j!} \\
&\overset{3}{=} \sum_{s=0}^{\infty} \sum_{m=0}^{s} |\alpha|^m T^m X^{s-m} |a_{k+s}| \frac{(k+s)!}{m!(s-m)!} \\
&\overset{4}{=} \sum_{s=0}^{\infty} (|\alpha|T + X)^s |a_{k+s}| \frac{(k+s)!}{s!},
\end{aligned}$$

where (1) is justified by the Contraction Inequalities (Lemma 1), (2) by equation (9), (3) by rearranging the sum and adding over diagonals $s = m + j$ and (4) by taking the binomial expansion of $(|\alpha|T + X)^s$.

By the root test, the above series is convergent, since

$$\limsup_{s\to\infty} \sqrt[s]{(|\alpha|T + X)^s |a_{k+s}| \frac{(k+s)!}{s!}} = (|\alpha|T + X) \cdot \limsup_{n\to\infty} \sqrt[n]{|a_n|} \cdot 1 < \frac{R}{R} = 1.$$

Since the series is convergent, it follows that, for $i < j$,

$$\|u_j - u_i\|_{T,X,x_0,k} \leq \sum_{m=i}^{j-1} \|u_{m+1} - u_m\|_{T,X,x_0,k} \leq \sum_{m=i}^{\infty} \|u_{m+1} - u_m\|_{T,X,x_0,k} \xrightarrow[i\to\infty]{} 0.$$

Hence (u_m) is a Cauchy sequence with respect to the pseudonorm $\|\cdot\|_{T,X,x_0,k}$. Since this holds for all $k \in \mathbb{N}$ and $C([0,T], C^\infty(x_0 - X, x_0 + X))$ is complete, it follows that (u_m) has a limit in $\mathbb{X}'$. Now, using continuity of $F_g\restriction_{\mathbb{X}'}$, we conclude that this limit must be a fixed point of $F_g\restriction_{\mathbb{X}'}$. □

Remark 1 The reader should distinguish between the following two concepts:

- the existence of a holomorphic function, defined in a disk of the complex plane $\mathbb{C}$, which coincides with g at the real axis $\{y = 0\}$;
- the convergence of the construction $u_m = F_g^m(0)$ to a fixed point u, defined in a rectangle of $\mathbb{T} \times \mathbb{R}$, which coincides with g at initial time $\{t = 0\}$.

As seen from Theorem 2, the existence of a holomorphic extension implies convergence to a fixed point. Both these functions (the holomorphic extension and the fixed point) can be depicted by planar diagrams, and both can be seen as extensions of g (see Fig. 3). However, these functions, and the domains which they inhabit, are substantially different.

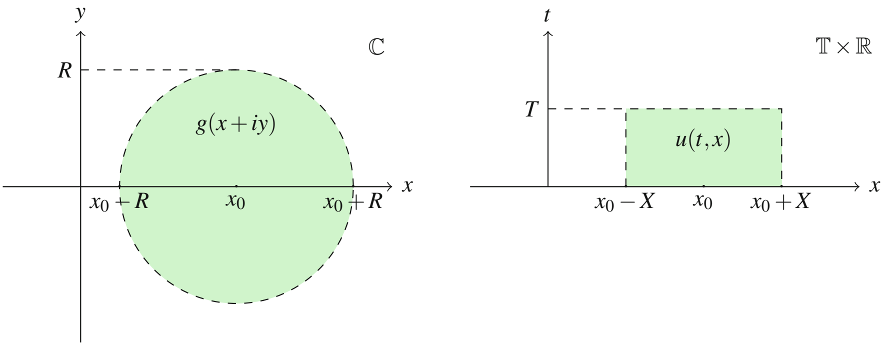


Fig. 3 On the *left*: a function $g(x + iy)$ of type $\mathbb{C} \to \mathbb{C}$, defined in a disk, that coincides with g at $\{y = 0, x_0 - R < x < x_0 + R\}$. On the *right*: a fixed point $u(t,x)$ of type $\mathbb{T} \times \mathbb{R} \to \mathbb{R}$, defined in a rectangle, that coincides with g at $\{t = 0, x_0 - X < x < x_0 + X\}$. The rectangle and disk dimensions follow the relation $|\alpha|T + X < R$

4 Global Convergence Theorems

As an immediate corollary of Theorem 2, we have:

Theorem 3 (First global Fréchet space convergence theorem) *Consider the Fréchet space* $C(\mathbb{T}, \mathscr{A})$ *with pseudonorms* $\|\cdot\|_{T,X,x_0,k}$ *given by* (6). *Take an initial input* $u_0 \in C(\mathbb{T}, \mathscr{A})$ *and initial condition* $g \in C^\infty(\mathbb{R})$. *Assume also that* $u_0 = 0$ *and* g *is entire (i.e. has a holomorphic extension to the complex plane). Let* $F_g : C(\mathbb{T}, \mathscr{A}) \to C(\mathbb{T}, \mathscr{A})$ *be given by* (3). *Then the sequence* (u_m) *given by* $u_m = F_g^m(u_0)$ *converges to a fixed point of* F.

Proof Since g is entire, it is analytic at any x_0 with any radius of convergence R. Thus, by Theorem 2, the sequence u_m converges to a fixed point on any compact rectangle $[0, T] \times [x_0 - X, x_0 + X]$. Therefore, we have convergence of u_m for any pseudonorm $\|\cdot\|_{T,X,x_0,k}$, so that we have convergence in $C(\mathbb{T}, \mathscr{A})$. □

The next step is to generalize Theorem 3 to a larger class of initial functions u_0 (other that $u_0 = 0$). We do that proof in two steps: first assume $g = 0$ to establish sufficient conditions on u_0; then consider the more general case $g \in C^\infty(\mathbb{R})$.

Definition 3 We say that a function $u \in C(\mathbb{T}, \mathscr{A})$ is *uniformly entire* if $u(t, x) = \sum_{j=0}^{\infty} a_j(t)x^j$ for some sequence of functions $(a_j) \in C(\mathbb{R})$ such that

$$\lim \sqrt[j]{\sup_{0 \le t \le T} |a_j(t)|} = 0 \text{ for all } T \in \mathbb{T}.$$

The motivation for this terminology is that, for such a function u, the section $x \mapsto u(t, x)$ is entire for all t, and the convergence $\sqrt[j]{|a_j(t)|} \to 0$ is uniform in t.

Theorem 4 *Consider the Fréchet space* $C(\mathbb{T}, \mathscr{A})$ *with the family of pseudonorms* $\|\cdot\|_{T,X,x_0,k}$ *given by* (6). *Let also* $u_0 \in C(\mathbb{T}, \mathscr{A})$ *be an initial input, and* $g \in C^\infty(\mathbb{R})$ *be an initial condition. We assume in addition that* u_0 *is uniformly entire and* $g = 0$. *Let* $F_0 : C(\mathbb{T}, \mathscr{A}) \to C(\mathbb{T}, \mathscr{A})$ *be given by*

$$F_0(u)(t, x) = \alpha \int_0^t \partial_x u(s, x) ds. \tag{10}$$

Then the sequence (u_m) *given by* $u_m = F_0^m(u_0)$ *converges to* 0.

Proof To ease the exposition we introduce the pseudonorms on a_j given by

$$\|a_j\|_T = \sup_{0 \le t \le T} |a_j(t)|, \text{ for } T \in \mathbb{T}. \tag{11}$$

We show that $\sum_m \|u_m\|_{T,X,0,k}$ is a convergent series for any pseudonorm $\|\cdot\|_{T,X,x_0,k}$ with $x_0 = 0$. We have that (see proof of Theorem 2)

$$\begin{aligned}
\sum_{m=0}^{\infty} \|u_m\|_{T,X,0,k} &= \sum_{m=0}^{\infty} \|F_0^m(u_0) - F_0^m(0)\|_{T,X,0,k} \\
&\leq \sum_{m=0}^{\infty} \frac{|\alpha|^m T^m}{m!} \|u_0\|_{T,X,0,k+m} \\
&= \sum_{m=0}^{\infty} \frac{|\alpha|^m T^m}{m!} \sup_{\substack{0\leq t\leq T \\ |x|\leq X}} \left| \sum_{j=0}^{\infty} \frac{(j+k+m)!}{j!} a_{j+k+m}(t) x^j \right| \\
&\leq \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} |\alpha|^m T^m X^j \frac{(j+k+m)!}{m! j!} \|a_{j+k+m}\|_T \\
&= \sum_{s=0}^{\infty} \sum_{m=0}^{s} |\alpha|^m T^m X^{s-m} \frac{(k+s)!}{m!(m-s)!} \|a_{k+s}\|_T \\
&= \sum_{s=0}^{\infty} (|\alpha| T + X)^s \frac{(k+s)!}{s!} \|a_{k+s}\|_T .
\end{aligned}$$

By the root test, the above series is convergent, since

$$\sqrt[s]{(|\alpha|T + X)^s \frac{(k+s)!}{s!}} \quad \underset{s\to\infty}{\longrightarrow} \quad |\alpha|T + X$$

and $\sqrt[s]{\|a_{k+s}\|_T} \underset{s\to\infty}{\longrightarrow} 0$ by assumption. Therefore $\sum \|u_m\|_{T,X,0,k}$ is convergent, so that $\|u_m\|_{T,X,0,k} \underset{m\to\infty}{\longrightarrow} 0$ and thus u_m converges to 0, as we wanted to prove. □

We now combine Theorems 3 and 4 to prove our most general result.

Theorem 5 (Second global Fréchet space convergence theorem) *Consider the Fréchet space $C(\mathbb{T}, \mathscr{A})$ with pseudonorms $\|\cdot\|_{T,X,x_0,k}$ given by (6). Take an initial input $u_0 \in C(\mathbb{T}, \mathscr{A})$ and initial condition $g \in C^\infty(\mathbb{R})$. Assume also that u is uniformly entire and that g is entire. Let $F_g : C(\mathbb{T}, \mathscr{A}) \to C(\mathbb{T}, \mathscr{A})$ be given by (3). Then the sequence (u_m) given by $u_m = F^m(u_0)$ converges to a fixed point of F.*

Proof Let $F_g, F_0 : C(\mathbb{T}, \mathscr{A}) \to C(\mathbb{T}, \mathscr{A})$ be given by (3), (10). We observe that, for any $u, v \in C^{0,\infty}(\mathbb{X})$ we have

$$F_g(u+v) = g + \alpha \int_0^t (u+v)_x ds = g + \alpha \int_0^t u_x ds + \alpha \int_0^t v_x ds = F_g(u) + F_0(v).$$

We can then infer that $u_1 = F_g(u_0) = F_g(0 + u_0) = F_g(0) + F_0(u_0)$. Also, $u_2 = F_g(u_1) = F_g(F_g(0) + F_0(u_0)) = F_g^2(0) + F_0^2(u_0)$, and, in general,

$$u_m = F_g^m(0) + F_0^m(u_0).$$

By Theorem 3, $(F_g^m(0))$ converges to a fixed point of F_g. By Theorem 4, $(F_0^m(u_0))$ converges to 0. Therefore, (u_m) and $(F_g^m(0))$ have the same limit. In particular, (u_m) converges to a fixed point of F_g. □

A nice consequence of the proof is that it allows us to also establish uniqueness in a certain class of functions.

Theorem 6 (Uniqueness of uniformly entire fixed points) *Consider the Fréchet space $C(\mathbb{T}, \mathscr{A})$ with pseudonorms $\| \cdot \|_{T,X,x_0,k}$ given by (6). Take an initial condition $g \in C^\infty(\mathbb{R})$ and assume also that g is entire. Let $F_g : C(\mathbb{T}, \mathscr{A}) \to C(\mathbb{T}, \mathscr{A})$ be given by (3). Then there is at most one uniformly entire fixed point of F_g.*

Proof Let u be any uniformly entire fixed point of F_g. By the proof of Theorem 5, we know that $u = F_g^m(u) = F_g^m(0) + F_0^m(u)$. Since $(F_0^m(u_0))$ converges to 0, we get that $(F_g^m(0))$ converges to u. Thus any uniformly entire fixed point of F_g must coincide with the limit of $(F_g^m(0))$. □

5 Conclusion and Further Research

In this paper we have seen how to study solutions to differential equations as outputs of analog networks, and how to obtain them using fixed point techniques. The example we have considered ($L = \alpha \partial_x$) is a well-known problem whose solution can be obtained analytically by taking a Fourier transform. However, the method presented in this paper provides a different perspective which is suitable for analog computability and the study of analog systems as in [6, 11–13], where Fréchet spaces clearly provide a natural framework.

We intend to investigate this approach (fixed points in Fréchet spaces) in more general settings. In fact, as a next step one can look at a more general operator $L : \mathscr{A} \to \mathscr{A}$ using higher-order derivatives, for example with bounds of the form

$$\|Lu\|_{T,X,x_0,k} \leq C \|\partial_x^\ell u\|_{T,X,x_0,k} \leq C \|u\|_{T,X,x_0,k+\ell}. \tag{12}$$

We observe that analyticity of the initial condition g is not enough to ensure existence of solutions. A counterexample is given by the heat equation $u_t = u_{xx}$ with initial condition $g(x) = \frac{1}{1-x}$. Even though g is analytic near zero, the solution fails to be analytic at a neighborhood of the origin (see [2]). Thus, the general case may require different tools such as explicit bounds on the pseudonorms of g.

We also plan to investigate properties of the fixed points, such as *computability*, *continuity* and *stability*, as functions of the parameters and initial conditions.

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