We prove a priori inequalities for the quasilinear equation
\[ Lw = \partial_x^2 w + \partial_y [k(x, w(x, y)) \partial_y w] = 0 \]
where \( k(x, y) \) is smooth and nonnegative, and positive for \( x \neq 0 \). This is then used to obtain a regularity theorem for the Dirichlet problem for the Monge-Ampère equation,
\[ u_{xx} u_{yy} - (u_{xy})^2 = k(x, y), \]
and the prescribed Gaussian curvature equation,
\[ u_{xx} u_{yy} - (u_{xy})^2 = k(x, y) (1 + u_x^2 + u_y^2)^2, \]
where \( k(x, y) \) is close to a function of one variable alone when \( k \) is small, but permitted to vanish to infinite order.

1. INTRODUCTION

The regularity of solutions to boundary value problems for second order partial differential equations,
\[
\begin{cases}
F(x, u, \nabla u, \nabla^2 u) = 0 & \text{in } \Omega \\
u = \phi & \text{on } \partial \Omega,
\end{cases}
\]
has been investigated primarily in two cases: \( F \) is elliptic, and \( F \) is linear. When \( F \) is elliptic, the problem (1.1) is typically hypoelliptic; the solution \( u \) is smooth if the data \( F, \phi \) and \( \partial \Omega \) are smooth (see e.g. Caffarelli and Cabré [3], Gilbarg and Trudinger [9] and the references there). When \( F \) is linear and ellipticity fails to only finite order - i.e. \( F \) is subelliptic - matters are well understood and (1.1) is again hypoelliptic (see e.g. Treves [28]). When \( F \) is linear but fails to be subelliptic, the situation is more complicated, and since this will provide a point of departure for our work, we give an example. In [7], Fedjii showed that the operator \( \partial_x^2 + f(x) \partial_y^2 \) is hypoelliptic on \( \mathbb{R}^2 \) if \( f(x) \) is smooth for all \( x \) and positive for \( x \neq 0 \), the point being that an arbitrary rate of vanishing of \( f \) at \( x = 0 \) is permitted. (However, the addition of an elliptic term in an extra variable, \( \partial_z^2 \), leads to an operator \( \partial_x^2 + f(x) \partial_y^2 + \partial_z^2 \) that fails to be hypoelliptic on \( \mathbb{R}^3 \) according to the rate of infinite order vanishing of \( f(x) \) at \( x = 0 \) - Kusuoka and Stroock [17], and see also Christ [6] and the references there). In [10], Guan considered the special case of (1.1) given in (1.2) below where both ellipticity and linearity fail, but assumed subellipticity as well as the nonvanishing of the rank of the Hessian of the solution \( u \).

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In this paper we will investigate the regularity of solutions to a special case of (1.1) where both subellipticity and linearity fail, in effect a nonlinear variation of Fediù’s example. In a sense, the simplest situation arises in two dimensions when \( F \) is of Monge-Ampère type,

\[
\det \nabla^2 u = u_{xx}u_{yy} - u_{xy}^2 = f(x, y, u, u_x, u_y), \quad (x, y) \in \Omega.
\]

(1.2)

Even here, regularity is victim to the simplest failure of ellipticity: if \( f(x, y, r, p, q) = x^2 + y^2 \) in the unit disk \( \Omega \) and \( \phi \equiv 0 \), then \( u = (x^2 + y^2)^{3/2} - 1 \) is not \( C^3 \) at the origin where \( f \) vanishes. While the equation (1.2) has a long history, which we discuss briefly in the next section, we will concentrate momentarily on a single thread here. Following [26], [10] and [24], we apply the partial Legendre transform (or semispherical mapping in [24])

\[
\begin{align*}
 s &= x \\
 t &= u_y(x, y)
\end{align*}
\]

which reduces the question of interior regularity of solutions to (1.2) to the regularity of solutions of the quasilinear second order equation

\[
\partial^2_s w + \partial_t f(s, w(s, t), r(s, t), z(s, t), t) \partial_t w = 0, \quad (s, t) \in T\Omega,
\]

(1.3)

where \( w = y(s, t) \) arises from inverting the partial Legendre transform \( T \), and \( r = u, z = u_x \) satisfy

\[
\begin{align*}
 r_s &= z + tw_s, \quad r_t = tw_t \\
 z_s &= fw_t, \quad z_t = -w_s
\end{align*}
\]

In the case \( f = 1 \), we thus have that \( z + iw \) is an analytic function of \( s + it \).

In [26], F. Schulz used the partial Legendre transform to establish the interior regularity of solutions to (1.3) in the elliptic case \( f > 0 \), and then the interior regularity for solutions of (1.2), also in the elliptic case (these results had been obtained much earlier by different methods in the work of Heinz, Lewy, Nirenberg and Pogorelov cited below). In [10], P. Guan partially extended this result to the case when (1.2) is subelliptic - \( f \) vanishes to at most finite order - and the solution \( u \) has one nonvanishing principle curvature, by using results of B. Franchi [8] on (1.3), which amount to a subelliptic version of J. Moser’s treatment of second order linear elliptic equations with bounded measurable coefficients.

In this paper, we consider the case \( f \geq 0 \) of (1.2) and (1.3) without subellipticity, or additional assumptions on the solution \( u \). In both the elliptic and subelliptic cases of (1.3), it is possible to deal directly with a weak solution \( w \) and increase its index of smoothness by applying techniques involving J. Moser’s iteration scheme. When subellipticity fails however, the corresponding integral estimates on derivatives of the solution \( w \) result in the quantity to be estimated appearing also on the right side of the inequality, albeit multiplied by a small constant. Thus the finiteness of these quantities must be assumed in advance in order to absorb them into the left side of the equation, and we must consequently rely on a priori estimates for solutions. In other words, we must assume in advance what we wish to prove, namely that a solution is smooth, and are only able to show that the sizes of its derivatives are under some sort of uniform control. It is then necessary to approximate a given weak solution by smooth solutions in an appropriate way.

Two difficulties now arise. First, even if we can obtain appropriate a priori estimates for (1.3) in \( T\Omega \), they may not translate into appropriate a priori estimates...
for (1.2) in all of $\Omega$, since $T$ depends on the approximating solution. The second and more profound difficulty lies in finding a sequence of smooth solutions that converges appropriately. For example, following [10], if $u$ is a $C^{1,1}(\overline{\Omega})$ generalized solution of (1.2) (see Appendix B at the end of the paper for a brief review of Alexandrov’s notion of generalized solutions) with $u_{yy} \geq c > 0$, then the partial Legendre transform is invertible and we may suspect that $u$ should be smooth in $\Omega$. Indeed, if $u$ is smooth, then we are able to obtain good a priori estimates on its derivatives depending on the constant $c > 0$ above. However, if we consider the elliptic problem with $f + \delta$ in place of $f$, a common approximation technique, there is no minimum principle to guarantee that the corresponding smooth solutions $u^\delta$ would also satisfy $u^\delta_{yy} \geq c > 0$ uniformly in $\delta > 0$, and so no way to obtain uniform a priori estimates. Alternatively, other approximation schemes for the quasilinear equation (1.3), such as convolution with approximate identities, multiplying by smooth cutoff functions whose derivatives are supported where $f > 0$, or using Hilbert space theory, all have their own peculiar breakdowns.

We succeed in establishing the smoothness in $\Omega$ (or rather $\Omega^*$ - see below) of $C^{1,1}(\overline{\Omega})$ solutions to (1.2) for the Monge-Ampère equation, $f(x, y, r, p, q) = k(x, y)$, and the prescribed Gaussian curvature equation,

$$f(x, y, r, p, q) = K(x, y)(1 + p^2 + q^2)^2,$$

in the case that the nonlinearities are mitigated by an assumption that $f$ is close to being independent of $y$ when $f$ is small, namely the condition

$$\frac{\partial}{\partial y} f \leq C f^\frac{3}{2}.$$  

(1.4)

If $f$ is actually dependent only on $x$, then (1.3) is linear and the theory of hypoelliptic linear equations mentioned above can be brought to bear; thus in a way, the above condition postulates a balance between the failure of ellipticity and the failure of linearity. An interesting special case arises when the prescribed Gaussian curvature $K$ depends only on $x$. Then $f(x, y, r, p, q)$ is $K(x)(1 + p^2 + q^2)^2$, so that (1.4) holds trivially while (1.3) remains nonlinear.

We emphasize that in obtaining smoothness of $C^{1,1}(\overline{\Omega})$ solutions across at least part of the degenerate set where $f$ vanishes, our results involve assumptions on only the data $f$ and $\phi$, and not the solution $u$ itself. Moreover, $f$ may vanish to infinite order, rendering the nonlinear problem nonsubelliptic, and therein lies both the novelty and the difficulty (of course there are additional restrictions in the theorems we prove). Examples of functions $k(x, y)$ or $K(x, y)$, with $f$ satisfying our condition (1.4), are those smooth functions of the form $F(x) \frac{1}{1 + \sqrt{F(x)h(x, y)}}$, and as indicated in Appendix B, essentially all such functions have this form.

We close this introduction with a few words on how the second of the difficulties mentioned above is overcome. The maximum principle shows that sup norm bounds hold for solutions $w$ to (1.3), with constants depending only on the data. In general, however, a breakdown occurs in passing to sup norm bounds for the gradient of $w$. As in either the elliptic or linear cases, we prove that we can bound higher order derivatives in terms of gradient bounds if $f > 0$ for $x \neq 0$, but this still leaves the gap between $w$ and $\nabla w$. The above hypothesis (1.4) on $f$, in conjunction with Moser iteration techniques, allows us to close this gap and thus "glue" the
approximating smooth solutions \( u^\delta \) to the weak solution \( u \). Remark 2.3 below comments on the exponent \( \frac{2}{3} \) in (1.4).

2. Statement of results, examples and prior work

In this section, we expand on the history of the Monge-Ampère type equations given in the introduction, and state our main theorems, along with some illustrative examples.

2.1. Monge-Ampère equation. We begin by considering the smoothness of solutions to the Dirichlet problem for the Monge-Ampère equation,

\[
\begin{aligned}
  u_{xx}u_{yy} - (u_{xy})^2 &= k(x, y), \quad (x, y) \in \Omega \\
  u &= \phi(x, y), \quad (x, y) \in \partial \Omega,
\end{aligned}
\]

where \( k \) and \( \phi \) are smooth (infinitely differentiable) and \( \Omega \) is a bounded convex planar domain with smooth positively curved boundary \( \partial \Omega \). Solutions \( u \) to (2.1) with \( k > 0 \) are either convex or concave according to whether \( u_{yy} \) is positive or negative. One may pass from one to the other by replacing \( u \) and \( \phi \) by \(-u\) and \(-\phi\). In [4], Caffarelli, Nirenberg and Spruck have shown that if \( k \) is positive on \( \overline{\Omega} \), then there is a unique convex solution \( u \) of (2.1) in \( C^\infty(\overline{\Omega}) \) (earlier results in two dimensions are in E. Heinz [13], [14], H. Lewy [19], [20], L. Nirenberg [22] and A. V. Pogorelov [23]). In [11] and [12], Guan, Trudinger and Wang have shown that for \( k \) merely nonnegative and smooth, there is a unique convex solution \( u \) to (2.1) in the generalized sense of Alexandrov (see the appendix) in \( C^{1,1}(\overline{\Omega}) \) with norm depending only on \( \|k\|_{C^{1,1}(\overline{\Omega})}, \|\phi\|_{C^{3,0}(\partial \Omega)} \) and \( \Omega \).

Constructions in Bedford and Fornaess [2], and an explicit example of Sibony reported in [10] and [12], show that \( C^{1,1}(\overline{\Omega}) \) cannot be improved (see Remark 2.5 below for a refinement of the Sibony example). In [2], a function \( u \in C^{1,1}(\overline{\Omega}) \setminus C^2(\overline{\Omega}) \) is constructed satisfying (2.1) with \( \phi = 0 \) and \( k \in C^\infty(\Omega) \). In Appendix A we show that in many cases, including those in which the data are symmetric about the \( y \)-axis, \( k = 0 \) on the \( y \)-axis and \( \phi \equiv 0 \), the \( C^{1,1}(\overline{\Omega}) \) solutions are never in \( C^2(\overline{\Omega}) \) (symmetry was also used in [2] to obtain counterexamples to regularity of the complex Monge-Ampère equation). Moreover, as the example

\[
\begin{aligned}
  u(x, y) &= \frac{3}{2}|(x, y)|^3 = \left(x^2 + y^2\right)^{\frac{3}{2}}, \\
  u_{xx}u_{yy} - (u_{xy})^2 &= 18 |(x, y)|^2 = 18 \left(x^2 + y^2\right),
\end{aligned}
\]

shows, we may have failure of smoothness even when \( k \) vanishes to second order at an isolated point. In fact, radial solutions to (2.1) in the unit disk \( \Omega \) are easily characterized - see Appendix B at the end of the paper. Here we only remark that if \( k(x, y) = r^{2N} \) (where \( r^2 = x^2 + y^2 \)), then \( u \approx C + r^{N+2} \), which fails to be smooth for \( N \) a positive odd integer. The case \( N = 1 \) is the example cited above.

In [10], Guan has shown that the \( C^{1,1}(\overline{\Omega}) \) solution \( u \) to (2.1) is smooth if \( k \) vanishes to finite order in the sense that

\[
k(x, y) \approx |x|^{2\ell} + By^{2m}, \quad (x, y) \in \Omega,
\]

for some \( B \geq 0 \) and positive integers \( \ell \leq m \), and if in addition,

\[
u_{yy} \geq c > 0.
\]
This suggests that the difficulty with the radial examples above may lie in the degeneracy of the rank of the Hessian (which vanishes at the origin for the radial examples, but not in Guan’s theorem).

Here we will give conditions on \( k \) and \( \phi \) alone that guarantee the smoothness of solutions to (2.1) - namely that \( k > 0 \) for \( x \neq 0 \) and \( k \) is close to a function of \( x \) alone when \( k \) is small. The precise notion we need is

\[
|k_2(x, y)| \leq C_L k(x, y)^{\frac{3}{2}}, \quad (x, y) \in L,
\]

for all compact subsets \( L \) of \( \Omega \) (see Appendix B for a discussion of how (2.3) relates to \( k \) being almost independent of \( y \)). By the rotation invariance of the Hessian, we may instead assume the analogous condition requiring \( k \) to be close to a function of \( ax + by \) alone when \( k \) is small (where \((a, b)\) denotes a unit vector). As motivation for considering the condition (2.3), we observe that if \( k(x, y) = k(x) \) is smooth, positive for \( x \neq 0 \) and independent of \( y \), then the quasilinear equation arising from the partial Legendre transform (see subsubsection 2.3.1 below for details),

\[
Lw = \left[ \partial^2_x + \partial_y k(x) \partial_y \right] w = 0,
\]

is the linear equation of Fediù mentioned in the introduction, and the known linear hypoelliptic theory implies that all solutions of (2.4) are smooth (see Fediù [7] and also [6] and [16]).

We prove that if (2.3) holds, then \( u \) is smooth except possibly on certain line segments of the \( y \)-axis that have at least one endpoint on the boundary of \( \Omega \). In order to make this statement precise, we introduce the quantities

\[
\omega^-(x) = \inf_{y : (x, y) \in \Omega} u_y(x, y),
\]

\[
\omega^+(x) = \sup_{y : (x, y) \in \Omega} u_y(x, y),
\]

for \( x \in P\Omega \), the projection of \( \Omega \) onto the \( x \)-axis, along with the following subdomain of \( \Omega \):

\[
\Omega^* = \\{(x, y) \in \Omega : \omega^-(x) < u_y(x, y) < \omega^+(x)\} = \Omega - \{(0, y) : u_y(0, y) = \omega^-(0) \text{ or } \omega^+(0)\}.
\]

Note that the inifimums in (2.5) are not attained when \( x \neq 0 \). Moreover, if \( u \in C^1(\overline{\Omega}) \), then

\[
\omega^-(x) = u_y(x, \alpha(x)), \omega^+(x) = u_y(x, \beta(x))
\]

where the lower and upper boundaries of \( \Omega \) are given as the graphs of \( \alpha \) and \( \beta \) respectively. The subdomain \( \Omega^* \) is obtained from \( \Omega \) by removing any segments from the \( y \)-axis that are projections onto the \( x, y \)-plane of maximal straight line segments in the graph of \( u \) that have at least one endpoint lying above the boundary \( \partial \Omega \). See [23], [5] and Appendix A below for a discussion of such segments in \( \Omega \), which we will refer to as Pogorelov segments. Note that we ignore any maximal line segments in the graph of \( u \) both of whose endpoints lie above points in \( \Omega \) - such segments will not exist under the hypotheses of our theorems. We emphasize again that in the theorem below, \( k \) may vanish to infinite order on the line \( x = 0 \), and it is this lack of subellipticity in conjunction with the nonlinearity that is the new element here.
Theorem 2.1. Suppose \( k(x,y) \) is smooth and nonnegative in \( \Omega \), is positive for \( x \neq 0 \), and satisfies (2.3), namely
\[
|\partial_x k(x,y)| \leq C_L k(x,y)^{\frac{3}{2}}, \quad (x,y) \in L \text{ compact } \subset \Omega.
\]
Suppose \( u \) is a \( C^{1,1}(\overline{\Omega}) \) convex solution to the Monge-Ampère boundary value problem
\[
(2.7) \quad \begin{cases}
 u_{xx}u_{yy} - (u_{xy})^2 = k(x,y), & (x,y) \in \Omega \\
 u = \phi(x,y), & (x,y) \in \partial \Omega,
\end{cases}
\]
where \( \phi \) is smooth and \( \partial \Omega \) is smooth with positive curvature. Then \( u \) is smooth in the subdomain \( \Omega^* \) as given in (2.6), and \( u_{yy} > 0 \) in \( \Omega^* \). Moreover, \( \Omega^* \) is connected provided
\[
(2.8) \quad \omega_-(0) < \omega^+(0).
\]

In many cases, conditions on the boundary data \( \phi \) prohibit line segments in the graph of the solution \( u \) from extending all the way across \( \Omega \), and thus force condition (2.8). Note that it is impossible for two Pogorelov segments to meet in \( \Omega \) if \( u \) is \( C^{1,1} \). An important example occurs when \( \phi \equiv 0 \) (but \( k \) is not identically zero). Indeed, a nontrivial convex function on \( \overline{\Omega} \) that vanishes on \( \partial \Omega \) is strictly negative in \( \Omega \). Thus \( \Omega^* \) is connected for convex \( C^{1,1} \) solutions to the homogeneous boundary value problem for the Monge-Ampère equation when \( k > 0 \) for \( x \neq 0 \). Alternatively, \( \Omega^* \) is connected if the convex hull of the boundary space curve
\[
(2.9) \quad \Gamma_\phi = \{(x,y,\phi(x,y)) : (x,y) \in \partial \Omega\}
\]
contains a point lying strictly below the line segment \( L_\phi \) that joins the two points of intersection of the curve \( \Gamma_\phi \) and the plane \( x = 0 \). Indeed, this prevents any supporting plane of a convex solution to (2.1) from containing \( L_\phi \), and so \( L_\phi \) cannot lie in the graph of \( u \).

In Appendix A below we show that for many of the boundary value problems (2.7) in Theorem 2.1, including the homogeneous ones, solutions are either smooth in all of \( \Omega \), or nearly as irregular as can be. In particular, we show that if \( \phi \equiv 0 \), \( k \) vanishes on the \( y \)-axis and \( u \) has a Pogorelov segment, then \( u \notin C^2(\overline{\Omega}) \cap C^3(\Omega) \).

Thus for the homogeneous case of (2.7) in Theorem 2.1, we conclude that \( u \in C^{1,1}(\overline{\Omega}) \cap C^\infty(\Omega^*) \), and that in addition, we have either
\[
u \in C^{1,1}(\overline{\Omega}) \cap C^\infty(\Omega)
\]
if there are no Pogorelov segments, or we have
\[
u \notin C^2(\overline{\Omega}) \cap C^3(\Omega)
\]
if there is a Pogorelov segment.

We close our discussion of Pogorelov segments here by indicating additional conditions on solutions \( u \) that prohibit their existence. For example, if \( u_y(0,\cdot) \) is not constant near the two points in \( \partial \Omega \cap \{x = 0\} \), then Pogorelov segments cannot exist and \( u \) is smooth in all of \( \Omega \). Less trivially, if \( u \in C^2(\Omega) \) solves (2.1) with data symmetric about the \( y \)-axis, and its Hessian has rank at least one everywhere, then again Pogorelov segments cannot exist and \( u \) is smooth in all of \( \Omega \) (this is a consequence of Theorem 6.3 in Appendix A; a shorter proof can be distilled from the arguments used there - see Remark 6.2). This lends support to the notion that a solution \( u \) should be smooth if the rank of its Hessian is positive everywhere. The
solution \( u \) in our theorem does in fact satisfy \( u_{yy} > 0 \) away from any exceptional segments, thus lending even further support.

2.2. Prescribed Gaussian curvature. We also consider more general Monge-Ampère equations and show that for \( K \) as above - smooth, positive for \( x \neq 0 \), and close to a function of \( x \) alone when \( K \) is small - the homogeneous boundary value problem for the equation of prescribed Gaussian curvature

\[
\frac{u_{xx}u_{yy} - (u_{xy})^2}{(1 + u_x^2 + u_y^2)^2} = K(x, y),
\]

has smooth solutions in \( \Omega \) except possibly on the at most two Pogorelov segments mentioned above (they are separated and have one endpoint each on \( \partial \Omega \) if they exist). Here we use the \( a \) priori estimates of Guan [11] for the equation of prescribed Gaussian curvature as well as imposing the additional condition of Ivochkina [15],

\[
0 \leq K(x, y) < \lambda, \quad (x, y) \in \overline{\Omega},
\]

where \( \lambda \) is the minimal curvature of \( \partial \Omega \), along with the condition \(|K_2| \leq CK^{\frac{3}{2}}\).

**Theorem 2.2.** Suppose \( K(x, y) \) is smooth, nonnegative and satisfies (2.10) in \( \overline{\Omega} \). Moreover, we suppose \( K(x, y) \) is positive for \( x \neq 0 \) and satisfies (2.3) with \( K \) in place of \( k \), namely

\[
|\partial_s K(x, y)| \leq C_L K(x, y)^{\frac{3}{2}}, \quad (x, y) \in \text{compact} \subset \Omega.
\]

Suppose \( u \) is the convex \( C^{1,1}(\overline{\Omega}) \) solution to the prescribed Gaussian curvature homogeneous boundary value problem

\[
\begin{cases}
  u_{xx}u_{yy} - (u_{xy})^2 &= K(x, y) \left(1 + u_x^2 + u_y^2\right)^2, & (x, y) \in \Omega \\
  u &= 0, & (x, y) \in \partial \Omega,
\end{cases}
\]

where \( \partial \Omega \) is smooth with positive curvature (as given in [11]). Then \( u \) is smooth in the connected subdomain \( \Omega^* \) given by (2.6).

**Remark 2.1.** Theorem 2.2 applies in particular when the Gaussian curvature \( K = K(x) \) is a function of \( x \) alone. In this case, unlike that of the Monge-Ampère equation, the partial Legendre transform results in a nonlinear system of equations (see (2.25) below):

\[
\begin{align*}
  \partial_s^2 + \partial_t K(s) \left(1 + z(s, t)^2 + t^2\right)^2 \partial_t w(s, t) &= 0, \\
  \partial_t z(s, t) &= K(s) \left(1 + z(s, t)^2 + t^2\right)^2 \partial_t w(s, t),
\end{align*}
\]

The second order equation (2.12) is the compatibility condition \( \partial_s \partial_t z = \partial_t \partial_s z \) for the first order Cauchy-Riemann system (2.13).

Remark 2.4 below addresses the possibility of theorems more general than Theorem 2.2.

2.3. Quasilinear equations. In this subsection we consider the quasilinear equation that results from an application of the partial Legendre transform to the Monge-Ampère equation.
2.3.1. Partial Legendre transform. As in [10] and [26], we will use the partial Legendre transformation \((s, t) = T(x, y)\) given by

\[
\begin{align*}
\begin{cases}
  s & = x \\
  t & = u_y(x, y)
\end{cases}
\end{align*}
\]

where \(u\) is a convex \(C^{1,1}(\Omega)\) solution of (2.1). We first note that if \(k(x_0, y_0) > 0\), then \(u_{yy}(x_0, y_0) > 0\) and \(u_y(x, y)\) is strictly increasing in \(y\) near \((x_0, y_0)\), making the partial Legendre transform \(T\) one-to-one on the set where \(k\) is nonvanishing.

Our goal here is to transform (2.1) into a quasilinear differential equation for \((s, t) \in T\Omega^*\) in the case \(k > 0\). Thus we now suppose that \(k\) is strictly positive on \(\Omega\) and thus that \(u\) is smooth. The function \(y(s, t)\), the second component of \(T^{-1}(s, t)\), is a weak solution of the quasilinear degenerate equation

\[
\mathcal{L}y(s, t) = \left[ \frac{\partial^2 \alpha}{\partial s^2} + \frac{\partial k}{\partial s} \right] y(s, t) = 0,
\]

in \(T\Omega\). Indeed, as in [10], the Jacobian of \(T\) is

\[
\begin{bmatrix}
  1 & 0 \\
  u_{xy} & u_{yy}
\end{bmatrix},
\]

and that of \(S = T^{-1}\) is

\[
\begin{bmatrix}
  -\frac{u_{xy}}{u_{yy}} & 0 \\
  1 & \frac{1}{u_{yy}}
\end{bmatrix}.
\]

Note that \(u_{yy} \geq c u_{xx} u_{yy} \geq c k > 0\). Thus

\[
\begin{align*}
\partial_s &= x_s \partial_x + y_s \partial_y = \partial_x - \frac{u_{yy}}{u_{yy}} \partial_y, \\
\partial_t &= x_t \partial_x + y_t \partial_y = \frac{1}{u_{yy}} \partial_y,
\end{align*}
\]

and for \(\eta \in C^\infty_c(T(\Omega))\), we have

\[
\int_{T(\Omega)} (y_s \eta_s + k y_y \eta_t) \, dsdt
\]

\[
= \int_{\Omega} \left\{ y_x - \frac{u_{xy}}{u_{yy}} y_y \right\} \left( \eta_x - \frac{u_{xy}}{u_{yy}} \eta_y \right) + k \left( \frac{1}{u_{yy}} \eta_y \right) \frac{1}{u_{yy}} \eta_y \, dxdy
\]

\[
= \int_{\Omega} \left\{ -\frac{u_{xy}}{u_{yy}} \eta_x + \left( \frac{u_{xy}}{u_{yy}} \right)^2 \eta_y + k \left( \frac{1}{u_{yy}} \right)^2 \eta_y \right\} u_{yy} \, dxdy
\]

\[
= \int_{\Omega} \left\{ -u_{xy} \eta_x + u_{xx} \eta_y \right\} \, dxdy = \int_{\Omega} \{ -u_{xy} \eta_x + u_{xy} \eta_y \} \, dxdy = 0,
\]

since

\[
u_{xx} = \frac{k + (u_{xy})^2}{u_{yy}}
\]

by equation (2.1).

We close this section by informally indicating how \textit{a priori} bounds on solutions to (2.15) with \(k > 0\) translate into \textit{a priori} bounds on solutions of (2.1). The \textit{a priori} bounds we refer to have the form that for solutions \(y(s, t)\) of (2.15) in \(T\Omega\), the quantity \(\|D^\alpha y(s, t)\|_{L^\infty(K)}\) is bounded by a constant depending only on \(\alpha\), \(\|g\|_{L^\infty(K)}\), the size of derivatives of \(k\), and the distance from the compact set \(K\) to \(\partial T\Omega\). Then we claim that \(u\) is smooth in \(\Omega\) with similar \textit{a priori} bounds on \(u\) derivatives on sets of the form \(T^{-1}K\). Indeed, if \(y(s, t) = u_y(s, \cdot)^{-1}(t)\) is smooth with \textit{a priori} bounds, then since \(x(s, t) = s\), we conclude that \(S = T^{-1}\) and hence
also $T$ is smooth with \textit{a priori} bounds since the Jacobian of $S$ is $y_t = \frac{1}{u_{yy}}$ which is bounded away from 0 (by [12]) and $\infty$ (by the estimates above). Then $u_y$ is smooth with \textit{a priori} bounds, and hence also $u_{yy}$ and $u_{xy}$. From (2.16) we then obtain that $u_{xx}$ is also smooth with \textit{a priori} bounds. Thus all the second order partial derivatives of $u$ are smooth with \textit{a priori} bounds, and so then $u$ is also smooth with \textit{a priori} bounds on its derivatives. In our application to the Monge-Ampère equation, we will apply this type of argument with $k + \delta$ in place of $k$, where $k$ is now permitted to vanish in $\Omega$ and $\delta > 0$. We can then conclude that \textit{a priori} bounds hold for solutions corresponding to \textit{a priori} bounds. In the case of partial derivatives of $u$ are smooth with \textit{a priori} bounds, since the in the Ampère equation, we will apply this type of argument with smooth with partial derivatives of $u$. We assume lies in $\Omega^*$ given in (2.6), namely

$$\Omega^* = \{(x, y) \in \Omega : \omega_-(x) < u_y(x, y) < \omega_+(x)\},$$

since the infima and suprema in (2.5) may now be attained inside $\Omega$ when $x = 0$ (since $k$ vanishes there). This accounts for the possible presence of Pogorelov segments as discussed above.

2.3.2. \textit{A priori estimates.} Here we consider the degenerate quasilinear equation

$$Lw = [\partial^2_x + \partial_y k(x, w(x, y)) \partial_y] w = 0, \quad (x, y) \in \Omega',$$

where $k(x, y)$ is smooth in a domain $\Omega$, and $w(x, y)$ and $\Omega'$ are such that

$$x, w(x, y) \in \Omega \text{ for all } (x, y) \in \Omega',$$

and where $k$ is positive for $x \neq 0$. Note that we have departed from the notation used in the previous section - $y(s, t)$ has been replaced by $w(x, y)$ and $T(\Omega)$ by $\Omega'$. Our first main theorem shows that if $w$ is a smooth solution of (2.17) in $\Omega'$, then all its derivatives are controlled on compact subsets of $\Omega'$ by the size of $w$ and $\nabla w$ (of course if $\Omega$ is bounded, then $w$ is \textit{a priori} bounded by the requirement (2.18)).

We will use nonnegative cutoff functions adapted to our operator $L$ as follows. Let $R = [-R_1, R_1] \times [-R_2, R_2]$ be a rectangle centred at the origin in the plane, which we assume lies in $\Omega'$, and let $\eta_i, \zeta_i, \theta_i \in C_c^\infty((-R_i, R_i))$ for $i = 1, 2$ satisfy

1. $\eta_i$ equals 1 in a neighbourhood of zero,
2. $\zeta_i$ equals 1 in a neighbourhood of zero,
3. $\theta_i = 1$ on the supports of both $\eta_i$ and $\zeta_i$,
4. 0 does not lie in the support of $\theta_i$.

Set

$$\eta(x, y) = \eta_1(x) \eta_2(y),$$

$$\zeta(x, y) = \zeta_1(x) \zeta_2(y),$$

$$\varphi_1(x, y) = \theta_1(x) \zeta_2(y),$$

$$\varphi_2(x, y) = \zeta_1(x) \theta_2(y).$$

Let $\xi, \varkappa \in C_c^\infty(R)$ satisfy

1. $\xi = 1$ on the support of all four functions $\eta, \zeta, \varphi_1$ and $\varphi_2$,
2. $\varkappa = 1$ on the support of $\xi$.

Before stating our first \textit{a priori} estimate, it will be convenient to recall the classical inequality

$$|\nabla k(x, y)| \leq B \sqrt{k(x, y)}, \quad (x, y) \in L,$$
Remark 2.2. The important point in the above is the restriction that the dependence of the derivatives of $k$ on the family of functions in the support of $k$ (2.20)

$$\|\nabla k(x,y)\| \leq C \left\{ \|\nabla^2 k\|_{\infty}^{\frac{1}{2}} + (\text{dist}((x,y),\partial\Omega))^{-\frac{1}{2}} \right\} \sqrt{k(x,y)}, \quad (x,y) \in \Omega,$$

if $k$ is merely nonnegative with bounded first and second derivatives on a domain $\Omega$. See the subsection on interpolation inequalities in Appendix B. We will also need some notation. Let $\mathcal{P}_c(\Omega)$ denote the collection of all compact subsets of $\Omega$. We will say that a real-valued function $f$ defined on $\mathcal{P}_c(\Omega)$ is increasing if $f(L_1) \leq f(L_2)$ whenever $L_1, L_2 \in \mathcal{P}_c(\Omega)$ with $L_1 \subset L_2$.

Theorem 2.3. Suppose $k(x,y)$ is smooth and nonnegative in a domain $\Omega$, and is positive for $x \neq 0$. Let $\zeta$ and $\varkappa$ be smooth cutoff functions supported in $\Omega'$ as above. Then for every multi-index $\alpha$, there is a real-valued function $C_\alpha(\sigma,L)$, defined for $(\sigma,L) \in [0,\infty) \times \mathcal{P}_c(\Omega)$ and increasing in each variable separately, depending only on $\Omega$, $\Omega'$, $\sum_{|\beta| \leq |\alpha| + 2} \{ \|D^\beta \zeta\|_{\infty} + \|D^\beta \varkappa\|_{L^\infty(L)} \}$, $\inf\{(x,y) \in L; |x| \geq \varepsilon, L \}$ and $\sum_{|\beta| \leq |\alpha| + 2} \sqrt{\|D^\beta k\|_{L^\infty(L)}}$ where

$$\varepsilon_\alpha = \varepsilon \left( \Omega, \|k\|_{C^{|\alpha|+2}(L)}; \frac{\|\nabla k\|}{k^{\frac{1}{2}}} \right)_{L^\infty(L)} > 0,$$

such that

$$\|\zeta D^\alpha w\|_{\infty} \leq C_\alpha (\|\varkappa \nabla w\|_{\infty}, L)$$

for all smooth solutions $w$ of (2.17) in $\Omega'$ such that $(x, w(x,y)) \in L$ for all $(x,y)$ in the support of $\varkappa$.

Note that the right hand side of (2.21) includes an implicit bound on $w$ through the restriction that $(x, w(x,y)) \in L$ when $\varkappa(x,y) \neq 0$.

Remark 2.2. The important point in the above a priori estimate is that the dependence of the final bound in (2.21) on the function $k$ involves only the size of derivatives of $k$ on $L$, the rate of decay of $k$ on $L$ as $x \to 0$, and the constant $B$ in (2.19), which also depends on $L$. In particular, these bounds are uniform over the family of functions $\{k + \theta\}_{0 < \theta \leq 1}$ for $k$ satisfying the hypotheses of the theorem. This observation provides the means of showing that the standard approximation procedure for the Monge-Ampère equation converges appropriately.

Throughout this paper we will use $C$ to denote a constant that may change from line to line, but is independent of any significant quantities. We will use a calligraphic $\mathcal{C}$ to denote a function of one or more variables, increasing in each variable separately, that may also change from line to line, but remains independent of any significant quantities apart from its variables.

Convention: When considering quasilinear equations, we introduce a small abuse of notation in order to greatly relieve congestion in subsequent complicated formulas. Our quasilinear equations all involve functions of the form $(D^\alpha k)(x, w(x,y))$ for a multiindex $\alpha$. We should of course write this as $(D^\alpha k) \circ \Phi$ where $\Phi(x,y) = (x, w(x,y))$, but we will instead write simply $D^\alpha k$ when it is clear that the derivative is evaluated at $\Phi(x,y)$. For example, using the standard notation that $k_i$ denotes partial differentiation of $k(x,y)$ with respect to $x$ if $i = 1$, and $y$ if $i = 2$, we will write $k_i$ and $k_{ij}$ to mean...
our second main theorem shows that we can improve the 
by removing independent
See Remark 5.2 below for a proof, and note also that
(2.22)
for all smooth solutions

where

We remark that throughout section 3 on linear equations, k always means
k(x,y), while in section 4 on quasilinear equations, k always means k(x,w(x,y)).

When there is the possibility of confusion, and especially in the present section,
we will write out k(x,y) or k(x,w(x,y)) explicitly.

In the event that the function k is nearly independent of y when k(x,y) is small,
our second main theorem shows that we can improve the a priori estimates above
by removing \(\|\nabla w\|_\infty\) from the right side, at least for those solutions w arising
from the Legendre transform. The a priori estimates \(u_{xx}, u_{yy} \leq C\) in [11] and
[12] for solutions u of (2.1) that are of relevance here translate into the following
estimates on w:

\[
\begin{align*}
|\nabla(x,y)k(x,w(x,y))| & \leq |k_1| + |k_2| |\nabla w| \\
\leq C\left(\sqrt{k} + k^{3/2}k^{-1}\right) &= C\sqrt{k(x,w(x,y))},
\end{align*}
\]

See Remark 5.2 below for a proof, and note also that \(kw^2 \leq C^2\) is a consequence
of the latter two lines.

Remark 2.3. Of crucial importance here is the fact that \(|k_2| \leq Ck^{3/2}\) implies the
analogue of (2.19) for \(k(x,w(x,y))\):

\[
(2.23) \quad |\nabla(x,y)k(x,w(x,y))| \leq |k_1| + |k_2| |\nabla w| \\
\leq C\left(\sqrt{k} + k^{3/2}k^{-1}\right) = C\sqrt{k(x,w(x,y))},
\]

independent of \(\nabla w\), since \(|\nabla w| \leq Ck^{-1}\) for solutions w to \(\partial^2_w + \partial_y k(x,w(x,y))\partial_y w = 0\)
of the type we consider (see (2.22)). This indicates that \(3/2\) is a natural exponent.

We do not know if it is sharp. Of course the radial examples u \(\approx C + (x^2 + y^2)^\frac{\alpha}{2N}\)
mentioned above fail to be \(C^{N+2}\) for N odd, while the corresponding k = \((x^2 + y^2)^N\)
satisfies \(|k_2| \leq Ck^{1-\frac{N}{2}}\), thus showing that the exponent \(\frac{3}{2}\) in (2.3) cannot be
replaced by any number less than one - leaving a gap from 1 to \(\frac{3}{2}\).

Theorem 2.4. Suppose k(x,y) is smooth and nonnegative in a domain \(\Omega\), is
positive for \(x \neq 0\) and satisfies (2.3). Let \(\zeta \) and \(\kappa\) be smooth cutoff functions
supported in \(\Omega'\) as above. Then for every multi-index \(\alpha\), there is an increasing
real-valued function \(C_\alpha(L)\), defined for \(L \in \mathcal{P}_c(\Omega)\), depending only on \(\Omega, \Omega'\),
\(\sum|\beta|\leq|\alpha|+2\left(\|D^3 \zeta\|_\infty + \|D^3 \alpha\|_\infty\right), \inf\{(x,y) \in \Omega:|x| \geq \varepsilon_\alpha\} k\) and \(\sum|\beta|\leq|\alpha|+2\|D^3 k\|_L^\infty(L)\)
where

\[
\varepsilon_\alpha = \varepsilon\left(\Omega, |k|_{C^{(\alpha+2)}(L)}, \left\|\frac{\partial_x^2 k}{k^{3/2}} + \frac{\partial_y k}{k^{3/2}}\right\|_{L^\infty(L)}\right) > 0,
\]
such that

\[
\|\zeta D^\alpha w\|_\infty \leq C_\alpha(L)
\]
for all smooth solutions w of the quasilinear equation (2.17) in \(\Omega'\) satisfying (2.22),
and such that \((x,w(x,y)) \in L\) for all \((x,y)\) in the support of \(\kappa\).
2.4. Generalized Monge-Ampère equations. Here we consider the Dirichlet problem for the generalized Monge-Ampère equation,

\[
\begin{aligned}
&\begin{cases}
  u_{xx}u_{yy} - (u_{xy})^2 = k(x, y, u, u_x, u_y), \\
  u = \phi(x, y),
\end{cases} \\
&\quad\quad (x, y) \in \Omega
\end{aligned}
\]

where \( k(x, y, v, p, q) \) is smooth and nonnegative on \( \mathbb{R}^3 \times \mathbb{R} \). As before, if \( k > 0 \), we apply the partial Legendre transform

\[
\begin{aligned}
&\begin{cases}
  s = x \\
  t = u_y(x, y)
\end{cases},
\end{aligned}
\]

for a smooth solution \( u \), but now also consider the functions

\[
\begin{aligned}
&w = y = y(s, t) \\
&z = u_x(x, y) = u_x(s, y(s, t)) \\
r = u(x, y) = u(s, y(s, t))
\end{aligned}
\]

where \((x, y) = (s, y(s, t))\) is the inverse partial Legendre transform. The same computation as before shows that the functions \( w, z \) and \( r \) satisfy the quasilinear divergence form equation

\[
\begin{aligned}
&\partial_s^2 w + \partial_t k(s, w(s, t), r(s, t), z(s, t), t) \partial_t w = 0, \\
&\quad\quad (s, t) \in \Omega',
\end{aligned}
\]

in the weak sense, where \( \Omega' \) is a domain satisfying \((s, w(s, t)) \in \Omega\) for all \((s, t) \in \Omega'\).

The key to proving \textit{a priori} estimates for this equation is the calculation of the \( s \) and \( t \) derivatives of \( r \) and \( z \); recall that \( \partial_s = \partial_x - \frac{u_{xx}}{u_{yy}} \partial_y \) and \( \partial_t = \frac{1}{u_{yy}} \partial_y \) so that

\[
\begin{aligned}
&w_s = \frac{u_{xx}}{u_{yy}} \\
&w_t = \frac{1}{u_{yy}}.
\end{aligned}
\]

We thus have

\[
\begin{aligned}
&r_s = \left( \partial_x - \frac{u_{xx}}{u_{yy}} \partial_y \right) u \\
r_t = \frac{u_{yy}}{u_{yy}} \partial_y u \\
z_s = \left( \partial_x - \frac{u_{xx}}{u_{yy}} \partial_y \right) u_x \\
z_t = \frac{1}{u_{yy}} \partial_y u_x
\end{aligned}
\]

which give the compatibility conditions

\[
\begin{aligned}
&r_s = z + tw_s \\
r_t = tw_t \\
z_s = kw_t \\
z_t = -w_s
\end{aligned}
\]

where \( k \) is evaluated at \((s, w(s, t), r(s, t), z(s, t), t)\). The equations (2.26) show that the \((s, t)\) derivatives of \( z \) and \( r \) satisfy the same or better size estimates as do those of \( w \), provided the sup norms of \( w, z \) and \( r \) are all \textit{a priori} bounded (of course, only the bound on \( z \) is needed for this purpose). This is indeed the case for solutions arising from the partial Legendre transform by the \textit{a priori} estimates in [4]. In estimating the higher derivatives of solutions to (2.25), we must differentiate the equation \( \mathcal{L}w = 0 \), where

\[
\mathcal{L}w = \partial_s^2 w + \partial_t \bar{k}(s, t) \partial_t w
\]
and \( \tilde{k} (s, t) = k (s, w (s, t), r (s, t), z (s, t), t) \). We obtain
\[
\mathcal{L} (\partial w) = -\partial_t \left( \partial \tilde{k} \right) \partial_t w
\]
and by using (2.26), we have
\[
\begin{align*}
\partial_s \tilde{k} &= k_1 + k_2 w_s + k_3 (z + tw_s) + k_4 kw_t, \\
\partial_t \tilde{k} &= k_2 w_t + k_3 tw_t - k_4 w_s + k_5,
\end{align*}
\]
where the partial derivatives \( k_j \) are evaluated at the point \((s, w (s, t), r (s, t), z (s, t), t)\). Moreover, (2.27) shows that the general case has the same form as the Monge-Ampère equation itself - namely, that the gradient of \( \tilde{k} \) is linear in the gradient of \( w \) with coefficients \( k_j \), except that in the Monge-Ampère equation, \( k_3 = k_4 = k_5 = 0 \).

It follows that Theorem 2.3 persists in this more general setting with essentially the same proof, as we show below.

Theorem 2.4 persists with \( k (x, y, v, p, q) \) satisfying
\[
\begin{align*}
|k_i| &\leq CK^{d(i)} , \quad 2 \leq i \leq 4, \\
|k_{55}| &\leq CK^{\frac{4}{5}},
\end{align*}
\]
on compact subsets of \( \Omega \times \mathbb{R}^3 \), where
\[
d (i) = \begin{cases} 
\frac{3}{2}, & i = 2, 3 \\
1, & i = 4
\end{cases}.
\]

Just as in Remark 2.3 above, these conditions on \( k_i \) are precisely those which together with (2.22), imply (2.19) for \( \tilde{k} \), namely \( |\nabla_{(s,t)} \tilde{k}| \leq C \sqrt{k} \). Note that \( d (4) = 1 \) is less than \( d (2) = d (3) = \frac{3}{2} \) since (2.26) yields
\[
|z_s| = kw_t \leq C, \\
|z_t| = |w_s| \leq Ck^{-\frac{4}{5}},
\]
by the \textit{a priori} estimates (2.22), and thus the term
\[
k_4 (s, w (s, t), r (s, t), z (s, t), t) |\nabla_{(s,t)} z|
\]
will be bounded by \( \sqrt{k} \) if \( |k_4| \leq CK \). This will be important in our application to prescribed Gaussian curvature as discussed in a previous paragraph. The reason for the special hypothesis on the second derivative \( k_{55} \), and not the others, is that the strong hypotheses on \( k_2, k_3 \) and \( k_4 \) actually turn out to imply that \( |\nabla k_i| \leq C \sqrt{k} \) for \( 2 \leq i \leq 4 \). Since the very last argument in the paper requires \( |k_{ij}| \leq C \sqrt{k} \) for \( 2 \leq i, j \leq 5 \), we see that only the bound on \( k_{55} \) is missing.

Note that on compact subsets of \( \Omega \times \mathbb{R}^3 \) we always have \( |k_i| \leq CK^{\frac{4}{5}} \) by (2.19) and \( |k_{ij}| \leq C \). However, in the case of the equation of prescribed Gaussian curvature \( k = K (x, y) \left( 1 + p^2 + q^2 \right)^2 \), all of the requirements (2.28) automatically hold if we assume only \( |k_2| \leq CK^{\frac{4}{5}} \) (see the previous paragraph); this requires \( K (x, y) \) to become more and more independent of \( y \) as \( K (x, y) \) tends to zero. Theorem 2.1 thus persists with homogeneous boundary data \( \phi \equiv 0 \), and
\[
k (x, y, v, p, q) = K (x, y) \left( 1 + p^2 + q^2 \right)^2,
\]
where \( K \) denotes the Gaussian curvature of the graph of \( u \) and satisfies the condition (2.10) of Ivochkina [15]. This was stated as Theorem 2.2 in subsection 2.2 above.
Remark 2.4. We can obtain an analogue of Theorem 2.2 for the more general boundary value problem (2.24) under the hypothesis (2.28) provided (i) there is an approximating family of smooth positive functions \( k \) such that the corresponding elliptic problems have smooth solutions, and (ii) there are \( C^2 \) a priori estimates for the boundary value problems in (i). See [12] in connection with (ii) and [9] and the references given there in connection with (i).

In the final section of this paper we will prove the following two results that summarize the above discussion on an a priori estimates for the generalized equation. Theorem 2.6 below will be used in the proof of Theorem 2.2 above. We consider here the collection \( \mathcal{P}_e (\Omega \times \mathbb{R}^3) \) of compact subsets of \( \Omega \times \mathbb{R}^3 \).

**Theorem 2.5.** Suppose \( k (x, y, v, p, q) \) is smooth and nonnegative in a domain \( \Omega \times \mathbb{R}^3 \) and is positive for \( x \neq 0 \). Let \( \zeta \) and \( \varphi \) be smooth cutoff functions supported in \( \Omega' \) as above. Then for every multi-index \( \alpha \), there is a real-valued function \( C_\alpha (\sigma, L) \), defined for \( (\sigma, L) \in [0, \infty) \times \mathcal{P}_e (\Omega \times \mathbb{R}^3) \) and increasing in each variable separately, depending only on \( (\sigma, L) \) and \( \zeta \), \( \sum_{\beta \leq |\alpha|+2} (\| D^3 \zeta \|_\infty + \| D^3 \varphi \|_\infty) \), \( \inf \{ (x, y, v, p, q) \in L : |x| \geq \varepsilon_0 \} \) \( k \) and \( \sum_{|\beta| \leq |\alpha|+2} \| D^3 k \|_{L^\infty (L)} \) where

\[
\varepsilon_\alpha = \varepsilon \left( \Omega, \| k \|_{C^{|\alpha|+2} (L)} \left\{ \frac{\| \nabla k \|}{k^2} \right\}_{L^\infty (L)} \right) > 0,
\]

such that

\[
\| \zeta D^3 w \|_\infty \leq C_\alpha (\| \varphi \nabla w \|_\infty, L)
\]

for all smooth solutions \( w, z \) and \( r \) of (2.25) and (2.26) in \( \Omega' \) such that

\[
(x, w (x, y), r (x, y), z (x, y), y) \in L
\]

for all \( (x, y) \) in the support of \( \varphi \).

**Theorem 2.6.** Suppose \( k (x, y, v, p, q) \) is smooth and nonnegative in a domain \( \Omega \times \mathbb{R}^3 \), is positive for \( x \neq 0 \) and satisfies (2.28). Let \( \zeta \) and \( \varphi \) be smooth cutoff functions supported in \( \Omega' \) as above. Then for every multi-index \( \alpha \), there is an increasing real-valued function \( C_\alpha (L) \), defined for \( L \in \mathcal{P}_e (\Omega \times \mathbb{R}^3) \), depending only on \( \Omega' \), \( \sum_{|\beta| \leq |\alpha|+2} (\| D^3 \zeta \|_\infty + \| D^3 \varphi \|_\infty) \), \( \inf \{ (x, y, v, p, q) \in L : |x| \geq \varepsilon_0 \} \) \( k \) and \( \sum_{|\beta| \leq |\alpha|+2} \| D^3 k \|_{L^\infty (L)} \) where

\[
\varepsilon_\alpha = \varepsilon \left( \Omega, \| k \|_{C^{|\alpha|+2} (L)} \left\{ \frac{|k_1| + |k_5| + |k_{55}|}{\sqrt{k}} + \sum_{i=2}^4 \frac{|k_i|}{k^{d(i)}} \right\}_{L^\infty (L)} \right) > 0,
\]

such that

\[
\| \zeta D^3 w \|_\infty \leq C_\alpha (L)
\]

for all smooth solutions \( w, z \) and \( r \) of (2.25) and (2.26) in \( \Omega' \) satisfying (2.22), and such that

\[
(x, w (x, y), r (x, y), z (x, y), y) \in L
\]

for all \( (x, y) \) in the support of \( \varphi \).
2.5. Examples. Our first example shows that we cannot simultaneously fatten the zero set of $k$ and relax the nondegeneracy condition corresponding to (2.8). Note that (2.8) is equivalent to requiring that the restriction of $u$ to the vertical line $x = 0$ not be linear. The following function $u$, that vanishes identically on two vertical lines, is a $C^{1,1}(\Omega)$ solution, and no better, of the Monge-Ampère boundary value problem (2.7) with smooth $\phi$, and smooth nonnegative $k$ satisfying all of the hypotheses of Theorem 2.1 except that $k$ vanishes on $[-\delta, \delta]$ and (2.8) fails with $x = \pm \delta$ in place of $x = 0$ - all the other hypotheses are uniform in $\delta > 0$. Let

$$
\varphi(x, y) = \frac{y^2}{2} \psi(x) + \frac{x^2}{2}
$$

for a smooth function $\psi$ to be chosen later. We observe that

$$
k = \varphi_{xx} \varphi_{yy} - (\varphi_{xy})^2 = \left(\frac{y^2}{2} \psi''(x) + 1\right) (\psi(x)) - (y \psi'(x))^2
$$

and

$$
k_2 = y \left(\psi(x) \psi''(x) - 2 \psi'(x)^2\right).
$$

Now it is possible to achieve

$$
\left|\psi(x) \psi''(x) - 2 \psi'(x)^2\right| \leq Ck^{2-\varepsilon},
$$

for example with $\psi(x) = \begin{cases} e^{-\frac{x^2}{2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. In fact, we then have $\psi'(x) = e^{-\frac{x^2}{2}} (\frac{4}{3})$ and $\psi''(x) = e^{-\frac{x^2}{2}} (\frac{16}{3} - \frac{12}{x^2})$ so that (2.3) holds with $|k_2| \leq Ck^{2-\varepsilon}$ for any $\varepsilon > 0$. We now employ a modification of an example of Sibony reported in [10]. Let $\Omega$ be the unit disk and for $\delta > 0$ small, define $u$ on $\overline{\Omega}$ by

$$
u(x, y) = \begin{cases} \varphi(x^2 - \delta^2, y), & |x| \geq \delta \\ (y^2 - (1 - \delta^2))^2, & |y| \geq \sqrt{1 - \delta^2} \\ 0, & |x| < \delta, |y| < \sqrt{1 - \delta^2} \end{cases}.
$$

Then in $\Omega$, $k = u_{xx}u_{yy} - (u_{xy})^2$ is smooth, nonnegative, positive for $|x| > \delta$, and satisfies (2.3). The boundary values $\varphi$ are also smooth (uniformly in $\delta$) since $\varphi(\cos^2 \theta - \delta^2, \sin \theta)$ and $(\sin^2 \theta - (1 - \delta^2))^2$ together with all their derivatives match up at the four angles $\theta_0$ where $\tan \theta_0 = \sqrt{1 - \delta^2}$ (since $\psi(x)$ vanishes to infinite order at $x = 0$). Also, the nondegeneracy condition holds except on the vertical lines $x = \pm \delta$. Finally, $u$ is no better than $C^{1,1}(\Omega)$ across these vertical lines since $u_{xx}$ has jump discontinuities there.

We now comment on the existence of line segments in the graphs of solutions to the Dirichlet problem (2.1). Of course, under the hypotheses of Theorem 2.1, all solutions $u$ satisfy $u_{yy} > 0$ in $\Omega^*$, and so a maximal line segment cannot exist in the graph of a solution $u$ unless it lies above the $y$-axis and at least one of its endpoints lies above $\partial \Omega$. In the case exactly one of the endpoints lies above $\partial \Omega$, we call the projection of such a maximal line segment onto the $x, y$-plane, a Pogorelov segment (see [23] and [5]). We first point out that, under the hypotheses of Theorem 2.1, it is possible to have a line segment in the graph of $u$ provided both endpoints lie above $\partial \Omega$, but we are unable to find any nonsmooth such solutions. For example,
Three examples above and Hessian of \( \Omega \) exist, and that consequently the solutions in Theorem 2.1 are smooth in all of any solutions, smooth or not, having a Pogorelov segment. We suspect they do not \( \phi \) relating
The homogeneous boundary condition may be relaxed to a more general condition
not connected due to a line segment in the graph of \( \Omega \) in [9]) yields an
hypotheses of Theorem 2.3. Indeed, while the maximum principle (see Theorem 9.1 in many cases the solutions are as irregular as can be.

An example with \( k \) vanishing to infinite order at the \( y \)-axis is given by

\[
 u(x, y) = e^{-\frac{x}{2}} + \frac{y^2}{2} e^{-\frac{1}{x^2}}.
\]

Then \( u(0, y) = 0 \) for all \( y \), and

\[
 k = u_{xx}u_{yy} - u_{xy}^2 = e^{-\frac{x}{2}} \left( \frac{4}{x^6} - \frac{6}{x^4} \right) - 18 y^2 e^{-\frac{1}{x^2}} \left( \frac{1}{x^6} + \frac{1}{2x^4} \right).
\]

Thus in a small disc about the origin, \( k \) is positive for \( x \neq 0 \) and \( |\partial_y k| \leq CK^{\frac{3}{2}} \).

Next we observe that Pogorelov segments can exist for the solutions \( u \) that arise in Theorem 2.1 if we omit the hypothesis (2.3), and moreover the conclusion \( u_{yy} > 0 \) in \( \Omega^* \) may fail. To see this, consider \( v(x, y) = u(x, y) + g(y) \) where \( u \) is one of the three examples above and \( g \) is smooth and convex. Then the determinant of the Hessian of \( v \) is

\[
 v_{xx}v_{yy} - v_{xy}^2 = u_{xx}u_{yy} - u_{xy}^2 + g''(y) u_{xx},
\]

which in a small disc about the origin, is smooth and positive for \( x \neq 0 \). However, \( v_{yy}(0, y) = g''(y) \) need not be positive in \( \Omega^* \), and of course \( v \) may have Pogorelov segments lying above the \( y \)-axis.

The problem with the above example is that \( k \) no longer vanishes on the \( y \)-axis. Indeed, as we show in Appendix A, if \( \phi \equiv 0 \), \( k \) vanishes on a line and \( u \) has a Pogorelov segment in the same line, then \( u \) must fail to be in \( C^2(\Omega) \). The homogeneous boundary condition may be relaxed to a more general condition relating \( \phi \), \( k \) and \( \partial \Omega \) (see (6.15) in Theorem 6.6 in Appendix A).

We emphasize that under the hypotheses of Theorem 2.1, we are unable to find any solutions, smooth or not, having a Pogorelov segment. We suspect they do not exist, and that consequently the solutions in Theorem 2.1 are smooth in all of \( \Omega \) when \( \Omega^* \) is connected. Perhaps the solutions are always smooth, even when \( \Omega^* \) is not connected due to a line segment in the graph of \( u \) extending all the way across \( \Omega \). On the other hand, Appendix A shows that if Pogorelov segments exist, then in many cases the solutions are as irregular as can be.

Finally we point out that the conclusion of Theorem 2.4 may fail under the hypotheses of Theorem 2.3. Indeed, while the maximum principle (see Theorem 9.1 in [9]) yields an \( a \) \textit{priori} bound for functions \( w \) that solve the quasilinear Dirichlet problem

\[
\begin{align*}
\{ \partial_t^2 + \partial_t k(s, w(s, t)) \partial_t \} w &= 0 & (s, t) \in \Omega' \\
w &= \phi(s, t) & (s, t) \in \partial\Omega'
\end{align*}
\]
(w is already a priori bounded if \( \Omega \) is bounded), the \( t \)-derivatives may not be bounded. For example, if \( k = 18 (s^2 + t^2) \) and
\[
w = \frac{1}{3\sqrt{2}} \sqrt{(9s^2)^2 + (6t)^2 - 9s^2},
\]
then \( w \) solves the first equation in (2.29), but \( w_t \) fails to belong to \( L^3 \). Indeed if we apply the partial Legendre transform
\[
\begin{cases}
s &= x \\
t &= u_y (x, y)
\end{cases}
\]
to the solution \( u = (x^2 + y^2)^\frac{3}{2} \) of the Monge-Ampère equation
\[
u_{xx}v_{yy} - (v_{xy})^2 = k (x, y) = 18 (x^2 + y^2),
\]
the above function \( w \) results. We remark that even though \( k \) vanishes here, the partial Legendre transform is one-to-one and a simple limiting argument justifies its use. Now \( u_{yy} \approx r = (x^2 + y^2)^\frac{3}{2} \) and so using \( \partial_t w = \frac{1}{u_{yy}} \) and \( \partial_t (u_{xy}) = u_{yy} \), we obtain
\[
\int \int |\partial_t w (s, t)|^p \, ds \, dt = \int \int \left| \frac{1}{u_{yy}} \right|^p u_{yy} \, dx \, dy \approx \int_0^1 r^{2-p} \, dr < \infty
\]
if and only if \( p < 3 \). We can of course come to the same conclusions by direct calculation using
\[
|w_t (s, t)| \approx \begin{cases}
|t|^{-\frac{1}{2}}, & |t| \geq |s|^2 \\
|s|^{-1}, & |t| \leq |s|^2
\end{cases},
\]
so that
\[
\int \int |w_t|^p \, ds \, dt \approx \int_0^1 |t|^{-\frac{p}{2}} |t|^\frac{p}{2} \, dt + \int_0^1 |s|^{-p} |s|^2 \, ds < \infty
\]
if and only if \( p < 3 \).

3. HYPOELLIPTICITY OF LINEAR EQUATIONS

We obtain Theorem 2.1, which is effectively a hypoellipticity result for the boundary value problem (2.7), by establishing the a priori estimates of Theorem 2.4. In this section we review the analogous linear theory of hypoelliptic a priori estimates, which will be used as part of our attack in the nonlinear case. We denote by \( \|f\|_s \) the usual Sobolev space norm given by
\[
\|f\|^2_s = \int_{\mathbb{R}^2} \left| \hat{f} (\xi) \right|^2 \left( 1 + |\xi|^2 \right)^{\frac{p}{2}} \, d\xi.
\]
One may obtain the hypoellipticity of the linear operator
\[
(3.1) \quad \mathcal{L} = \partial_x^2 + \partial_y k (x, y) \partial_y
\]
in the case that \( k \in C^\infty (\mathcal{R}) \), where \( \mathcal{R} \) is a rectangle \([-R_1, R_1] \times [-R_2, R_2] \), by first establishing an a priori inequality of the form
\[
(3.2) \quad \|\eta u\|^2_s \leq C \|\eta \mathcal{L} u\|^2_s + C \|\xi \mathcal{L} u\|^2_{s-\varepsilon} + C \|\varepsilon u\|^2_{s-\varepsilon}, \quad \varepsilon > 0,
\]
where the cutoff functions \( \eta, \xi \) and \( \varepsilon \) are adapted to the operator \( \mathcal{L} \) as above. Note that the cutoff function \( \eta \) is replaced by a larger cutoff \( \xi \), which is 1 on the support of \( \eta \), when \( \mathcal{L} u \) is measured in a Sobolev space of smaller order. This is important
Lemma 3.4 and Corollary 3.5 below). The commutator and estimating the commutator to there is $L$ by exploiting the fact that the weight is better behaved, actually elliptic by hypothesis, and term $R$ when $\varepsilon > 0$ such that

$$\|u\|_\varepsilon^2 \leq C \left( \left\| (Lu) (u) \right\| + \|u\|^2 \right),$$

for all smooth compactly supported $u$. Since the function $k(x, y) = k(x, w(x, y))$ arising in the quasilinear equation has bounds on its derivatives depending on those of the solution $w$, we will restrict attention to the case $s = \varepsilon = 1$,

$$\|\eta u\|^2 \leq C \|\eta Lu\|^2 + C \|\xi Lu\|^2 + C \|\varepsilon u\|^2,$$

in order to avoid difficult remainder terms arising from the pseudodifferential calculus when $s$ is not integral.

The basic idea of the proof, following J. Kohn [16], is to estimate $\|\eta u\|^2$ by the Poincaré inequality in the $x$ - variable (which requires no information on the degenerate function $k$),

$$(3.3) \quad \|\eta u\|^2 = \|\nabla \eta u\|^2 \leq CR^2 \|\partial_x (\nabla \eta u)\|^2,$$

and then use the $k$ - gradient estimate (compare Corollary 3.3 below),

$$\int_\mathcal{R} \left( |\zeta \partial_\zeta \nabla \eta u|^2 + k |\zeta \partial_y \nabla \eta u|^2 \right) \leq -2 \int_\mathcal{R} (\zeta \nabla \eta u) \cdot (\zeta \nabla u) + 4 \int_\mathcal{R} |\zeta_\zeta \nabla \eta u|^2 + 4 \int_\mathcal{R} k |\zeta_y \nabla \eta u|^2 = I + II + III,$$

exploiting the fact that $R_1$ is small. In the subelliptic case, where $k$ vanishes to finite order in $x$, there are Poincaré inequalities that actually improve the $L^p$ integrability of solutions. These are not available here, and the small constant is our only improvement. Term $I$ is handled by writing

$$[L, \nabla \eta] = \nabla [L, \eta] + [L, \nabla] \eta$$

and estimating the commutator $[L, \eta]$ with the help of even and odd operators (see Lemma 3.4 and Corollary 3.5 below). The commutator $[L, \nabla] = -\partial_y (\nabla k) \partial_y$ can be suitably estimated using inequality (2.19), $|\nabla k| \leq C \sqrt{\mathcal{R}}$. In both cases, terms of the form $C \|\eta u\|^2$ arise in the estimates, but can be absorbed into the left side of (3.3) since they are multiplied by $R_1$, which we can take sufficiently small. We remark that $[L, \nabla]$ has no remainder term while $[L, \Lambda^s]$ for $s$ not an integer, has a remainder that requires too much smoothness of $k(x, w(x, y))$. Term $II$ is supported where $L$ is better behaved, actually elliptic by hypothesis, and term $III$ can be handled by exploiting the fact that the weight $k$ in the norm is the least eigenvalue of the operator $L$; this permits us to replace the identity $v = \nabla \cdot I_1 v$, $I_1 = \nabla \cdot \Delta^{-1}$, with the pointwise inequality

$$k |v|^2 \leq |\partial_x I_1 v|^2 + k |\partial_y I_1 v|^2.$$

We then turn to the $L^p$ estimates of J. Moser, and establish $a$ priori inequalities with an improvement in the integrability of derivatives of the solution, similar to

$$(3.4) \quad \left( \int_\mathcal{R} |\zeta D^\alpha u^\beta|^2 \, dx dy \right)^{\frac{1}{2}} \leq C_\beta \left( \int_\mathcal{R} |\zeta D^\alpha u|^p \, dx dy \right)^{\frac{1}{p}},$$
for some $p < 2$. This will be useful in estimating the nonlinearities in $L u$ in the quasilinear case. Next, we consider the quasilinear degenerate elliptic equation (2.17),

$$L w = \left[ \partial_x^2 + \partial_y k(x, w(x, y)) \partial_y \right] w = 0,$$

where $k$ is smooth and nonnegative on $\mathcal{R}$ and $w$ is smooth. We alternately apply the a priori inequalities (3.3) and (3.4) to obtain that the derivatives of $w$ are controlled by $\|w\|_{\infty}$ and $\|\nabla w\|_{\infty}$.

Finally, it might be helpful to keep the following points in mind while reading the estimates in subsequent sections. Since $u$ is a solution of $L u = 0$, the operator $L$ behaves better than an operator of order 2 when applied to $u$. However, when $L$ is commuted with an operator $P$ of order $\alpha$, then $L$ loses its special status in $[L, P]$, and the commutator has order only $2 + \alpha - 1$. In order to compensate for this loss, we need to exploit special properties of $[L, P]$:

3.1. The gradient estimate. Let $k$ be nonnegative and smooth on $\mathcal{R}$ (we remind the reader that throughout this section $k = k(x, y)$). We begin with the well known

Caccioppoli inequality estimating the energy of the $L$-gradient of a function $u$ in terms of $u$ and $L u$. For this it is convenient to introduce the inner product

$$\langle v, w \rangle_k = v_1 w_1 + k v_2 w_2 = v_1 w_1 + k(x, y) v_2 w_2,$$

as well as the matrix

$$A = A(x, y) = \begin{bmatrix} 1 & 0 \\ 0 & k(x, y) \end{bmatrix},$$

so that $L = \nabla \cdot A \nabla$. The operator $L$ is a sum of “squares” of the two vector fields $\partial_x$ and $\sqrt{k} \partial_y$, usually called the unit vector fields associated with $L$. Later it will be important to observe that the vector fields $k \partial_y$ are subunit in the sense that $|k_1| \leq C \sqrt{k}$ by (2.19).

Lemma 3.1. Suppose $L$ is as in (3.1) with $k$ nonnegative and smooth. For $u \in C^\infty(\mathcal{R})$, we have the identity

$$\int_{\mathcal{R}} (\zeta \nabla u, \zeta \nabla u)_k dxdy = - \int_{\mathcal{R}} (\zeta L u)(\zeta u) dxdy - 2 \int_{\mathcal{R}} (u \nabla \zeta, \zeta \nabla u)_k dxdy,$$

and the inequality

$$\int_{\mathcal{R}} \left( |\zeta \partial_x u|^2 + k |\zeta \partial_y u|^2 \right) dxdy \leq -2 \int_{\mathcal{R}} (\zeta L u)(\zeta u) dxdy$$

$$+ 4 \|\zeta_x\|^2 \int_{\mathcal{R}} |\partial_1 u|^2 dxdy + 4 \|\zeta_y\|^2 \int_{\mathcal{R}} k |\partial_2 u|^2 dxdy.$$
Proof. Integration by parts yields the identity above:
\[
\int_{\mathbb{R}} \langle \zeta \nabla u, \zeta \nabla u \rangle_k = \int_{\mathbb{R}} (\zeta \nabla u) \cdot A (\zeta \nabla u) \\
= - \int_{\mathbb{R}} u \nabla \cdot \zeta^2 A \nabla u \\
= - \int_{\mathbb{R}} u \zeta^2 \nabla \cdot A \nabla u - \int_{\mathbb{R}} u 2 \zeta (\nabla \zeta) \cdot A \nabla u \\
= - \int_{\mathbb{R}} (\zeta u) (\zeta L u) - 2 \int_{\mathbb{R}} (u \nabla \zeta) \cdot (A \zeta \nabla u).
\]
Using
\[
2 |(u \nabla \zeta, \zeta \nabla u)_k| \leq \frac{1}{2} \langle \zeta \nabla u, \zeta \nabla u \rangle_k + 2 \langle u \nabla \zeta, u \nabla \zeta \rangle_k
\]
in the identity yields
\[
\int_{\mathbb{R}} \langle \zeta \nabla u, \zeta \nabla u \rangle_k \leq - \int_{\mathbb{R}} (\zeta L u) (\zeta u) + \frac{1}{2} \int_{\mathbb{R}} \langle \zeta \nabla u, \zeta \nabla u \rangle_k + 2 \int_{\mathbb{R}} \langle u \nabla \zeta, u \nabla \zeta \rangle_k,
\]
and so by absorbing the second term on the right,
\[
\int_{\mathbb{R}} \langle \zeta \nabla u, \zeta \nabla u \rangle_k \leq -2 \int_{\mathbb{R}} (\zeta L u) (\zeta u) + 4 \int_{\mathbb{R}} \langle u \nabla \zeta, u \nabla \zeta \rangle_k,
\]
and (3.5) now follows from \( \langle \nabla \zeta, \nabla \zeta \rangle_k = \zeta_x^2 + k \zeta_y^2 \leq ||\zeta_x||_\infty^2 + ||\zeta_y||_\infty^2 k \rho_2^2 \).

**Corollary 3.2.** Suppose \( \mathcal{L} \) is as in (3.1) with \( k \) nonnegative and smooth. For \( u \in C^\infty(\mathbb{R}) \), we have
\[
\int_{\mathbb{R}} \left( |\partial_x \zeta u|^2 + k |\partial_y \zeta u|^2 \right) dxdy \\
\leq -4 \int_{\mathbb{R}} (\zeta \mathcal{L} u) (\zeta u) dxdy \\
+ 10 ||\zeta_x||_\infty^2 \int_{\mathbb{R}} |\varphi_1 u|^2 dxdy + 10 ||\zeta_y||_\infty^2 \int_{\mathbb{R}} k |\varphi_2 u|^2 dxdy.
\]

**Proof.** Using \( \partial_x \zeta u = \zeta \partial_x u + \zeta_x u \) and \( \partial_y \zeta u = \zeta \partial_y u + \zeta_y u \), we obtain
\[
\int_{\mathbb{R}} \left( |\partial_x \zeta u|^2 + k |\partial_y \zeta u|^2 \right) \\
\leq 2 \int_{\mathbb{R}} \left( |\zeta \partial_x u|^2 + k |\zeta \partial_y u|^2 \right) \\
+ 2 ||\zeta_x||_\infty^2 \int_{\mathbb{R}} |\varphi_1 u|^2 + 2 ||\zeta_y||_\infty^2 \int_{\mathbb{R}} k |\varphi_2 u|^2,
\]
and the corollary now follows from (3.5).

3.1.1. Gradients and commutators. It will be convenient to set
\[
A^6 = 1 + ||\nabla \eta||_\infty^6 + ||\nabla \zeta||_\infty^6 + ||\nabla \varphi_1||_\infty^6 + ||\nabla \varphi_2||_\infty^6 \\
+ ||\nabla^2 \eta||_\infty^3 + ||\nabla^2 \zeta||_\infty^3 + ||\nabla^3 \eta||_\infty^2,
\]
in order to collect constants in front of the lower order terms in what follows. It is important to observe that since \( A \geq \mathcal{R}_1^{-1} \), if we wish to show that a certain term is small by applying the one-dimensional Poincaré inequality in the \( x \)-variable in
order to gain a factor of $R_1$ (as in (3.3) above), we must ensure that the term to be shown small is not multiplied by a constant which increases with $A$.

**Corollary 3.3.** Suppose $\mathcal{L}$ is as in (3.1) with $k$ nonnegative and smooth. Let $\partial$ denote either $\partial_x$ or $\partial_y$. For $u \in C^\infty (\mathbb{R})$, we have

\[
\int_{\mathbb{R}} \left( |\partial_x (\zeta \partial u)|^2 + k |\partial_y (\zeta \partial u)|^2 \right) dxdy \leq -4 \int_{\mathbb{R}} (\zeta \partial \eta \mathcal{L} u) (\zeta \partial \eta u) dxdy - 4 \int_{\mathbb{R}} (\zeta [\mathcal{L}, \partial \eta] u) (\zeta \partial \eta u) dxdy + CA^2 \|\partial_1 u\|_1^2 + CA^2 \|\sqrt{k} \partial_2 u\|_0^2 + CA^4 \|\xi u\|_0^2.
\]

**Proof.** Replace $u$ with $\partial \eta u$ in (3.6) and then use

\[
\mathcal{L} \partial \eta = \partial \eta \mathcal{L} + [\mathcal{L}, \partial \eta]
\]
and

\[
\partial_i \partial \eta = [\partial_i, \partial] \eta + [\partial, \eta] \partial_i + \eta \partial_i
\]
for $i = 1, 2$ to obtain

\[
\int_{\mathbb{R}} |\partial_1 \partial \eta u|^2 \leq C \int_{\mathbb{R}} |\partial_1 \partial \eta u|^2 + CA^2 \|\xi u\|_0^2
\]
\[
\leq C \|\partial_1 u\|_1^2 + CA^2 \|\xi u\|_0^2,
\]
and

\[
\int_{\mathbb{R}} k |\partial_2 \partial \eta u|^2 \leq C \int_{\mathbb{R}} k |\partial_2 \partial \eta u|^2 + CA^2 \|\xi u\|_0^2
\]
\[
\leq C \|\sqrt{k} \partial_2 u\|_0^2 + CA^2 \|\xi u\|_0^2.
\]

The above corollary leads us to consideration of the commutator

(3.8) \[
[\mathcal{L}, \partial \eta] = \partial [\mathcal{L}, \eta] + [\mathcal{L}, \partial] \eta.
\]

In order to handle the commutator $[\mathcal{L}, \eta]$, we will need a standard lemma regarding even and odd operators (see e.g. [16]). Let $\partial$ denote a partial derivative of order one, either $\partial_x$ or $\partial_y$, so that $\partial + \partial^t = 0$.

**Lemma 3.4.** Let $P$ and $Q$ denote arbitrary operators, and let $\partial$ be as above. Then for $u \in C^\infty (\mathbb{R}^2)$,

(3.9) \[
\int (P \partial \xi u) (Q \xi u) + \int (P^t \partial \xi u) (Q^t \xi u)
\]
\[
= \int ([P, \partial] \xi u) (Q \xi u) + \int (P \xi u) ([Q, \partial] \xi u) + \int ([P, Q^t] \xi u) (\partial \xi u).
\]
Proof. Using $\partial + \partial^t = 0$ we obtain
\[
\int (P\partial_\zeta u) (Q\zeta u) = \int ([P, \partial] \zeta u) (Q\zeta u) + \int (\partial P\zeta u) (Q\zeta u)
\]
\[
= \int ([P, \partial] \zeta u) (Q\zeta u) - \int (P\zeta u) (\partial Q\zeta u)
\]
\[
= \int ([P, \partial] \zeta u) (Q\zeta u) + \int (P\zeta u) ([Q, \partial] \zeta u)
\]
\[
- \int (P\zeta u) (Q\partial_\zeta u),
\]
and by definition of transpose,
\[
\int (Q^t \zeta u) (P^t \partial_\zeta u) = \int (PQ^t \zeta u) (\partial_\zeta u)
\]
\[
= \int ([P, Q^t] \zeta u) (\partial_\zeta u) + \int (Q^t P\zeta u) (\partial_\zeta u)
\]
\[
= \int ([P, Q^t] \zeta u) (\partial_\zeta u) + \int (P\zeta u) (Q\partial_\zeta u).
\]
Adding the two equalities yields (3.9).

Let $P$ and $Q$ denote classical pseudodifferential operators such that either of the following holds:

- $P - P^t$ and $Q - Q^t$ have order one less than the order $\mathit{ord}(P)$ and $\mathit{ord}(Q)$ of $P$ and $Q$ respectively. For example this will be the case if both $P$ and $Q$ have real principal symbol.
- $P + P^t$ and $Q + Q^t$ have order one less than the order $\mathit{ord}(P)$ and $\mathit{ord}(Q)$ of $P$ and $Q$ respectively. For example this will be the case if both $P$ and $Q$ have imaginary principal symbol.

Corollary 3.5. Let $P, Q$ and $\partial$ be as above and suppose that
\[
\mathit{ord}(P) + \mathit{ord}(Q) \leq 2s
\]
for some $s > 0$. Then for $u \in C^\infty (\mathbb{R}^2)$,
\[
\left| \int (P\partial_\zeta u) (Q\zeta u) \right| \leq C (s, P, Q) \| \zeta u \|^2_s. \tag{3.10}
\]

Proof. We have
\[
2 \int (P\partial_\zeta u) (Q\zeta u) = \int (P\partial_\zeta u) (Q\zeta u) + \int (P^t \partial_\zeta u) (Q^t \zeta u)
\]
\[
+ \int ((P - P^t) \partial_\zeta u) (Q\zeta u) + \int (P^t \partial_\zeta u) ((Q - Q^t) \zeta u).
\]
In the case $P - P^t$ and $Q - Q^t$ have order one less than $\mathit{ord}(P)$ and $\mathit{ord}(Q)$ respectively, Lemma 3.4 implies that the above is a sum of terms of the form
\[
\int (U \zeta u) (V \zeta u)
\]
where $U$ and $V$ are operators satisfying
\[
\mathit{ord}(U) + \mathit{ord}(V) \leq 2s.
\]
Let $\Lambda^s$ be the operator with symbol $\left(1 + |\xi|^2 \right)^{\frac{s}{2}}$. Now $\Lambda^{s-\text{ord}(U)}$ has order $s$, and $\Lambda^{\text{ord}(U) - s} V$ has order $\text{ord}(U) - s + \text{ord}(V) \leq s$, and so

$$\left| \int (U \zeta u)(V \zeta u) \right| = \left| \int (\Lambda^{s-\text{ord}(U)} U \zeta u) \left( \Lambda^{\text{ord}(U) - s} V \zeta u \right) \right| \leq C \|\zeta u\|_2^2.$$

On the other hand, in the case $P + P^t$ and $Q + Q^t$ have order one less than $\text{ord}(P)$ and $\text{ord}(Q)$ respectively, we use instead the identity

$$2 \int (P \partial \zeta u)(Q \zeta u) = \int (P \partial \zeta u)(Q \zeta u) + \int (P^t \partial \zeta u)(Q^t \zeta u) + \int ((P + P^t) \partial \zeta u)(Q \zeta u) - \int (P^t \partial \zeta u)((Q + Q^t) \zeta u).$$

This completes the proof of the lemma.

**Remark 3.1.** In the case $P + P^t$ and $Q + Q^t$ have order one less than $\text{ord}(P)$ and $\text{ord}(Q)$ respectively, we have the following identity for $u \in C^\infty$ which will be useful later on. The point is that the sum of the orders of the operators appearing in the terms on the right is one less than the sum on the left.

$$2 \int (P \partial \zeta u)(Q \zeta u) = \int ([P, \partial] \zeta u)(Q \zeta u) + \int (P \zeta u)([Q, \partial] \zeta u) + \int ((P + P^t) \partial \zeta u)(Q \zeta u) - \int (P^t \partial \zeta u)((Q + Q^t) \zeta u).$$

We can now handle the first term $\partial [\mathcal{L}, \eta]$ on the right side of (3.8) using inequality (2.19).

**Lemma 3.6.** Suppose $\mathcal{L}$ is as in (3.1) with $k$ nonnegative, smooth and satisfying (2.19). Let $\partial$ denote either $\partial_x$ or $\partial_y$. For $u \in C^\infty(\mathcal{R})$ and $0 < \alpha < 1$, we have with $B$ as in (2.19),

$$\left| \int_\mathcal{R} (\zeta \partial [\mathcal{L}, \eta] u)(\zeta \partial \eta u) \, dxdy \right| \leq C \alpha (B^2 + 1) \left( \|\partial_x \zeta \partial \eta u\|_0^2 + \|\sqrt{\kappa} \partial_y \zeta \partial \eta u\|_0^2 \right) + C \|\eta u\|_1^2$$

$$+ C \left( A^4 \|\partial_1 u\|_1^2 + \left(A^4 + \frac{A^2}{\alpha}\right) \|\sqrt{\kappa} \partial_y \partial_1 u\|_0^2 + A^4 \left( \frac{1}{\alpha} + B^2 + A^2 \right) \|\xi u\|_0^2 \right)$$

$$+ C \alpha (B^2 + 1) A^2 \left| \int_\mathcal{R} (\eta \mathcal{L} u)(\eta u) \, dxdy \right|.$$

**Proof.** We compute that

$$[\mathcal{L}, \eta] = (\partial_x^2 + \partial_y k \partial_y) \eta - \eta (\partial_x^2 + \partial_y k \partial_y)$$

$$= 2\eta_x \partial_x + \eta_{xx} + \partial_y k \eta_y + k \eta_y \partial_y$$

$$= 2 (\eta_x \partial_x + k \eta_y \partial_y) + (\eta_{xx} + k \eta_{yy} + k \eta_y \partial_y).$$
and so we have
\[
\left| \int_{\mathbb{R}} (\zeta \partial [L, \eta] \nu) (\zeta \partial \eta \nu) \right| \\
\leq 2 \int_{\mathbb{R}} (\zeta \partial \eta_x \partial_x \nu) (\zeta \partial \eta \nu) \\
+ \int_{\mathbb{R}} (\zeta \partial \eta_x + k \eta_y u) \partial \eta \nu \right| \\
= I + II + III.
\]

To estimate I, we note that since \( \eta_x = \eta_x \partial_x^2 \), we have
\[
- \int_{\mathbb{R}} (\zeta \partial \eta_x \partial_x u) (\zeta \partial \eta \nu) = \int_{\mathbb{R}} (\partial \zeta^2 \partial_\eta \eta_x \partial_x \nu) (\eta \nu) \\
= \int_{\mathbb{R}} (\partial_\eta \partial \zeta^2 \partial \eta_x \partial_x \nu) (\eta \nu) \\
- \int_{\mathbb{R}} (\partial \zeta^2 \partial_\eta \eta_x \partial_x \nu) (\eta \nu) \\
- \int_{\mathbb{R}} \left[ \partial_\eta, \partial \zeta^2 \partial \eta \right] \eta_x \partial_x \nu \right) (\eta \nu),
\]
and so, as we shall show,
\[
(3.12) \quad \left| \int_{\mathbb{R}} (\zeta \partial \eta_x \partial_x u) (\zeta \partial \eta \nu) \right| \\
\leq \int_{\mathbb{R}} (\zeta \partial \eta_x \partial_x \nu) (\zeta \partial \eta \nu) \\
+ C \left( \| \eta \nu \|^2 + A^4 \| \eta_1 u \|^2 + A^6 \| \xi u \|^2 \right).
\]

Note that the first term on the right side of (3.12) is the absolute value of the first term on the right side of the previous display. Indeed, we have
\[
\left| \int_{\mathbb{R}} (\partial \zeta^2 \partial_\eta \eta_x \partial_x \nu) (\eta \nu) \right| = \left| \int_{\mathbb{R}} (\zeta^2 \partial_\eta \eta_x \partial_x \nu) (\eta \nu) \right| \\
\leq \left| \int_{\mathbb{R}} (\zeta^2 \eta_x \partial_x \nu) (\eta \nu) \right| \\
+ \left| \int_{\mathbb{R}} (\zeta^2 \partial_\eta \partial_\eta \eta_x \partial_x \nu) (\eta \nu) \right| \\
\leq C \left( \| \eta \nu \|^2 + A^4 \| \eta_1 u \|^2 + A^6 \| \xi u \|^2 \right),
\]
since \([ \partial_\eta, \eta_x \partial_x \nu] \) has order 0 and norm bounded by \( A^3 \). Similarly,
\[
\left| \int_{\mathbb{R}} \left[ \partial_\eta, \partial \zeta^2 \partial \eta \right] \eta_x \partial_x \nu \right) (\eta \nu) \right| \leq C \left( \| \eta \nu \|^2 + A^4 \| \eta_1 u \|^2 + A^6 \| \xi u \|^2 \right),
\]
since \([ \partial_\eta, \partial \zeta^2 \partial \eta \] is the sum of a zero order operator of norm \( A^2 \) and a first order operator of norm \( A \), upon expanding the commutator. Now apply Corollary 3.5,
or more precisely Remark 3.1, with \( P = \zeta \partial \eta_z \) and \( Q = \zeta \partial \eta \) to obtain from (3.12) that
\[
|I| \leq C \left( \|\eta u\|_1^2 + A^4 \|\rho_1 u\|_1^2 + A^6 \|\xi u\|_0^2 \right).
\]

We remark that since \( P + P^t = \zeta \partial \eta_z - \eta_z \partial \zeta \) has order 0 (and similarly for \( Q + Q^t \)), all the terms on the right side of Remark 3.1 have less total order than the left side, and after much computation we have the desired result. Note also the tradeoff of order for powers of \( A \) in Remark 3.1 - if a derivative hits a cutoff function, the order is reduced but an additional factor of \( A \) arises in the norm.

For \( II \) we write
\[
|II| = 2 \left| \int_{\mathcal{R}} \left( \sqrt{k} \eta_y \partial_y u \right) \left( \sqrt{k} \partial \zeta^2 \partial \eta u \right) \right|
\]
\[
\leq C \frac{1}{\alpha} \left\| \sqrt{k} \eta_y \partial_y u \right\|_0^2 + C \alpha \left\| \sqrt{k} \partial \zeta \partial \eta u \right\|_0^2.
\]

We may assume \( \varrho_2 = 1 \) on the support of \( \eta_y \) if we assume that \( \zeta_1 = 1 \) on the support of \( \eta_1 \), since \( \varrho_2 = \zeta_1(x) \theta_2(y) \) and \( \eta_y = \eta_1(x) \eta'_2(y) \).

**Cautionary Note:** We initially defined the cutoff functions \( \zeta_i \) and \( \eta_i \) to be independent for \( i = 1, 2 \). We caution the reader that while we will now assume that \( \zeta_1 = 1 \) on the support of \( \eta_1 \), in later sections we will want to choose just the opposite, namely \( \eta_i = 1 \) on the support of \( \zeta_i \). This will not be circular as in the iterations of our inequalities, we replace our existing complement of cutoff functions with a completely new collection supported in a much larger set, and often without notice.

So with \( \varrho_2 = 1 \) on the support of \( \eta_y \) we have
\[
\left\| \sqrt{k} \eta_y \partial_y u \right\|_0^2 \leq A^2 \left\| \sqrt{k} \partial \varrho_2 u \right\|_0^2,
\]
and
\[
\left\| \sqrt{k} \partial \zeta^2 \partial \eta u \right\|_0^2 \leq C \left\| \zeta \sqrt{k} \partial \zeta \partial \eta u \right\|_0^2 + C \left\| \sqrt{k} \partial \zeta \partial \eta u \right\|_0^2
\]
\[
\leq C \left\| \partial_x \zeta \partial \eta u \right\|_0^2 + C \left\| \sqrt{k} \partial y \zeta \partial \eta u \right\|_0^2 + C A^2 \left\| \partial_x \eta u \right\|_0^2 + C A^2 \left\| \sqrt{k} \partial y \eta u \right\|_0^2
\]
upon considering the cases \( \partial = \partial_x \) and \( \partial = \partial_y \) separately, throwing away the \( \sqrt{k} \) when \( \partial = \partial_x \). Thus we obtain
\[
|II| \leq C A \left\| \partial_x \zeta \partial \eta u \right\|_0^2 + C A \left\| \sqrt{k} \partial y \zeta \partial \eta u \right\|_0^2
\]
\[
+ C A \left\| \partial_x \eta u \right\|_0^2 + C A \left\| \sqrt{k} \partial y \eta u \right\|_0^2
\]

We now apply Corollary 3.2 to estimate the middle line above by
\[
C A^2 \left\| \partial_x \eta u \right\|_0^2 + C A \left\| \sqrt{k} \partial y \eta u \right\|_0^2
\]
\[
\leq C A^2 \left( \int_{\mathcal{R}} (\eta \mathcal{L} u) (\eta u) \right) + C A \left\| \xi u \right\|_0^2.
\]
Finally, for III, we have

\[ |III| \leq \int_R (\zeta \partial \eta_{xx} u) (\zeta \partial \eta) \\
+ \int_R (\zeta \partial k \eta_{yy} u) (\zeta \partial \eta) \\
+ \int_R (\zeta \partial k \eta_y u) (\zeta \partial \eta). \]

Using \( \eta_{xx} = \eta_{xx} \theta_1 \), we see that the first of the three terms is dominated by

\[ C \left( ||\eta u||_1^2 + A^4 ||\eta_1 u||_1^2 + A^6 ||\xi u||_0^2 \right). \]

Using \( \eta_{yy} = \eta_{yy} \theta_2 \) and our hypothesis (2.19), we see that in the case \( \partial = \partial_y \), the second term is dominated by

\[ C \left( ||\eta u||_1^2 + A^4 ||k \partial_y \theta_2 u||_0^2 + A^4 (A^2 + B^2) ||\xi u||_0^2 \right). \]

In the case \( \partial = \partial_x \), we have

\[ \int_R (\zeta \partial k \eta_{yy} u) (\zeta \partial \eta) = \int_R (k \eta_{yy} u) (\partial_x \zeta^2 \partial_x \eta) \\
\leq \frac{C}{\alpha} A^4 ||\xi u||_0^2 + C\alpha \||\partial_x \zeta^2 \partial_x \eta\||_0^2 \\
\leq \frac{C}{\alpha} A^4 ||\xi u||_0^2 + C\alpha \||\partial_x \zeta \partial_x \eta\||_0^2 + C\alpha A^2 \||\partial_x \eta\||_0^2, \]

and we can apply (3.14) to the last term here. Finally, by using (3.13) and our hypothesis (2.19), the third term satisfies

\[ \int_R (\zeta \partial k \eta_y u) (\zeta \partial \eta) = \int_R (\eta_y u) (k \partial \zeta^2 \partial \eta) \\
\leq \frac{C}{\alpha} A^2 ||\xi u||_0^2 + C\alpha B^2 \||\partial \zeta \partial \eta||_0^2 \\
\leq \frac{C}{\alpha} A^2 ||\xi u||_0^2 + B^2 C\alpha \||\partial \zeta \partial \eta||_0^2 + B^2 C\alpha \||\partial \partial \zeta \partial \eta||_0^2 \\
+ C\alpha A^2 B^2 \||\partial \zeta \partial \eta||_0^2 + C\alpha A^2 B^2 \||\partial \partial \zeta \partial \eta||_0^2. \]

Now use (3.14) on the final two terms on the right side to complete the proof of the lemma.

**Lemma 3.7.** Suppose \( \mathcal{L} \) is as in (3.1) with \( k \) nonnegative, smooth and satisfying (2.19). Let \( \partial \) denote either \( \partial_x \) or \( \partial_y \). For \( u \in C^\infty (\mathcal{R}) \) and \( 0 < \alpha < 1 \), we have

\[ \left| \int_R (\zeta [\mathcal{L}, \partial] \eta u) (\zeta \partial \eta u) \, dx \, dy \right| \leq \frac{C}{\alpha} ||\eta u||_1^2 + CB^2 \alpha \int_R k \| \partial_y \zeta \partial \eta u \|^2 \, dx \, dy \\
+ C\alpha A^2 B^2 \left| \int_R (\eta \mathcal{L} u) (\eta u) \, dx \, dy \right| + C\alpha A^2 B^2 \||\xi u||_0^2. \]
Proof. We have \([L, \partial] = \partial_y [k, \partial] \partial_y = -\partial_y (\partial_k) \partial_y\) and so using (2.19),
\[
\left| \int_R (\zeta [L, \partial] \eta u) (\zeta \partial \eta u) \right| = \left| \int_R (\zeta \partial_y (\partial_k) \partial_y \eta u) (\zeta \partial \eta u) \right|
\]
\[
= \left| \int_R (\partial_y \eta u) ((\partial_k) \partial_y \zeta^2 \partial \eta u) \right|
\]
\[
\leq C \frac{1}{\alpha} \left\| \eta u \right\|_2^2 + CB^2 \alpha \left( \int_R |\partial_x \zeta \partial_y \eta u|^2 + \int_R k |\partial_y \zeta \partial_y \eta u|^2 \right)
\]
\[
+ C \alpha^2 A^2 B^2 \left( \left\| \partial_x \eta u \right\|_0^2 + \left\| \sqrt{k} \partial_y \eta u \right\|_0^2 \right).
\]
We now use (3.14) to bound \(C \alpha^2 A^2 B^2 \left( \left\| \partial_x \eta u \right\|_0^2 + \left\| \sqrt{k} \partial_y \eta u \right\|_0^2 \right)\) by
\[
C \alpha^2 A^2 B^2 \left\| \int_R (\eta \partial u) (\eta u) \right\| + C \alpha^4 A^2 \left\| \eta u \right\|_0^2,
\]
and this completes the proof of the lemma.

**Corollary 3.8.** Suppose \(L\) is as in (3.1) with \(k\) nonnegative, smooth and satisfying (2.19). Let \(\partial\) denote either \(\partial_x\) or \(\partial_y\). For \(u \in C^\infty (R)\), we have
\[
\int_R \left( |\partial_x (\zeta \partial \eta u)|^2 + k |\partial_y (\zeta \partial \eta u)|^2 \right) dxdy
\]
\[
\leq -4 \int_R (\zeta \partial \eta Lu) (\zeta \partial \eta u) dxdy + C A^2 \int_R (\eta \partial u) (\eta u) dxdy
\]
\[
+ C A^4 \left\| \partial_1 u \right\|_1^2 + C A^2 (A^2 + B^2) \int_R k |\partial_y \partial_2 u|^2 dxdy + C A^4 \int_R k |\partial_y \partial_2 u|^2 dxdy
\]
\[
+ C (1 + B^2) \left\| \eta u \right\|_1^2 + C A^4 (A^2 + B^2) \left\| \eta u \right\|_0^2.
\]
Proof. We plug the identity
\[
[L, \partial \eta] = \partial [L, \eta] + [L, \partial] \eta,
\]
into the second term on the right side of Corollary 3.3,
\[
\int_R \left( |\partial_x (\zeta \partial \eta u)|^2 + k |\partial_y (\zeta \partial \eta u)|^2 \right)
\]
\[
\leq -4 \int_R (\zeta \partial \eta Lu) (\zeta \partial \eta u) - 4 \int_R (\zeta [L, \partial \eta] u) (\zeta \partial \eta u)
\]
\[
+ C A^2 \left\| \partial_1 u \right\|_1^2 + C A^2 \sqrt{k} |\partial_y \partial_2 u|^2 + C A^4 \left\| \eta u \right\|_0^2,
\]
and then estimate the resulting terms with Lemma 3.6 for \(0 < \alpha < 1\) to be chosen,
\[
\left| \int_R (\zeta \partial [L, \eta] u) (\zeta \partial \eta u) \right|
\]
\[
\leq C \alpha (B^2 + 1) \left( \left\| \partial_x \zeta \partial \eta u \right\|_0^2 + \sqrt{k} |\partial_y \zeta \partial \eta u|^2 \right) + C \left\| \eta u \right\|_0^2
\]
\[
+ C \left( A^4 \left\| \partial_1 u \right\|_1^2 + \left( A^4 + \frac{A^2}{\alpha} \right) \left\| \sqrt{k} \partial_y \partial_2 u \right\|_0^2 \right) + A^4 \left( 1 + B^2 + A^2 \right) \left\| \eta u \right\|_0^2
\]
\[
+ C A^2 (B^2 + 1) \left| \int_R (\eta \partial u) (\eta u) \right|,
\]
and Lemma 3.7,

\[
\int_{\mathcal{R}} (\zeta [L, \partial] \eta u) (\zeta \partial \eta u) \leq C \frac{1}{\alpha} \|\eta u\|_1^2 + CB^2 \int_{\mathcal{R}} k |\partial_x \zeta \partial \eta u|^2
\]

\[
+ C \alpha A^2 B^2 \int_{\mathcal{R}} (\eta Lu) (\eta u) + C \alpha A^2 B^2 \|u\|_0^2.
\]

Then choose \(\alpha = \frac{1}{2C(1 + B^2)}\) so that the term

\[C \alpha(1 + B^2) \|\partial_x \zeta \partial \eta u\|_0^2 + C \alpha(1 + B^2) \int_{\mathcal{R}} k |\partial_y \zeta \partial \eta u|^2\]

can be absorbed into the left side.

3.2. The Moser Iteration. In this section we establish local \(L^p\) improvement for solutions \(u\) of \(Lu = 0\), where \(\mathcal{L} = \partial_x^2 + \partial_y k(x,y) \partial_y\). Whenever we use \(\beta\) to denote a positive real number, we assume that \(\beta = \frac{m}{n}\) is rational with \(n\) odd, so that expressions such as \(u(x)^\beta\) make sense. Let \(\mathcal{R} = [-R_1, R_1] \times [-R_2, R_2]\) be a rectangle in the plane, and let \(\eta, \zeta, \vartheta, \xi, \kappa\) be as in section 1. Let \(k\) be nonnegative and smooth in a neighbourhood of \(\mathcal{R}\).

3.2.1. The gradient estimate for powers. We begin by generalizing Lemma 3.1 to powers of \(u\) as in (3.1). Recall that \((v, w)_k = v_1 w_1 + kv_2 w_2\) and \(\mathcal{A} = \begin{bmatrix} 1 & 0 \\ 0 & k(x, y) \end{bmatrix}\).

**Lemma 3.9.** Suppose \(\mathcal{L}\) is as in (3.1) with \(k\) nonnegative and smooth. For \(u \in C^\infty(\mathcal{R})\) and \(\beta > \frac{1}{2}\), we have

\[
\int_{\mathcal{R}} \langle \zeta \nabla u^\beta, \zeta \nabla u^\beta \rangle_k dxdy
\]

\[
= -\frac{\beta^2}{2\beta - 1} \int_{\mathcal{R}} (\zeta Lu)(\zeta u^{2\beta - 1}) dxdy - \frac{2\beta}{2\beta - 1} \int_{\mathcal{R}} \langle u^\beta \nabla \zeta, \zeta \nabla u^\beta \rangle_k dxdy,
\]

and

\[
\int_{\mathcal{R}} \left( |\zeta \partial_x u^\beta|^2 + k |\zeta \partial_y u^\beta|^2 \right) dxdy
\]

\[
\leq \frac{2\beta^2}{2\beta - 1} \int_{\mathcal{R}} (\zeta Lu)(\zeta u^{2\beta - 1}) dxdy
\]

\[
+ \left( \frac{2\beta}{2\beta - 1} \right)^2 \|\zeta x\|_\infty^2 \int_{\mathcal{R}} |\zeta_1 u^\beta|^2 dxdy + \left( \frac{2\beta}{2\beta - 1} \right)^2 \|\zeta y\|_\infty^2 \int_{\mathcal{R}} k |\zeta_2 u^\beta|^2 dxdy.
\]
Proof. For $\beta \geq 1$, integration by parts yields the identity above;
\[
\int_R \langle \zeta \nabla u^\beta, \zeta \nabla u^\beta \rangle \; \mathrm{d}x = -\frac{\beta^2}{2\beta - 1} \int_R \nabla (\zeta u^{2\beta - 1}) \cdot A (\zeta \nabla u)
\]
\[
= -\frac{\beta^2}{2\beta - 1} \int_R u^{2\beta - 1} \nabla \cdot (\zeta^2 A u)
\]
\[
= -\frac{\beta^2}{2\beta - 1} \int_R \left( u^{2\beta - 1} \zeta^2 \nabla \right) \cdot A (\zeta \nabla u)
\]
For $\frac{1}{2} < \beta < 1$, the third equality above requires the elementary lemma below, applied in each variable separately, after writing the term in the second line as
\[
\beta^2 \int_R \zeta^2 u^{2\beta - 2} \nabla u \cdot A \nabla u.
\]
Now use the inequality $2ab \leq a^2 + \frac{1}{n}b^2$ to obtain
\[
2 \left| \langle u^\beta \nabla \zeta, \zeta \nabla u^\beta \rangle \right| \leq \frac{2\beta - 1}{2\beta} \langle \nabla u^\beta, \nabla u^\beta \rangle + \frac{2\beta}{2\beta - 1} \langle \nabla \zeta, u^\beta \nabla \zeta \rangle,
\]
and combining this with $\langle \nabla \zeta, \nabla \zeta \rangle = k \zeta_\rho^2 \leq \|\zeta_x\|_\infty \rho_1^2 + \|\zeta_x\|_\infty \rho_2^2$ as in Lemma 3.1, we obtain the desired inequality.

Lemma 3.10. If $u$ and $\varphi$ are smooth and $\varphi$ has compact support in the unit interval $(0, 1)$, and $\delta = \frac{\delta_0}{n} > 0$ with $n$ odd, then $\int_0^1 |u(x)|^{\delta - 1} |\varphi(x)| |u'(x)|^2 \; \mathrm{d}x < \infty$ and
\[
\int_0^1 \delta u(x)^{\delta - 1} \varphi(x) |u'(x)|^2 \; \mathrm{d}x = -\int_0^1 u(x)^{\delta - 1} (\varphi(x) u'(x))' \; \mathrm{d}x.
\]

Proof. Suppose first that there is a subinterval $(a, b)$ of $(0, 1)$ such that $u(a) = u(b) = 0$ and $u(x) \varphi(x) \neq 0$ for $a < x < b$. Then the fundamental theorem of calculus yields
\[
0 = \lim_{\varepsilon \to 0} \left[ u(b - \varepsilon)^\delta \varphi(b - \varepsilon) u'(b - \varepsilon) - u(a + \varepsilon)^\delta \varphi(a + \varepsilon) u'(a + \varepsilon) \right]
\]
\[
= \lim_{\varepsilon \to 0} \int_{a + \varepsilon}^{b - \varepsilon} \left[ u(x)^\delta \varphi(x) u'(x) \right]' \; \mathrm{d}x
\]
\[
= \lim_{\varepsilon \to 0} \int_{a + \varepsilon}^{b - \varepsilon} \delta u(x)^{\delta - 1} \varphi(x) |u'(x)|^2 \; \mathrm{d}x + \int_{a + \varepsilon}^{b - \varepsilon} \delta u(x)^{\delta - 1} \varphi(x) |u'(x)|^2 \; \mathrm{d}x
\]
\[
= \int_a^b \delta u(x)^{\delta - 1} \varphi(x) |u'(x)|^2 \; \mathrm{d}x + \int_a^b \delta u(x)^{\delta - 1} \varphi(x) |u'(x)|^2 \; \mathrm{d}x,
\]
where the limit in the second summand exists because the integrand is bounded, and the monotone convergence theorem yields the existence of the first (since $u$ and $\varphi$ do not change sign in $(a, b)$). Note also that since $u^\delta \varphi$ has constant sign in $(a, b)$,
\[
\delta \int_a^b |u(x)|^{\delta - 1} \varphi(x) |u'(x)|^2 \; \mathrm{d}x \leq \int_a^b |u(x)|^\delta \varphi(x) u'(x) \; \mathrm{d}x.
\]
Proposition 3.11. Suppose \( \nu \) is a Fourier multiplier operator, then the cut-off \( \psi \) is dominated by \( 30 \) and \( \nu' \) vanishes almost everywhere that \( \nu \) vanishes. Now write \( \{ x \in (0,1) : \psi(x) \neq 0 \} \) as an at most countable union of intervals \((a_j, b_j)\) with \( u \psi \) nonzero on \((a_j, b_j)\) and vanishing at the endpoints for all \( j \). Then

\[
\delta \int_0^1 |u(x)|^{\delta-1} |\varphi(x)| |u'(x)|^2 \, dx = \delta \sum_j \int_{a_j}^{b_j} |u(x)|^{\delta-1} |\varphi(x)| |u'(x)|^2 \, dx
\]

\[
\leq \sum_j \int_{a_j}^{b_j} |u(x)|^\delta |(\varphi(x) u'(x))'| \, dx
\]

\[
\leq \int_0^1 |u(x)|^\delta |(\varphi(x) u'(x))'| \, dx < \infty,
\]

and

\[
\int_0^1 \delta u(x)^{\delta-1} \varphi(x) |u'(x)|^2 \, dx = \sum_j \int_{a_j}^{b_j} \delta u(x)^{\delta-1} \varphi(x) |u'(x)|^2 \, dx
\]

\[
= - \sum_j \int_{a_j}^{b_j} u(x)^\delta (\varphi(x) u'(x))' \, dx
\]

\[
= - \int_0^1 u(x)^\delta (\varphi(x) u'(x))' \, dx.
\]

3.2.2. The subunit estimate. While the integral \( \int_\mathbb{R} |\varphi u|^2 \) in Lemma 3.9 can be handled since it is supported where \( L \) is elliptic, the integral \( \int_\mathbb{R} k |\varphi u|^2 \) requires further work. We will use the following fractional integral result repeatedly in this effort.

**Proposition 3.11.** Suppose \( T \) is a pseudodifferential operator of order \( \alpha \in (-2,0] \). Then

\[
\| \xi T \xi f \|_{L^q(\mathbb{R}^2)} \leq C \| \xi f \|_{L^p(\mathbb{R}^2)}, \quad \frac{1}{q} \geq \frac{1}{p} + \frac{\alpha}{2},
\]

provided \( 1 \leq p \leq q < \infty \), and \( q < \frac{2}{2 + \alpha} \) in the case \( p = 1 \). If \( T \) is in addition a Fourier multiplier operator, then the cutoff function \( \xi \) can be omitted.

We do not need (2.19), the inequality \( |\nabla k| \leq B |\sqrt{k}| \) for the next result.

**Lemma 3.12.** Suppose \( L \) is as in (3.1) with \( k \) nonnegative and smooth. Then for each \( \nu > 0 \), there is \( p < 2 \) such that for all \( u \in C_c^\infty (\mathbb{R}) \) and all \( \beta > 1 \),

\[
\left( \int_\mathbb{R} k |\varphi u|^\beta \right)^{\frac{1}{\beta}}
\]

is dominated by

\[
C \sqrt{\beta} \left| \int_\mathbb{R} (\xi I_1 \varphi u^{\beta-1} Lu) \left( \frac{\beta}{\beta - 1} \right)^{\frac{\beta}{2}} \right|
\]

\[
+ C \left( \frac{\beta}{\beta - 1} \right)^{\frac{\beta}{2}} \left( \int_\mathbb{R} \left( \varphi u^{\beta-1} Lu \right) \left( \varphi u^\beta \right) \right)
\]

\[
+ C_p \left( \frac{\sqrt{\beta (\beta - 1)}}{\beta (\beta - 1)^{\frac{\beta}{2}}} + \| \nu k \|_{C^{\nu}} + A \| \nu k \|_{C^{\nu}} + A^2 \right) \left( \int_\mathbb{R} |\xi u^\beta|^p \right)^{\frac{1}{p}},
\]
where $I_1$ is a Fourier multiplier operator of order $-1$.

Proof. Denote by $\Lambda^*$ the multiplier operator with symbol \((1 + |\cdot|^2)^\frac{\alpha}{2}\). We use the identity,

$$Id = (I - \nabla^2) \Lambda^{-2} = \Lambda^{-2} - \nabla \cdot (\nabla \Lambda^{-2}) ,$$

to write

$$\int_{\mathbb{R}} k |\varphi_2 u^\beta|^2 = \int_{\mathbb{R}} k |\xi \varphi_2 u^\beta|^2$$

$$\leq C \int_{\mathbb{R}} k |\xi \Lambda^{-2} \varphi_2 u^\beta|^2 + C \int_{\mathbb{R}} k |\xi \nabla \cdot (\nabla \Lambda^{-2}) \varphi_2 u^\beta|^2$$

$$\leq C \int_{\mathbb{R}} |\xi \Lambda^{-2} \varphi_2 u^\beta|^2 + C \int_{\mathbb{R}} |\xi \partial_x (I_1 \varphi_2 u^\beta)|^2 + C \int_{\mathbb{R}} k |\xi \partial_y (I_1 \varphi_2 u^\beta)|^2$$

where $I_1 = \partial_x \Lambda^{-2}$ in the second integral on the right, and $I_1 = \partial_y \Lambda^{-2}$ in the third integral. Both operators $I_1$ have order $-1$, and this small abuse of notation should cause no problems. Now the first term on the right satisfies

$$\int_{\mathbb{R}} |\xi \Lambda^{-2} \varphi_2 u^\beta|^2 \leq C_p \left( \int_{\mathbb{R}} |\xi u^\beta|^p \right)^\frac{2}{p}$$

for any $1 \leq p < 2$, by Proposition 3.11 on fractional integration ($\Lambda^{-2}$ has order $\alpha$ for all $\alpha > -2$). By Lemma 3.1, and with $\mathcal{L}$ as in (3.1), the last two terms on the right are dominated by

$$C \int_{\mathbb{R}} |\xi \mathcal{L} I_1 \varphi_2 u^\beta| \left( \xi I_1 \varphi_2 u^\beta \right) + CA^2 \int_{\mathbb{R}} |\mathcal{L} I_1 \varphi_2 u^\beta|^2$$

upon replacing $u$ by $I_1 \varphi_2 u^\beta$ in (3.5). Strictly speaking, we should replace $I_1$ in (3.15) by $\partial_x \Lambda^{-2}$, and then by $\partial_y \Lambda^{-2}$, and finally add the two expressions. Now the last term in (3.15) satisfies

$$\int_{\mathbb{R}} |I_1 \varphi_2 u^\beta|^2 \leq C_p \left( \int_{\mathbb{R}} |\xi u^\beta|^p \right)^\frac{2}{p}$$

for any $1 < p < 2$, by Proposition 3.11 again. It remains to estimate the first term on the right side of (3.15) given by

$$\int_{\mathbb{R}} (\xi \mathcal{L} I_1 \varphi_2 u^\beta) \left( \xi I_1 \varphi_2 u^\beta \right) = \int_{\mathbb{R}} (\xi I_1 \varphi_2 \mathcal{L} u^\beta) \left( \xi I_1 \varphi_2 u^\beta \right)$$

$$+ \int_{\mathbb{R}} (\xi [\mathcal{L}, I_1 \varphi_2] u^\beta) \left( \xi I_1 \varphi_2 u^\beta \right) .$$

Noting that

$$\mathcal{L} u^\beta = (\partial_x^2 + \partial_y k \partial_y) u^\beta$$

$$= \beta u^{\beta-1} \mathcal{L} u + \beta (\beta - 1) u^{\beta-2} \left( \partial_x u^2 + k |\partial_y u|^2 \right) ,$$
we have
\[
\int_{\mathcal{R}} (\xi \mathcal{L} I_1 \varrho_2 u^\beta) (\xi I_1 \varrho_2 u^\beta) = \beta \int_{\mathcal{R}} (\xi I_1 \varrho_2 u^{\beta-1} \mathcal{L} u) (\xi I_1 \varrho_2 u^\beta) \\
+ \beta (\beta - 1) \int_{\mathcal{R}} (\xi I_1 \varrho_2 u^{\beta-2} \left( |\partial_x u|^2 + k |\partial_y u|^2 \right)) (\xi I_1 \varrho_2 u^\beta) \\
+ \int_{\mathcal{R}} (\xi [\mathcal{L}, I_1 \varrho_2] u^\beta) (\xi I_1 \varrho_2 u^\beta) \\
= \mathcal{I} + \mathcal{J} + \mathcal{K}.
\]

Now the term $|\mathcal{I}|^{\frac{3}{4}}$ is the first term in the conclusion of the lemma. For the second term, we write $I_1 = \partial \Lambda^{-2} = \left( \partial \Lambda^{-\frac{3}{2}} \right) \left( \Lambda^{-\frac{1}{2}} \right) = I_\frac{3}{2} I_\frac{1}{2}$ where $\partial$ is either $\partial_x$ or $\partial_y$ (we continue to abuse notation by writing $I_\frac{3}{2}$ for the three different operators $\partial_x \Lambda^{-\frac{3}{2}}, \partial_y \Lambda^{-\frac{3}{2}}$ and $\Lambda^{-\frac{3}{2}}$, each of order $-\frac{1}{2}$). We then obtain

\[
|\mathcal{J}| = \beta (\beta - 1) \int_{\mathcal{R}} \left( I_\frac{3}{2} \xi^2 I_1 \varrho_2 u^{\beta-2} \left( |\partial_x u|^2 + k |\partial_y u|^2 \right) \right) (I_\frac{3}{2} \varrho_2 u^\beta) \\
\leq C \beta (\beta - 1) \int_{\mathcal{R}} \left| I_\frac{3}{2} \xi^2 I_1 \varrho_2 u^{\beta-2} \left( |\partial_x u|^2 + k |\partial_y u|^2 \right) \right|^2 \\
+ C \beta (\beta - 1) \int_{\mathcal{R}} \left| I_\frac{3}{2} \varrho_2 u^\beta \right|^2 \\
\leq C \beta (\beta - 1) \left\{ \int_{\mathcal{R}} |\varrho_2 u^{\beta-2} (|\partial_x u|^2 + k |\partial_y u|^2) |^2 \right\}^\frac{3}{2} + C \beta (\beta - 1) \left\{ \int_{\mathcal{R}} |\varrho_2 u^\beta |^2 \right\}^\frac{3}{2},
\]

by Proposition 3.11 with first $T = I_\frac{3}{2} \xi^2 I_1$, $\alpha = -\frac{3}{2}$, $p = 1$ and $q = 2$, and then with $T = I_\frac{3}{2}$, $\alpha = -\frac{1}{2}$, $p = \frac{3}{2}$ and $q = 2$. Using Lemma 3.9 with $\frac{3}{2}$ in place of $\beta$ in the case $\beta > 1$ ($|\mathcal{J}| = 0$ when $\beta = 1$), the first integral above satisfies

\[
\int_{\mathcal{R}} \left| \varrho_2 u^{\beta-2} \left( |\partial_x u|^2 + k |\partial_y u|^2 \right) \right| \\
= C \frac{1}{\beta^2} \int_{\mathcal{R}} \left( |\sqrt{\varrho_2} \partial_x u^{\frac{\beta}{2}}|^2 + k |\sqrt{\varrho_2} \partial_y u^{\frac{\beta}{2}}|^2 \right) \\
\leq C \frac{1}{\beta - 1} \left| \int_{\mathcal{R}} \left( \sqrt{\varrho_2} u^{\frac{\beta}{2}-1} \mathcal{L} u \right) \left( \sqrt{\varrho_2} u^{\frac{\beta}{2}} \right) \right| \\
+ CA^2 \frac{1}{(\beta - 1)^2} \int_{\mathcal{R}} \left| u^\beta \right|.
\]

So altogether, we have

\[
|\mathcal{J}| \leq C \frac{\beta}{\beta - 1} \left| \int_{\mathcal{R}} \left( \sqrt{\varrho_2} u^{\frac{\beta}{2}-1} \mathcal{L} u \right) \left( \sqrt{\varrho_2} u^{\frac{\beta}{2}} \right) \right|^2 \\
+ C \left[ \frac{A^4 \beta}{(\beta - 1)^2} + \beta (\beta - 1) \left( \int_{\mathcal{R}} \left| u^\beta \right| \right)^\frac{3}{2} \right],
\]

where the first term here leads to (by taking the square root) the second term in the conclusion of the lemma.
Finally, to estimate the third term $K$, we write $I_1 = \partial \Lambda^{-2} = (\partial \Lambda^{\alpha-2}) (\Lambda^{-\alpha}) = I_{1-\alpha} I_0$ for any $0 < \alpha < 1$ to obtain
\[
|K| = \left| \int_R (I_{1-\alpha} \xi^2 [\mathcal{L}, I_1 \varrho_2] u^\beta) (I_0 \varrho_2 u^\beta) \right|
\leq C \int_R |I_{1-\alpha} \xi^2 [\mathcal{L}, I_1 \varrho_2] u^\beta|^2 + C \int_R |I_0 \varrho_2 u^\beta|^2.
\]
As before,
\[
\int_R |I_0 \varrho_2 u^\beta|^2 \leq C_p \left( \int_R |\varrho_2 u^\beta|^p \right)^{\frac{2}{p}}
\]
for $\frac{1}{2} = \frac{1}{p} - \frac{\alpha}{p}$. We now write $[\mathcal{L}, I_1 \varrho_2] = [\mathcal{L}, I_1] \varrho_2 + I_1 [\mathcal{L}, \varrho_2]$ and consider the two terms
\[
(3.16) \quad \int_R |I_{1-\alpha} \xi^2 [\mathcal{L}, I_1] \varrho_2 u^\beta|^2 \quad \text{and} \quad \int_R |I_{1-\alpha} \xi^2 I_1 [\mathcal{L}, \varrho_2] u^\beta|^2
\]
separately.

To estimate the first term in (3.16), we note that
\[
[I, I_1] = \partial_y [k, I_1] \partial_y
\]
\[
= \partial_y (k I_1 \partial_y - I_1 k \partial_y)
\]
\[
= \partial_y (k (I_1 \partial_y) - (I_1 \partial_y) k + I_1 k_y)
\]
\[
= \partial_y (k, I_1 \partial_y) + I_1 k_y).
\]

Following [25], we denote by $O^p$ the collection of rough pseudodifferential operators mapping $H^{s+p, p}_{\text{compact}}$ to $H^{s, p}_{\text{loc}}$ for $1 < p < \infty$ and $s \in I$, where $H^{s, p}$ denotes the Sobolev space of functions whose fractional derivatives up to order $s$ lie in $L^p$. Now for $0 < \mu < 1$ and $\varepsilon > 0$ we have
\[
\xi [k, I_1 \partial_y] \in O_\varepsilon^{\mu} \quad \text{for } \mu + \varepsilon < \nu,
\]
with norm $\|\xi k\|_{C^\infty((-\varepsilon, \varepsilon))}$ (see Taylor [27], or Theorem 4 in [25]). Since $I_{1-\alpha} \xi \partial_y$ has order $\alpha$, and since $\xi^2 \partial_y = \xi \partial_y \xi - \xi_y \xi$, we thus have
\[
I_{1-\alpha} \xi^2 \partial_y [k, I_1 \partial_y] \in O_\varepsilon^{\mu-\varepsilon} \quad \text{for } 0 < \alpha < \min \{\mu, \varepsilon\}, \mu + \varepsilon < \nu.
\]

Thus $I_{1-\alpha} \xi^2 \partial_y [k, I_1 \partial_y]$ maps $P_{\text{compact}}^{p_1} = H^{0, p_1}_{\text{compact}}$ to $H^{\mu-\alpha, p_1}_{\text{loc}}$ provided $\mu - \alpha \in (\varepsilon + \alpha, \varepsilon - \alpha)$, i.e. $\mu \in (2\alpha - \varepsilon, \varepsilon)$, which is in turn embedded in $L^2_{\text{loc}}$ by the Sobolev embedding theorem with $\frac{1}{2} = \frac{1}{p_1} - \frac{\mu - \alpha}{2}$. Note that given $\nu > 0$, we can first choose $\varepsilon$ and $\alpha$ such that $0 < \frac{\mu}{2} < \alpha < \varepsilon < \frac{\mu}{2}$, and then choose $\mu$ such that $\alpha < \mu < \varepsilon$, in order that all of the above parameter restrictions hold. So,
\[
\int_R |I_{1-\alpha} \xi^2 [\mathcal{L}, I_1] \varrho_2 u^\beta|^2 \leq C \int_R |I_{1-\alpha} \xi^2 \partial_y [k, I_1 \partial_y] \varrho_2 u^\beta|^2
\]
\[+ C \int_R |I_{1-\alpha} \xi^2 \partial_y I_1 k \varrho_2 u^\beta|^2 \leq C \|\xi k\|^2_{C^\infty} \left( \int_R |\varrho_2 u^\beta|^{p_1} \right)^{\frac{2}{2}} + C \|\xi k\|_{C^\infty}^2 \left( \int_R |\varrho_2 u^\beta|^{p_2} \right)^{\frac{2}{2}},
\]
for $\frac{1}{2} = \frac{1}{p_1} - \frac{\mu - \alpha}{2}$ and $\frac{1}{2} = \frac{1}{p_2} - \frac{\mu - \alpha}{2}$ by Proposition 3.11.
To estimate the second term in (3.16), we observe that if $T$ is defined by $T = I_1([\mathcal{L}, \varrho_2])$, then by using (3.11),
$$T = 2 \left( I_1(\varrho_2), \partial_x + I_1 k(\varrho_2), \partial_y \right) + I_1 \left( \varrho_2 \right)_{xx} + k(\varrho_2)_{yy} + k_y(\varrho_2)_y,$$
and then $T$ is a bounded operator on $L^p_{loc}$ with norm at most $CA(A + \|\varphi k_2\|_{\infty})$ for all $1 < p < \infty$, and satisfies $T = T\xi$. Thus we have
$$\int_{\mathbb{R}} \left| I_{1-\alpha} \xi^2 I_1 [\mathcal{L}, \varrho_2] u^\beta \right|^2 = \int_{\mathbb{R}} \left| I_{1-\alpha} \xi^2 T u^\beta \right|^2 \leq C \left( \int_{\mathbb{R}} |T\xi u^\beta| |p_z| \right)^{\frac{\alpha}{p_2}} \leq CA^2 \left( A^2 + \|\varphi k_2\|_{\infty}^2 \right) \left( \int_{\mathbb{R}} |\xi u^\beta| |p_z| \right)^{\frac{\alpha}{p_2}},$$
where $\frac{1}{\alpha} = \frac{1}{p_2} - \frac{1-\alpha}{3}$ as above. This completes the proof of the lemma if we take $p = \max \{p_1, p_2, \frac{4}{3} \}$.

4. A Nonlinear Degenerate Elliptic Equation

In this section we begin the proof of the a priori estimates (2.21) for smooth solutions of the quasilinear equation (2.17), which we recall here as
\begin{equation}
\|\zeta D^\alpha w\|_{\infty} \leq C_\alpha \left( \|\varphi \nabla w\|_{\infty}, L \right),
\end{equation}
where $C_\alpha (\cdot, \cdot)$ is finite and increasing on $[0, \infty) \times \mathcal{P}_c(\Omega)$, and $w$ is smooth and satisfies
\begin{equation}
\mathcal{L}w = \left[ \partial_x^2 + \partial_y k(x, w(x, y)) \partial_y \right] w = 0, \quad (x, y) \in \Omega',
\end{equation}
and also
\begin{equation}
(x, w(x, y)) \in L \text{ for all } (x, y) \in \text{support}(\varphi).
\end{equation}
Throughout this section, $w$ will be a smooth solution of (4.2) satisfying (4.3), and for convenience, we will say that an expression involving derivatives of $w$ is under control if it is dominated by the right side of (4.1). Similarly we will make statements to the effect that some derivative $D^\alpha w$ is in a Banach space $\mathcal{X}$ with control, meaning that $\|\zeta D^\alpha w\|_{\mathcal{X}}$ is under control for an appropriate cutoff function $\zeta$.

We attack the problem by differentiating (4.2), to obtain the equations
\begin{align*}
0 &= \mathcal{L}w_x + \partial_y \left[ \{ k_1(x, w(x, y)) + k_2(x, w(x, y)) w_x \} w_y \right], \\
0 &= \mathcal{L}w_y + \partial_y \left[ \{ k_2(x, w(x, y)) w_x^2 \} \right],
\end{align*}
or
\begin{align*}
\mathcal{L}w_x &= -\partial_y k_1 w_y - \partial_y k_2 w_x w_y, \\
\mathcal{L}w_y &= -\partial_y k_2 w_y^2,
\end{align*}
for $w_x$ and $w_y$. Note that we use $\partial_y$ as an operator acting on everything to its right, unless parentheses indicate otherwise. Recall also Convention 2.3.2 concerning the expressions $k, k_1$ etc. in this and subsequent sections: $k$ denotes $k(x, w(x, y))$ and $k_i$ denotes $k_i(x, w(x, y))$ etc., except in section 5 where $k$ has more variables and the convention is modified accordingly.

We will apply Corollary 3.8 in the section on gradient estimates to the components of $\nabla w$, and using the facts that both $w$ and $\nabla w$ are bounded with control,
we will show that in fact \( w \in H^2 \), i.e., \( \nabla^2 w \in L^2 \) with control. Note that this does not increase the index of smoothness of \( w \) that is under control, but only reverses the Sobolev embedding theorem \( H^2(\mathbb{R}^2)^s \subset \text{"Lip}_1(\mathbb{R}^2)\). Recall that the index of smoothness of an \( n \)-dimensional \( L^p \) Sobolev space \( H^s_p(\mathbb{R}^n) \) is the quantity \( s - \frac{n}{p} \). Since the equations (4.4) are not homogeneous, we must handle with care the terms arising from \( \mathcal{L}\nabla w \) in applying Corollary 3.8. We then apply the results of the section on gradient estimates to obtain that \( \nabla^2 w \in L^q \) with control for \( q \) large depending on how small \( R_1 \) is chosen, again handling with care the terms arising from \( \mathcal{L}\nabla w \). Note that the Moser iteration actually increases the index of smoothness another \( \frac{2}{q} \), for a total of 1. From now on, it turns out that due to the nature of the quasilinear systems satisfied by higher order gradients of \( w \), which become progressively less nonlinear, we can continue to alternately apply the reverse Sobolev embedding and the Moser iteration to increase the index of smoothness of \( w \) that is under control by 1 with each repetition. Thus we obtain the a priori estimates (4.1).

4.1. Reverse Sobolev embedding. Here we show that if \( \nabla w \in L^\infty \) with control, and satisfies the system (4.4), then \( \nabla^2 w \in L^2 \) with control. The following lemmas will be crucial in handling the nonhomogeneous terms in (4.4).

**Lemma 4.1.** Suppose \( w \) is a smooth solution of (4.2) in a compact rectangle \( \mathcal{R} \) in \( \Omega' \), where \( k(x, y) \) is smooth and nonnegative in \( \Omega \), so that \( u = w_x \) and \( v = w_y \) are smooth solutions in \( \mathcal{R} \) of the nonlinear system (4.4). Then we have

\[
\int_{\mathcal{R}} \left( |\partial_x \xi u|^2 + k |\partial_y \xi u|^2 \right) dxdy + \int_{\mathcal{R}} \left( |\partial_x \xi v|^2 + k |\partial_y \xi v|^2 \right) dxdy \\
\leq CA^2 \left( ||\xi u||^2_{L^2} + ||\xi v||^2_{L^2} \right) + CB^2 \left( ||\xi u||^4_{L^4} + ||\xi v||^4_{L^4} \right).
\]

Alternatively, we have a bound in terms of at most \( ||\xi u||_{L^2} \) and \( ||\xi v||_{L^\infty} \):

\[
\int_{\mathcal{R}} \left( |\partial_x \xi u|^2 + k |\partial_y \xi u|^2 \right) dxdy + \int_{\mathcal{R}} \left( |\partial_x \xi v|^2 + k |\partial_y \xi v|^2 \right) dxdy \\
\leq CA^2 \left( ||\xi u||^2_{L^2} + ||\xi v||^2_{L^2} \right) + CB^2 \left( ||\xi u||^2_{L^2} ||\xi v||^2_{L^\infty} + ||\xi v||^4_{L^4} \right).
\]

**Proof.** From Corollary 3.2, applied with \( k(x, w(x, y)) \) in place of \( k(x, y) \) there, we have

\[
(4.5) \quad \int_{\mathcal{R}} \left( |\partial_x \xi u|^2 + k |\partial_y \xi u|^2 \right) + \int_{\mathcal{R}} \left( |\partial_x \xi v|^2 + k |\partial_y \xi v|^2 \right) \\
\leq -4 \int_{\mathcal{R}} (\xi \mathcal{L} u) (\xi u) - 4 \int_{\mathcal{R}} (\xi \mathcal{L} v) (\xi v) \\
+ CA^2 ||\xi u||^2_0 + CA^2 ||\xi v||^2_0.
\]
For the integral involving $Lv$, we have by (4.4)
\[- \int_{\mathcal{R}} (\zeta L v) (\zeta v) = \int_{\mathcal{R}} (\zeta \partial_y k_2 v^2) (\zeta v) \]
\[= - \int_{\mathcal{R}} (v^2) (k_2 \partial_y \zeta v) \]
\[= - \int_{\mathcal{R}} (v^2) (k_2 \zeta \partial_y v) - \int_{\mathcal{R}} (v^2) (k_2 \zeta_y \zeta v) \]
\[= - \int_{\mathcal{R}} (\zeta v^2) (k_2 \partial_y \zeta) - \int_{\mathcal{R}} (\zeta v^2) (k_2 \zeta_y v) . \]

The first term on the right is dominated by
\[B^2 C \varepsilon \int_{\mathcal{R}} k |\partial_y \zeta v|^2 + \frac{C}{\varepsilon} \|\xi v\|^4_{L^4} , \]
while the second term is at most
\[CB^2 \|\xi v\|^4_{L^4} + CA^2 \|\xi v\|^2_{L^2} . \]

Choosing $\varepsilon = \frac{1}{4CB^2}$, we can absorb the term $B^2 C \varepsilon \int_{\mathcal{R}} k |\partial_y \zeta v|^2$ into the left side of (4.5). The same argument yields the appropriate estimate for
\[- \int_{\mathcal{R}} (\zeta L u) (\zeta u) = \int_{\mathcal{R}} (\zeta \partial_y k_1 v) (\zeta u) + \int_{\mathcal{R}} (\zeta \partial_y k_2 uv) (\zeta u) . \]

To obtain the alternate bound, we estimate the last integral above by
\[\left| \int_{\mathcal{R}} (\zeta \partial_y k_2 uv) (\zeta u) \right| = \left| - \int_{\mathcal{R}} (\zeta uv) (k_2 \partial_y \zeta u) - \int_{\mathcal{R}} (k_2 \zeta uv) (\zeta_y u) \right| \]
\[\leq \frac{C}{\varepsilon} \|\xi v\|^2_{L^2} \|\xi u\|^2_{L^2} + B^2 C \varepsilon \int_{\mathcal{R}} k |\partial_y \zeta u|^2 \]
\[+ CA^2 \|\xi u\|^2_{L^2} + CB^2 \|\xi u\|^2_{L^2} \|\xi v\|^2_{L^2} , \]
and similarly for the other integral. Choosing $\varepsilon = \frac{1}{4CB^2}$ again completes the proof of the lemma.

Up to this point we have been keeping precise track of all the constants. This will prove increasingly difficult from now on, and we will instead only keep close track of the critical constants - typically those which are involved in subsequent absorptions.

**Lemma 4.2.** Suppose $w$ is a smooth solution of (4.2) in $\mathcal{R}$, where $k (x, y)$ is non-negative and smooth in $\Omega$ and satisfies (2.19) so that $u = w_x$ and $v = w_y$ are smooth solutions in $\mathcal{R}$ of the nonlinear system (4.4). Then we have with $\partial = \partial_x$ or $\partial = \partial_y$,
\[\int_{\mathcal{R}} \left( |\partial_x (\zeta \partial_y u)|^2 + k |\partial_y (\zeta \partial_y u)|^2 \right) \, dx \, dy \]
\[+ \int_{\mathcal{R}} \left( |\partial_x (\zeta \partial_y v)|^2 + k |\partial_y (\zeta \partial_y v)|^2 \right) \, dx \, dy \]
\[\leq \mathcal{C} (B, \|x \nabla w\|_{\infty}) \left( \|\zeta u\|^2_{L^1} + \|\zeta v\|^2_{L^1} \right) + \mathcal{C} (A, B, \|x \nabla w\|_{\infty}) , \]
where the functions $\mathcal{C} (\cdot, \cdot)$ and $\mathcal{C} (\cdot, \cdot, \cdot)$ are finite and increasing in each variable separately.
Proof. We wish to apply Corollary 3.8 with \( k(x, y) \) replaced by \( k = k(x, w(x, y)) \).
Now by (2.19), we have

\[
|\nabla k| = |(k_1 + k_2 w_x, k_2 w_y)| \leq |k_1| + |k_2 \nabla w| \\
\leq CB\sqrt{k(1 + |\nabla w|)},
\]

and thus we can apply Corollary 3.8 if we replace \( B \) by

\[
\tilde{B} = CB(1 + \|x \nabla w\|_{\infty}).
\]

We obtain

\[
\begin{align*}
(4.6) \quad |\nabla k| &= |k_1 + k_2 w_x, k_2 w_y| \\
\leq CB\sqrt{k(1 + |\nabla w|)},
\end{align*}
\]

for \( \partial = \partial_x \) or \( \partial_y \). We first estimate

\[
\begin{align*}
-\int_{\mathcal{R}} (\zeta \partial \eta \mathcal{L}v) (\zeta \partial \eta v) &= \int_{\mathcal{R}} (\zeta \partial \eta \partial_y k_2 \mathcal{L}v^2) (\zeta \partial \eta v) \\
&= \int_{\mathcal{R}} (v^2) (k_2 \partial_y \eta \partial \zeta \partial \eta v) \\
&= \int_{\mathcal{R}} (v^2) (\eta \partial \zeta k_2 \partial \eta \partial \eta v) + \int_{\mathcal{R}} (v^2) ([k_2 \partial_y, \eta \partial \zeta^2] \partial \eta v) \\
&= -\int_{\mathcal{R}} (\zeta \partial \eta v^2) (k_2 \partial_y \zeta \partial \eta v) + \int_{\mathcal{R}} (v^2) ([k_2 \partial_y, \eta \partial \zeta^2] \partial \eta v) \\
&\quad + \int_{\mathcal{R}} (\zeta \partial \eta v^2) (k_2 \zeta \partial \eta v) \\
&= I + II + III.
\end{align*}
\]

For term \( I \) we use

\[
|I| \leq \frac{C}{\varepsilon} \int_{\mathcal{R}} |\zeta \partial \eta v|^2 + \tilde{B}^2 \varepsilon \int_{\mathcal{R}} k |\partial_y \zeta \partial \eta v|^2,
\]

and thus we can apply Corollary 3.8 if we replace \( B \) by

\[
\tilde{B} = CB(1 + \|x \nabla w\|_{\infty}).
\]
since $|k_2| \leq \tilde{B} \sqrt{k}$ by (2.19), and absorb the second term $\tilde{B}^2 \varepsilon \int_\mathbb{R} k |\partial_y \zeta \partial \eta v|^2$ into the left side of (4.7) upon choosing $\varepsilon = \frac{1}{2 \tilde{B}^2}$. As for the first term, since

$$\partial \eta v^2 = 2 v \partial \eta v - (\partial \eta) v^2,$$

we have

$$\int_\mathbb{R} |\partial \eta v^2|^2 \leq C \int_\mathbb{R} |v \partial \eta v|^2 + C \int_\mathbb{R} |(\partial \eta) v|^2 \leq C \|\xi v\|^4_{L^\infty} \|\eta v\|_{L^2}^2 + C A^2 \|\xi v\|^4_{L^4}.$$

Now use $\|\xi v\|^4_{L^4} \leq |\mathbb{R}| \|\xi v\|^4_{L^\infty} \leq C \|\xi v\|^4_{L^\infty}$ and multiply the resulting terms above by $\frac{C}{\varepsilon} = 2CB^2$ to obtain an expression which is bounded by the right side of the conclusion of Lemma 4.2.

For term $III$, we use

$$|III| = \left| \int_\mathbb{R} (\zeta \partial \eta v^2) (k_2 \zeta \partial \eta v) \right| \leq C \int_\mathbb{R} |\partial \eta v^2|^2 + C A^2 \tilde{B}^2 \int_\mathbb{R} k |\partial \eta v|^2.$$

The first term is handled by the previous inequality, and the second is at most

$$C (A^2 \tilde{B}^2) (A^2 + \tilde{B}^2 \|\xi v\|^2_{L^\infty}) \|\xi v\|^2_{L^\infty}$$

by Lemma 4.1.

For term $II$, 

$$[k_2 \partial_y, \eta \partial \zeta^2] = k_2 \eta \partial \zeta^2 + k_2 \eta \partial \zeta \partial \eta \zeta - \eta \zeta^2 \partial k_2 \partial_y,$$

implies

$$|(II)| \leq \left| \int_\mathbb{R} (v^2) (k_2 \eta \partial \zeta \partial \eta v) \right| + \left| \int_\mathbb{R} (v^2) (k_2 \eta \partial \zeta \partial \eta v) \right| + \left| \int_\mathbb{R} (v^2) (\eta \zeta \partial k_2 \partial_y \eta v) \right|.$$

Now the first of the terms in (4.8) satisfies

$$\left| \int_\mathbb{R} (v^2) (k_2 \eta \partial \zeta \partial \eta v) \right| \leq \int_\mathbb{R} (v^2) (k_2 \eta \partial \zeta \partial \eta v) + \int_\mathbb{R} (v^2) (k_2 \eta \partial \zeta \partial \eta v) \leq \tilde{B}^2 C \varepsilon \int_\mathbb{R} k |\partial \zeta \partial \eta v|^2 + \frac{C}{\varepsilon} \int_\mathbb{R} |\eta v^2|^2 + \tilde{B}^2 C \int_\mathbb{R} k |\eta v^2|^2 + C \int_\mathbb{R} |\partial \eta v|^2 \leq \tilde{B}^2 C \varepsilon \int_\mathbb{R} k |\partial \zeta \partial \eta v|^2 + \frac{C}{\varepsilon} A^2 \|\xi v\|^2_{L^4} + C A^4 \tilde{B}^2 \|\xi v\|^4_{L^4} + C \|\eta v\|^2,$$

and the first term on the right above can be absorbed into the left side of (4.7) with $\varepsilon = \frac{1}{2C \tilde{B}^2}$. The second term in (4.8) can be handled in exactly the same way. The
third term in (4.8) is handled as follows:

\[
\left| \int_{\mathcal{R}} (v^2) \left( \eta \zeta^2 (\partial k_2) \partial_y \partial \eta v \right) \right| \leq \left| \int_{\mathcal{R}} (v^2) (\eta \zeta^2 (k_{21} + k_{22}u) \partial_y \partial_x \eta v) \right| \\
+ \left| \int_{\mathcal{R}} (v^2) (\eta \zeta^2 (k_{22}v) \partial_y \partial \eta v) \right| \\
\leq C \int_{\mathcal{R}} |\partial_y k_{21} \zeta^2 \eta v^2|^2 + C \int_{\mathcal{R}} |\partial_y k_{22} u \zeta^2 \eta v^2|^2 \\
+ C \int_{\mathcal{R}} |\partial_y k_{22} v \zeta^2 \eta v^2|^2 + C \|\eta v\|_1^2.
\]

The first three integrals on the right are now easily dominated by

\[
C \|\xi \nabla w\|_{L^\infty}^4 \left( \|\xi \nabla w\|_{L^\infty}^4 + A^2 \|\xi \nabla w\|_{L^\infty}^2 + \|\eta u\|_1^2 + \|\eta v\|_1^2 \right)
\]

using

\[
\partial \zeta^2 k_{22} \eta v^2 = (\partial \zeta^2 k_{22}) \eta v^2 + \zeta^2 k_{22} [v^2 \partial \eta u + u 2v \partial \eta v - 2uv^2 \partial \eta] 
\]

and

\[
\partial \zeta^2 k_{22} = 2 \zeta (\partial \zeta) k_{22} + \zeta^2 \partial k_{22} = \begin{cases} \ 2 \zeta_x k_{22} + \zeta^2 (k_{221} + k_{222}u), & \partial = \partial_x, \\ \ 2 \zeta_y k_{22} + \zeta^2 k_{222}v, & \partial = \partial_y, \end{cases}
\]

along with similar formulas for the terms involving \(k_{21} v^2\) and \(k_{22} v^3\). This completes the estimates for the second term on the right side of (4.7). Similar arguments handle the term

\[
\int_{\mathcal{R}} (\zeta \partial \eta \mathcal{L} u) (\zeta \partial \eta u)
\]

in (4.7).

Next, we turn to estimating \( |\int_{\mathcal{R}} (\eta \mathcal{L} u) (\eta u) | + |\int_{\mathcal{R}} (\eta \mathcal{L} v) (\eta v) | \). For this we have

\[
\left| \int_{\mathcal{R}} (\eta \mathcal{L} v) (\eta v) \right| = \left| \int_{\mathcal{R}} (\eta \partial_y k_{22} v^2) (\eta v) \right| \\
= \left| - \int_{\mathcal{R}} (\eta v^2) (k_{22} \partial_y \eta v) - \int_{\mathcal{R}} (k_{22} \eta v^3) \right| \\
\leq \mathcal{B}^2 \int_{\mathcal{R}} k |\partial_y \eta v|^2 + C \int_{\mathcal{R}} |\eta v|^2 + CA \|\zeta v\|_\infty^3,
\]

and the first term on the right is controlled by Lemma 4.1. A similar argument applies to \( |\int_{\mathcal{R}} (\eta \mathcal{L} u) (\eta u) | \).

Finally, we turn to the remaining terms in (4.7) that arose from the application of Corollary 3.8. The terms

\[
\int_{\mathcal{R}} k |\partial_y \theta_2 u|^2 + \int_{\mathcal{R}} k |\partial_y \theta_2 v|^2
\]

and

\[
\int_{\mathcal{R}} k |\partial \theta_2 u|^2 + \int_{\mathcal{R}} k |\partial \theta_2 v|^2
\]

are handled by Lemma 4.1, while the terms \( \| \theta_1 u \|_1^2 + \| \theta_1 v \|_1^2 \) are handled by elliptic theory (since \( \theta_1 \) is supported where \( k > 0 \)) as given in the Proposition below.
The penultimate term in (4.7) is included in the first term on the right side of the conclusion, while the final term in (4.7) is included in the second term. This completes the proof of the theorem.

**Proposition 4.3.** Suppose \( k \geq c > 0 \) is smooth and \( \zeta, \xi \) are smooth cutoff functions with \( \xi = 1 \) on the support of \( \zeta \). For each multiindex \( \alpha \), there is a finite increasing function \( C_\alpha (\cdot) \) on \([0, \infty)\), such that

\[
\| \zeta D^\alpha w \|_{L^\infty} \leq C_\alpha \left( \| \xi w \|_{L^\infty} + \| \xi \nabla w \|_{L^\infty} \right),
\]

for all smooth solutions \( w \) of

\[
(4.9) \quad \partial_x^2 w + \partial_y k (x, w) \partial_y w = 0.
\]

**Proof.** We write (4.9) in nondivergence form as follows:

\[
(4.10) \quad \partial_x^2 w + k (x, w) \partial_y^2 w = -k_2 (x, w) (\partial_y w)^2 = f.
\]

Then \( k (x, w) \) and \( f \) are bounded functions with \( k (x, w) \geq c > 0 \), and so by Theorem 12.4 in [9], we conclude that for some \( \delta > 0 \),

\[
\| \zeta w \|_{C^{2+\delta}} \leq C \left( \| \xi w \|_{L^\infty} + \| \xi \nabla w \|_{L^\infty} \right) \leq C_1 \left( \| \xi w \|_{L^\infty} + \| \xi \nabla w \|_{L^\infty}^2 \right).
\]

Now return to (4.10) and note that \( f \in C^6 \) and \( k (x, w) \in C^{1+\delta} \) with control. By the Schauder estimates, Theorem 6.2 in [9], we obtain

\[
\| \zeta w \|_{C^{2+\delta}} \leq C_2 \left( \| \xi w \|_{L^\infty}, \| \xi \nabla w \|_{L^\infty} \right),
\]

and so also \( k (x, w) \in C^{2+\delta} \) and \( f \in C^{1+\delta} \) with control. We can now differentiate (4.10) with respect to \( \partial \) and apply Schauder theory again to obtain

\[
\| \zeta w \|_{C^{3+\delta}} \leq C_3 \left( \| \xi w \|_{L^\infty}, \| \xi \nabla w \|_{L^\infty} \right).
\]

Iterating this process yields the conclusion of the proposition.

**Theorem 4.4.** Suppose \( w \) is a smooth solution of (2.17),

\[
\partial_x^2 w + \partial_y k (x, w (x, y)) \partial_y w = 0
\]

in \( \mathcal{R} \), where \( k \) is nonnegative, smooth and satisfies (2.19). Then \( w \in H^2_{\text{loc}} \) with control, \( i.e. \), \( \partial_x^2 w \in L^2_{\text{loc}} \) with control.

**Proof.** The Poincaré inequality (3.3) and Lemma 4.2 yield with \( u = \partial_x w \) and \( v = \partial_y w \),

\[
\| \eta u \|_1^2 + \| \eta v \|_1^2 \leq CR_1^2 \int_{\mathcal{R}} \left( |\partial_x (\zeta \nabla \eta u)|^2 + |\partial_x (\zeta \nabla \eta v)|^2 \right) \leq CR_1^2 \left( 1 + (B^2 + 1) \| \xi \nabla w \|_{L^\infty}^4 \right) \left( \| \eta u \|_1^2 + C \| \eta v \|_1^2 \right) + CR_1^2 C (A, B, \| \xi \nabla w \|_{L^\infty}).
\]

Choosing \( R_1 \leq \left\{ 2C \left( 1 + (B^2 + 1) \| \xi \nabla w \|_{L^\infty}^4 \right) \right\}^{-\frac{1}{2}} \) (note that \( A \) is not involved here) permits the first term on the right above to be absorbed into the left hand side, and this completes the proof of the theorem.
4.2. An $L^p$ improvement. In this subsection, we improve the index of smoothness of $w$ that is under control by showing that $\nabla^2 w \in L^q$ with control for large $q > 2$. Let us first compute the equations satisfied by the $L^2$ functions $\nabla^2 w$. Differentiating (4.4), and continuing to set $u = w_x$ and $v = w_y$, yields

$$0 = \mathcal{L} u_x + \partial_y \{(k_1 + k_2 u) u_y \} + \partial_y \{(k_{11} + k_{12} u) v + k_{11} v_x + (k_{12} + k_{22} u) uv + k_{22} w_x \}$$
$$0 = \mathcal{L} u_y + \partial_y \{k_2 u u_y \} + \partial_y \{(k_{12} v) v + k_{11} v_y + (k_{22} v) w + k_{22} w v \}$$
$$0 = \mathcal{L} v_x + \partial_y \{(k_1 + k_2 u) v_y \} + \partial_y \{(k_{12} + k_{22} u) v^2 + k_{22} v v_x \}$$
$$0 = \mathcal{L} v_y + \partial_y \{k_2 v v_y \} + \partial_y \{(k_{22} v) v^2 + k_{22} v v_y \}$$

or

$$-\mathcal{L} u_x = \partial_y \left\{ k_1 (u_y + v_x) + k_2 (u u_y + u_x v + w v_x) \right\} + k_{11} u + 2 k_{12} w + k_{22} u^2 v$$
$$-\mathcal{L} u_y = \partial_y \{k_1 v_y + k_2 (w u + 2 u_y v) + k_{12} v^2 + k_{22} w u^2 \}$$
$$-\mathcal{L} v_x = \partial_y \{k_1 v_x + k_2 (u u_x + 2 v_x) + k_{12} v^2 + k_{22} w u^2 \}$$
$$-\mathcal{L} v_y = \partial_y \{3 k_{22} v v_y + k_{22} v^3 \}.$$

The key feature of this system is that the right hand side is a combination of terms involving either the operator $\partial_y k_1 = (k_i \partial_y)^j$, the transpose of the subunit vector field $k_i \partial_y$, or the identity operator acting on an expression which is affine in the components of $\nabla u$ and $\nabla v$ with bounded coefficients. We rewrite this system so as to exploit this feature as follows:

$$-\mathcal{L} u_x = (k_1 \partial_y)^j (u_y + v_x) + (k_2 \partial_y)^j (u u_y + u_x v + w v_x)$$
$$-\mathcal{L} u_y = (k_1 \partial_y)^j v_y + (k_2 \partial_y)^j (w u + 2 u_y v)$$
$$-\mathcal{L} v_x = (k_1 \partial_y)^j v_x + (k_2 \partial_y)^j (u u_x + 2 v_x)$$
$$-\mathcal{L} v_y = (k_2 \partial_y)^j 3 v v_y + \{ 2 k_{22} v^4 + k_{22} 3 u^2 v \},$$

where we recall that the derivatives of $k$ are evaluated at $(x, w (x, y))$. The following lemma is crucial for estimating the nonlinear terms above. We recall that by Lemma 3.10, expressions like $\partial_y u^\beta_y = \beta u^{\beta-1} u_y$ are square integrable for $\beta > \frac{1}{2}$ (and not just $\beta > 1$).

**Lemma 4.5.** Suppose that $u_x, u_y, v_x, v_y$ give a smooth solution of the system (4.11) in $\mathcal{R}$ with $k = k (x, w (x, y))$. Then for $\beta > \frac{1}{2}$, the $k$-gradient integrals

$$\int_{\mathcal{R}} \left( |\zeta \partial_x u_x| \right)^2 + k \left( |\zeta \partial_y u_y| \right)^2 + \int_{\mathcal{R}} \left( |\zeta \partial_x v_x| \right)^2 + k \left( |\zeta \partial_y v_y| \right)^2$$
$$+ \int_{\mathcal{R}} \left( |\zeta \partial_x v_x| \right)^2 + k \left( |\zeta \partial_y v_y| \right)^2 + \int_{\mathcal{R}} \left( |\zeta \partial_y v_y| \right)^2 + k \left( |\zeta \partial_y v_y| \right)^2$$
are dominated by
\[
C_1 \left( \beta, \frac{1}{\beta - \frac{1}{2}}, B, \|\nabla w\|_\infty \right) \int_\mathbb{R} \left\{ |\zeta u_x^\beta|^2 + |\zeta u_y^\beta|^2 + |\zeta v_x^\beta|^2 + |\zeta v_y^\beta|^2 \right\} \\
+ C \left( \frac{\beta}{2\beta - 1} \right)^2 A^2 \int_\mathbb{R} \left\{ |\varrho_1 u_x^\beta|^2 + |\varrho_1 u_y^\beta|^2 + |\varrho_1 v_x^\beta|^2 + |\varrho_1 v_y^\beta|^2 \right\} \\
+ C_2 \left( \beta, \frac{1}{\beta - \frac{1}{2}}, A, B, \|\nabla w\|_\infty \right) \int_\mathbb{R} k \left\{ |\varrho_2 u_x^\beta|^2 + |\varrho_2 u_y^\beta|^2 + |\varrho_2 v_x^\beta|^2 + |\varrho_2 v_y^\beta|^2 \right\} \\
+ C_3 \left( \beta, \frac{1}{\beta - \frac{1}{2}}, \|\nabla w\|_\infty \right),
\]
where \( C_1, C_2 \) and \( C_3 \) are finite and increasing in each variable separately.

A crucial point is that \( C_1 \) in the above lemma does not depend on \( A \), so that in applying the one dimensional Poincaré inequality in the next theorem, the product \( R_1^2 C_1 \) can be made less than one for \( R_1 \) sufficiently small. This would be impossible if \( A^2 \) were present since \( A \geq R_1^{-1} \) - recall (3.7).

**Proof.** We see from Lemma 3.9 applied to the four functions \( u_x, u_y, v_x, v_y \), that it suffices to prove that
\[
\left( \int_\mathbb{R} (\zeta L u_x) (\zeta u_x^{2\beta - 1}) \right) + \left( \int_\mathbb{R} (\zeta L u_y) (\zeta u_y^{2\beta - 1}) \right) \\
+ \left( \int_\mathbb{R} (\zeta L v_x) (\zeta v_x^{2\beta - 1}) \right) + \left( \int_\mathbb{R} (\zeta L v_y) (\zeta v_y^{2\beta - 1}) \right)
\]
is dominated by
\[
C \alpha B^2 \int_\mathbb{R} k \left\{ |\zeta \partial_y u_x^\beta|^2 + |\zeta \partial_y u_y^\beta|^2 + |\zeta \partial_y v_x^\beta|^2 + |\zeta \partial_y v_y^\beta|^2 \right\} \\
+ C \left( \beta, \frac{1}{\beta - \frac{1}{2}}, \|\nabla w\|_\infty, \frac{1}{\alpha} \right) \int_\mathbb{R} \left\{ |\zeta u_x^\beta|^2 + |\zeta u_y|^2 + |\zeta v_x^\beta|^2 + |\zeta v_y|^2 \right\} \\
+ C \left( \beta, \frac{1}{\beta - \frac{1}{2}}, A, B, \|\nabla w\|_\infty \right) \int_\mathbb{R} k \left\{ |\varrho_2 u_x^\beta|^2 + |\varrho_2 u_y^\beta|^2 + |\varrho_2 v_x^\beta|^2 + |\varrho_2 v_y^\beta|^2 \right\} \\
+ C \left( \|\nabla w\|_\infty \right),
\]
for any \( 0 < \alpha < 1 \), and where the function \( C \) is finite and increasing in each variable separately. Indeed, then the terms
\[
C \alpha B^2 \frac{\beta^2}{2\beta - 1} \int_\mathbb{R} k \left\{ |\zeta \partial_y u_x^\beta|^2 + |\zeta \partial_y u_y^\beta|^2 + |\zeta \partial_y v_x^\beta|^2 + |\zeta \partial_y v_y^\beta|^2 \right\}
\]
can be absorbed into the left side of (4.13) for \( \alpha = \frac{2\beta - 1}{2CB^2 \beta} \). Let us illustrate the bound for the term \( \int_\mathbb{R} (\zeta L u_y) (\zeta v_y^{2\beta - 1}) \), which is given by
\[
\left( \int_\mathbb{R} \zeta \left( (k_2 \partial_y^t)^3 3v_y + (k_{222} v^4 + k_{222} v^2 v_y) \right) (\zeta v_y^{2\beta - 1}) \right) \\
\leq C \left( \int_\mathbb{R} \zeta (k_{222} v^4 + k_{222} v^2 v_y) (\zeta v_y^{2\beta - 1}) \right) + C \left( \int_\mathbb{R} \zeta \left( (k_2 \partial_y^t)^3 3v_y \right) (\zeta v_y^{2\beta - 1}) \right).
\]
Now the first term here satisfies
\[
\left| \int_\mathcal{R} \zeta \left( (k_2 \partial_y)^4 3v v_y \right) (\zeta v_y^{2\beta-1}) \right| = \left| \int_\mathcal{R} (3v v_y) (k_2 \partial_y \zeta^2 v_y^{2\beta-1}) \right| \\
\leq C \left| \int_\mathcal{R} (3v v_y) (\zeta v_y^{\beta-1} k_2 \partial_y v_y^\beta) \right| \\
+ C \left| \int_\mathcal{R} (3v v_y) (\zeta v_y^{\beta-1} k_2 \zeta v_y^\beta) \right|,
\]
since \(2\beta-1\) \(\beta\) is bounded. Estimating these two terms separately, we have
\[
\left| \int_\mathcal{R} (3v v_y) (\zeta v_y^{\beta-1} k_2 \partial_y v_y^\beta) \right| = \left| \int_\mathcal{R} (3v \zeta v_y^\beta) (k_2 \partial_y v_y^\beta) \right| \\
\leq \frac{\|\xi v\|_L^2}{\alpha} \int_\mathcal{R} |\zeta v_y^\beta|^2 + \alpha \tilde{B}^2 \int_\mathcal{R} k |\zeta \partial_y v_y^\beta|,
\]
and
\[
\left| \int_\mathcal{R} (3v v_y) (\zeta v_y^{\beta-1} k_2 \zeta v_y^\beta) \right| = \left| \int_\mathcal{R} (3v \zeta v_y^\beta) (\zeta v_y^\beta) \right| \\
\leq C \|\xi v\|_L^2 \int_\mathcal{R} |\zeta v_y^\beta|^2 + C A^2 \tilde{B}^2 \int_\mathcal{R} k |t_2 v_y^\beta|.
\]

As for the second term in (4.15), we have
\[
\left| \int_\mathcal{R} \zeta (k_2 k_3 v_2^4 + k_23v_2 v_y) (\zeta v_y^{2\beta-1}) \right| \\
\leq C \|\xi v\|_L^2 \left( \int_\mathcal{R} |\zeta v_y^\beta|^2 + |\mathcal{R}| \right) + C \|\xi v\|_L^2 \int_\mathcal{R} |\zeta v_y^\beta|^2 \\
\leq C \|\xi v\|_L^2 \left( \int_\mathcal{R} |\zeta v_y^\beta|^2 + |\mathcal{R}| \right) + C \|\xi v\|_L^2 \int_\mathcal{R} |\zeta v_y^\beta|^2.
\]

The remaining terms in (4.14) are handled similarly. Indeed, from (4.12), we see that the only differences in the remaining terms are that some powers of \(v\) are replaced by the same or smaller powers of \(u, y\)-derivatives by \(x\)-derivatives, and partial derivatives of \(k\) by others of the same or smaller order. This completes the proof of the lemma.

**Theorem 4.6.** Suppose that \(w\) solves (4.9) so that with \(u = w_x\) and \(v = w_y\), the four functions \(u_x, u_y, v_x, v_y\) give a smooth solution of the system (4.11) in \(\mathcal{R}\). Then for \(q > 2\), we have \(u_x, u_y, v_x, v_y \in L^q\), i.e. \(\nabla^2 w \in L^q\) with control provided \(R_1\) is sufficiently small, depending on \(q\).

**Proof.** Using the one-dimensional Poincaré inequality, we have for \(\beta > 1\),
\[
\int_\mathcal{R} \left\{ |\zeta u_x^\beta| + |\zeta u_y^\beta| + |\zeta v_x^\beta| + |\zeta v_y^\beta| \right\} \\
\leq C R_1^2 \int_\mathcal{R} \left\{ |\partial_x \zeta u_x^\beta| + |\partial_x \zeta v_x^\beta| + |\partial_x \zeta v_y^\beta| \right\} \\
\leq C R_1^2 \int_\mathcal{R} \left\{ |\zeta \partial_x u_x^\beta| + |\zeta \partial_x v_x^\beta| + |\zeta \partial_x v_y^\beta| \right\} \\
+ C A^2 R_1^2 \int_\mathcal{R} \left\{ |\zeta \partial_y u_y^\beta| + |\zeta \partial_y v_y^\beta| \right\}.
\]
since $|\partial_x \zeta| \leq A \theta_1$. Now using the above lemma on the first term on the right side above, and then absorbing the term

$$CR^2C_1 \left( \beta, \frac{1}{\beta - \frac{1}{2}}, B, \| \xi \nabla w \|_{\infty} \right) \int_{\mathcal{R}} \left\{ |\zeta u_x^\beta|^2 + |\zeta u_y^\beta|^2 + |\zeta v_x^\beta|^2 + |\zeta v_y^\beta|^2 \right\}$$

into the left side for $R_1$ sufficiently small, we have

$$\left( \frac{\beta}{2\beta - 1} \right)^2 \int_{\mathcal{R}} \left\{ |\theta_1 u_x^\beta|^2 + |\theta_1 u_y^\beta|^2 + |\theta_1 v_x^\beta|^2 + |\theta_1 v_y^\beta|^2 \right\}$$

$$\leq CA^2 \left( \frac{\beta}{2\beta - 1} \right)^2 \int_{\mathcal{R}} \left\{ |\theta_2 u_x^\beta|^2 + |\theta_2 u_y^\beta|^2 + |\theta_2 v_x^\beta|^2 + |\theta_2 v_y^\beta|^2 \right\}$$

$$\leq CA^2 \left( \frac{\beta}{2\beta - 1} \right)^2 \int_{\mathcal{R}} \left\{ |\theta_2 u_x^\beta|^2 + |\theta_2 u_y^\beta|^2 + |\theta_2 v_x^\beta|^2 + |\theta_2 v_y^\beta|^2 \right\}$$

Now the first term on the right is under control by our assumption that $\mathcal{L}$ is elliptic on the support of $\theta_1$ (see Proposition 4.3 above). The second term is clearly under control. We will next show that the third term on the right hand side above is dominated by

$$\mathcal{C} \left( \beta, \frac{1}{\beta - \frac{1}{2}}, A, B, \| \xi \nabla w \|_{\infty} \right)$$

$$\times \left\{ 1 + \| \xi u_x^\beta \|_{L^p}^2 + \| \xi u_y^\beta \|_{L^p}^2 + \| \xi v_x^\beta \|_{L^p}^2 + \| \xi v_y^\beta \|_{L^p}^2 \right\}$$

$$\times \left\{ 1 + \| \sigma u_x \|_{L^2}^2 + \| \sigma u_y \|_{L^2}^2 + \| \sigma v_x \|_{L^2}^2 + \| \sigma v_y \|_{L^2}^2 \right\}$$

for some $p < 2$. Recall that we extend the usual convention regarding constants $\mathcal{C}$ to the functions $\mathcal{C} \left( \beta, \frac{1}{\beta - \frac{1}{2}}, A, B, \| \xi \nabla w \|_{\infty} \right)$ - they may change from line to line, while remaining increasing in each variable separately. Indeed, with this done, we can then choose $\beta = \frac{4}{p}$ and conclude that $\nabla u, \nabla v \in L^q$ with control for $q = \frac{4}{p} > 2$. In fact, we can continue to iterate this inequality as long as $R_2^2$ is sufficiently small. Thus we end up with $\nabla u, \nabla v \in L^q$ with control for $q$ large provided $R_1$ is small enough.

We will handle the integrals involving $\theta_2$, namely

$$\int_{\mathcal{R}} k \left\{ |\theta_2 u_x^\beta|^2 + |\theta_2 u_y^\beta|^2 + |\theta_2 v_x^\beta|^2 + |\theta_2 v_y^\beta|^2 \right\},$$
with Lemma 3.12 as follows. For $\beta > 1$,

\[(4.19)\]

\[
\int_R k \left| \varrho_2 v_y^\beta \right|^2 \, dxdy \\
\leq C\beta \left| \int_R \left( \xi_1 \varrho_2 v_y^{\beta-1} L_{vy} \right) \left( \xi_1 \varrho_2 v_y^\beta \right) \right| \\
+ C\frac{\beta}{\beta-1} \left| \int_R \left( \sqrt{\varrho_2 v_y^\beta} \right)^{\frac{\beta}{\beta-1}} L_{vy} \left( \sqrt{\varrho_2 v_y^\beta} \right)^2 \right| \\
+ C(p, \beta, A, k)^2 \left( \int_R \left| \xi v_y^\beta \right| \right)^\frac{2}{p} \\
= C\beta I + C\frac{\beta}{\beta-1} II + C(p, \beta, A, k)^2 \left( \int_R \left| \xi v_y^\beta \right| \right)^\frac{2}{p},
\]

where

\[
C(p, \beta, A, k) = C_p \left( \sqrt{\beta (\beta - 1)} + \frac{A^2 \sqrt{\beta}}{\beta - 1} + \|\kappa k\|_{C^0} + A \|\kappa \partial k\|_{\infty} + A^2 \right),
\]

and with similar estimates for $u_x, u_y, v_x$ in place of $v_y$. We remind the reader of Convention 2.3.2 here: $k$ means $k(x, w(x, y))$ and $\partial_y k$ means $k_2(x, w(x, y)) w_y$. We thus note that both $\|\kappa k\|_{C^0}$ and $\|\kappa \partial k\|_{\infty}$ are under control. It follows that the last term on the right side of (4.19) has the desired form.

We first consider the simpler term $II$, and plugging in the nonlinear term for $L v_y$, we have

\[
\sqrt{II} \leq \left| \int_R \left( \sqrt{\varrho_2 v_y^{\frac{\beta}{2}}} \left( (k_2 \partial_y) v_x v_y^{\beta-1} \right) \right) \left( \sqrt{\varrho_2 v_y^\beta} \right) \right| \\
+ \left| \int_R \left( \sqrt{\varrho_2 v_y^{\frac{\beta}{2}}} \left( k_222 v^4 + k_2223 v^2 v_y \right) \right) \left( \sqrt{\varrho_2 v_y^\beta} \right) \right| \\
= III + IV.
\]

Now using Lemma 3.10 to justify the needed formal manipulations (recall that $\beta > 1$ here), we have

\[
III = \left| \int_R (3v v_y) \left( k_2 \partial_y \varrho_2 v_y^{\beta-1} \right) \right| \\
\leq C \left| \int_R \left( 3v v_y^\beta \right) \left( k_2 \partial_y \varrho_2 v_y \right) \right| + CA \|\xi v\|_{L^\infty} \int_R \left| \xi v_y^\beta \right| \\
\leq C \tilde{B}^2 \int_R k \left| \partial_y v_y \right|^2 + CA \left( \|\xi v\|_{L^\infty} + \|\xi v\|_{L^\infty}^2 \right) \int_R \left| \xi v_y^\beta \right| \\
\leq C \left( \beta, \frac{1}{\beta - 1}, A, B, \|\kappa \nabla w\|_{\infty} \right) \left( 1 + \int_R \left\{ \left| \xi u_x^\beta \right|^2 + \left| \xi u_y^\beta \right|^2 + \left| \xi v_x^\beta \right|^2 + \left| \xi v_y^\beta \right|^2 \right\} \right)
\]

by (4.13) with $\frac{\beta}{2}$ in place of $\beta$, and with $\varrho_2$ in place of $\zeta$, upon combining all three integrals there under the common cutoff function $\xi$. This shows that term $III$ is
dominated by (4.17) with \( p = 1 \) upon using
\[
\int_{\mathbb{R}} f \leq 1 + \left( \int_{\mathbb{R}} f \right)^2,
\]
valid for any \( f \geq 0 \). The estimate for \( IV \) is
\[
IV \leq C \| \xi v \|_{L^{\infty}}^4 \left( \int_{\mathbb{R}} |\xi v|^{\beta} + |\Re| \right) + C \| \xi v \|_{L^{\infty}}^2 \int_{\mathbb{R}} |\xi v|^{\beta}.
\]

Turning now to term \( I \), and plugging in the nonlinear term for \( L v_y \), we have
\[
I \leq C \left[ \int_{\mathbb{R}} \left\{ \xi I_1 \partial_2 v_y^{\beta-1} \left( (k_2 \partial_y)^4 3v y \right) \right\} \left( \xi I_1 \partial_2 v_y^{\beta} \right) \right]
+ C \left[ \int_{\mathbb{R}} \left\{ \xi I_1 \partial_2 v_y^{\beta-1} \left( k_2 v^4 + k_223v^2 v_y \right) \right\} \left( \xi I_1 \partial_2 v_y^{\beta} \right) \right]
= V + VI.
\]

We can quickly dispense with term \( VI \) using that \( I_1 \) maps \( L^p \) to \( L^2 \) for \( 1 < p < 2 \). We handle term \( V \) with the identity
\[
v_y^{\beta-1} \partial_y k_2 v v_y = \frac{1}{\beta} \partial_y k_2 v v_y + \left( 1 - \frac{1}{\beta} \right) v_y^{\beta} \partial_y k_2 v
= \frac{1}{\beta} \partial_y k_2 v v_y + \left( 1 - \frac{1}{\beta} \right) v_y^{\beta} (k_2 v_y + k_22v^2)
\]
to get
\[
V = C \left[ \int_{\mathbb{R}} \left( \xi I_1 \partial_2 v_y^{\beta-1} \partial_y k_2 v v_y \right) \left( \xi I_1 \partial_2 v_y^{\beta} \right) \right]
+ C \left[ \int_{\mathbb{R}} \left( \xi I_1 \partial_2 \left[ \frac{1}{\beta} \partial_y k_2 v v_y + \left( 1 - \frac{1}{\beta} \right) v_y^{\beta} (k_2 v_y + k_22v^2) \right] \right) \left( \xi I_1 \partial_2 v_y^{\beta} \right) \right]
\leq C \left[ \int_{\mathbb{R}} \left( \xi I_1 \partial_2 \partial_y k_2 v v_y \right) \left( \xi I_1 \partial_2 v_y^{\beta} \right) \right]
+ C \left[ \int_{\mathbb{R}} \left( \xi I_1 \partial_2 v_y^{\beta} k_2 v_y \right) \left( \xi I_1 \partial_2 v_y^{\beta} \right) \right]
+ C \left[ \int_{\mathbb{R}} \left( \xi I_1 \partial_2 v_y^{\beta} k_22v^2 \right) \left( \xi I_1 \partial_2 v_y^{\beta} \right) \right]
= VII + VIII + IX.
\]

For term \( VII \), we commute \( \partial_2 \) and \( \partial_y \) so that we can exploit the \( L^p \) boundedness of \( I_1 \partial_y \) as follows:
\[
VII \leq C \left[ \int_{\mathbb{R}} \left( \xi I_1 \partial_y \partial_2 k_2 v v_y \right) \left( \xi I_1 \partial_2 v_y^{\beta} \right) \right]
+ \left[ \int_{\mathbb{R}} \left( \xi (\partial_y \partial_2) k_2 v v_y \right) \left( \xi I_1 \partial_2 v_y^{\beta} \right) \right].
\]
The first integral is
\[
\left| \int_R \left( I_2 \xi^2 I_1 \partial_y \partial_2 k_2 v_y^\beta \right) (I_2 \partial_2 v_y^\beta) \right| \leq C \int_R \left| I_2 \xi^2 I_1 \partial_y \partial_2 k_2 v_y^\beta \right|^2 + C \int_R \left| I_2 \partial_2 v_y^\beta \right|^2
\]
\[
\leq C \left( \int_R \left| \xi^2 I_1 \partial_y \partial_2 k_2 v_y^\beta \right|^p \right)^{\frac{1}{p}} + C \left( \int_R \left| \partial_2 v_y^\beta \right|^p \right)^{\frac{1}{p}}
\]
\[
\leq C \left( 1 + \|v\|_{L^\infty}^2 \right) \left( \int_R \left| \xi v_y^\beta \right|^p \right)^{\frac{1}{p}},
\]
where \(\frac{1}{2} = \frac{1}{p} - \frac{1}{4}\), and the second is dominated by
\[
C \int_R \left| I_1 \partial_y \partial_2 k_2 v_y^\beta \right|^2 + C \int_R \left| I_1 \partial_2 v_y^\beta \right|^2
\]
\[
 \leq C \left( 1 + A^2 \|v\|_{L^\infty}^2 \right) \left( \int_R \left| \xi v_y^\beta \right|^p \right)^{\frac{1}{p}},
\]
for any \(1 < p < 2\). Term IX is dominated by
\[
C \int_R \left| I_1 \partial_2 v_y^\beta \right|^2 + C \int_R \left| I_1 v_y^\beta \right|^2
\]
\[
 \leq C \left( 1 + \|v\|_{L^\infty} \right) \left( \int_R \left| \xi v_y^\beta \right|^p \right)^{\frac{1}{p}},
\]
for any \(1 < p < 2\) also.

In term VIII, the most problematic, we have an additional power of \(v_y\) to deal with. We write using \(\varphi_2 = \varphi_2 \xi\),
\[
VIII = C \int_R \left( \varphi_2 k_2 v_y^\beta \right) (I_1 \xi^2 I_1 \varphi_2 v_y^\beta) \left| \varphi_2 \partial_2 v_y^\beta \right|
\]
\[
 \leq C \left\{ \int_R \left| \xi I_1 \xi^2 I_1 \varphi_2 v_y^\beta \right| \left| v_y k_2 \varphi_2 v_y^\beta \right| \right\}
\]
\[
 \leq C \left( \int_R \left| \xi v_y^\beta \right|^p \right)^{\frac{1}{p}} \left\{ \frac{C}{\varepsilon} \int_R \left| \xi v_y \right|^2 + \tilde{B}^2 \varepsilon \int_R \left| \varphi_2 v_y^\beta \right|^2 \right\},
\]
for any \(1 < p < 2\) (since \(I_2 : L^p_{\text{compact}} \to L^\infty_{\text{loc}}\) for such \(p\)) and \(\varepsilon > 0\). We can choose \(\varepsilon > 0\) sufficiently small, in fact \(\varepsilon \approx \tilde{B}^2 (\int_R \left| v_y^\beta \right|^p)^{-\frac{1}{p}}\), so that the term \(\left( \int_R \left| \xi v_y^\beta \right|^p \right)^{\frac{1}{p}} \approx \tilde{B}^{-2} \left( \int_R \left| v_y^\beta \right|^p \right)^{\frac{1}{p}}\) can be absorbed into the left side of (4.19). Then term VIII is dominated by
\[
\left( \int_R \left| \xi v_y^\beta \right|^p \right)^{\frac{1}{p}} \frac{C}{\varepsilon} \int_R \left| \xi v_y \right|^2 \approx C \tilde{B} \left( \int_R \left| \xi v_y^\beta \right|^p \right)^{\frac{1}{p}} \int_R \left| v_y \right|^2,
\]
as required. The remaining terms in (4.18) are handled similarly and this completes the proof of the theorem.

4.3. The iteration. We can now obtain our a priori inequality (4.1) for the quasilinear equation (2.17). We briefly restate Theorem 2.3 as follows.

**Theorem 4.7.** Suppose that \(k\) is smooth and positive for \(x \neq 0\). Then with \(C_\alpha\) as in Theorem 2.3, we have
\[
\|\zeta D^\alpha w\|_{\infty} \leq C_\alpha (\|\zeta \nabla w\|_{\infty} \cdot L), \quad |\alpha| \geq 0.
\]
for all smooth solutions $w$ of (2.17) in $\Omega'$ such that $(x, w(x, y)) \in L$ for all $(x, y)$ in the support of $\kappa$.

Proof. Recall that in the subsection on "Reverse Sobolev Embedding", we used the fact that $\nabla w \in L^\infty$ with control together with the fact that $\nabla w = (w_x, w_y)$ satisfies the system (4.4) to conclude that $\nabla^2 w \in L^2$ with control. We now wish to deduce that $\nabla^3 w \in L^2$ with control from a somewhat weaker integrability assumption on $\nabla^2 w$, namely that $\nabla^2 w \in L^4$ with control rather than $L^\infty$ with control, plus the fact that $\nabla^2 w$ satisfies an appropriate system. We continue to use the phrase "under control" to mean bounded by an increasing function $C_{\alpha} (\|\nabla w\|_\infty, L)$ of $\|\nabla w\|_\infty \in [0, \infty)$ and $L \in P_c (\Omega)$, etc. At this point, we already know that if $w$ is a smooth solution of (2.17) in $\mathcal{R}$, then

$$
\nabla^2 w \in L^q \text{ with control, for } q \text{ large depending on } R_1,
$$

and that $\nabla^2 w$ satisfies (4.12), a system of the form

$$(4.20) \quad \mathcal{L} (\nabla^2 w) = P (\nabla w) (\nabla^2 w) + Q (\nabla w) + T^4 [R (\nabla w) (\nabla^2 w) + S (\nabla w)],$$

where $P (\nabla w)$, $Q (\nabla w)$, $R (\nabla w)$ and $S (\nabla w)$ (at this early stage in the iteration, the polynomial $S$ vanishes) are polynomials in the components of the bounded vector field $\nabla w$ with partial derivatives of $k$ as coefficients. Also, the expression $P (\nabla w) (\nabla^2 w)$ means sums of such polynomials times some second order derivatives of $w$. Finally, $T$ is a subunit vector field of the form $k_i \partial_y$. We can now apply the methods of the subsection "Reverse Sobolev embedding", since the components of $\nabla^2 w$ appear only to the first power multiplied by components of $\nabla w$, which are bounded. To estimate the $L^2_{loc}$ norm $\nabla^3 w$ by the technique of the proof of Theorem 4.4, we need to estimate

$$
\int_{\mathcal{R}} |\partial_x (\zeta \partial_y \nabla^2 w)|^2.
$$

To do this, we will use the analogue of Lemma 4.2 to estimate

$$(4.21) \quad \int_{\mathcal{R}} \left( |\partial_x (\zeta \partial_y \nabla^2 w)|^2 + k |\partial_y (\zeta \partial_y \nabla^2 w)|^2 \right)$$

for $\partial = \partial_x$ or $\partial = \partial_y$ by applying Corollary 3.8. The main terms to be estimated are of the form

$$(4.22) \quad \int_{\mathcal{R}} (\zeta \partial_y \mathcal{L} \nabla^2 w) (\zeta \partial_y \nabla^2 w).$$

Replacing $\mathcal{L} \nabla^2 w$ by one of the terms in (4.20), say $T^4 [R (\nabla w) (\nabla^2 w)]$, we can decompose the resulting expression into three pieces $I + II + III$ as in the proof of Lemma 4.2. Term $I$ has the form

$$(4.23) \quad \int_{\mathcal{R}} (\zeta \partial_y R (\nabla w) (\nabla^2 w)) (k_j \partial_y \zeta \partial_y \nabla^2 w),$$

which is dominated by

$$
\frac{C}{\alpha} \int_{\mathcal{R}} |\zeta \partial_y R (\nabla w) (\nabla^2 w)|^2 + C \alpha \beta^2 \int_{\mathcal{R}} k |\partial_y \zeta \partial_y \nabla^2 w|^2.
$$

Now the second term here can be absorbed, while in the first term we use

$$
\partial R (\nabla w) \eta (\nabla^2 w) = R (\nabla w) \partial \eta (\nabla^2 w) + \eta \nabla^2 w (\partial R (\nabla w)).
$$
Now $R(\nabla w)$ is bounded, and so we can use the one-dimensional Poincaré inequality to get
\[
\int_R \left| R(\nabla w) \partial_\eta (\nabla^2 w) \right|^2 \leq CR_1^2 \|\partial R(\nabla w)\|_{L,\infty}^2 \int_R \left| \partial_\zeta \partial_\eta (\nabla^2 w) \right|^2,
\]
which can be absorbed for $R_1$ small enough. Finally, $\partial R(\nabla w)$ consists of bounded terms times components of $\nabla^2 w$, plus bounded terms, and we simply use that $\nabla^2 w \in L^4$ with control. Terms $II$ and $III$ are also handled just as in Lemma 4.2.

The result of all this is that $\nabla^3 w \in H^1_{loc}$ with control, or $\nabla^3 w \in L^2_{loc}$ with control. Moreover, $\nabla^3 w$ solves a system of equations obtained by differentiating (4.12), and thus has the form
\[
(4.24) \quad -\mathcal{L}(\nabla^3 w) = (k_2 \partial_y)^{\alpha} \left\{ \left( \nabla^2 w \right)^2 + (\nabla w) \left( \nabla^3 w \right) \right\},
\]

where $\mathcal{K}_j$ denotes a derivative of $k$ of order $j$, and $\mathcal{K}_j(\nabla w)^m$ represents a sum of products of such derivatives times $m^{th}$ order products of first order derivatives of $w$. For example, $v_{yy} = w_{yy}$ satisfies
\[
-\partial_y \mathcal{L} v_y = \partial_y \left[ (k_2 \partial_y)^{\alpha} \left( 3 v_{yy} + \left\{ k_{2222} v^4 + k_{22} v^2 w_y \right\} \right) \right],
\]
or
\[
-\mathcal{L} v_{yy} = (k_2 \partial_y)^{\alpha} \left( 3 v_{yy}^2 + 4 v_{yy} w_y + k_{2222} v^3 w_y + k_{22} v^2 w_y + 6 v_{yy}^2 + 3 v^2 w_{yy} \right) + \left\{ k_{22222} v^5 + k_{2222} v^3 v_y + k_{2222} v^3 w_y + k_{222} v^2 v_y + k_{222} v^2 w_y \right\}.
\]

Here $\mathcal{K}_4 = k_{2222}$, etc.

Note that this system has the form
\[
\mathcal{L}(\nabla^3 w) = P(\nabla w)(\nabla^3 w) + Q(\nabla w, \nabla^2 w) + T^q \left[ R(\nabla w)(\nabla^3 w) + S(\nabla w, \nabla^2 w) \right],
\]
where $P, Q, R$ and $S$ are polynomials with partial derivatives of $k$ as coefficients.

Altogether, we have
\[
\nabla w, \nabla^2 w, \nabla^3 w \in L^2 \text{ with control},
\]
\[
\mathcal{L}(\nabla^3 w) = P(\nabla w)(\nabla^3 w) + Q(\nabla w, \nabla^2 w) + T^q \left[ R(\nabla w)(\nabla^3 w) + S(\nabla w, \nabla^2 w) \right].
\]

Note that the Sobolev embedding theorem shows that we actually have $\nabla^2 w \in L^q$, for all $q < \infty$ (prior to this we only had $\nabla^2 w \in L^q$ for $q$ depending on $R_1$) and $\nabla w \in L^\infty$ with control (the latter assertion is of course redundant at this point). We can now apply the methods of the previous subsection “An $L^p$ improvement”, since the unknowns $\nabla^3 w$ appear only to the first power and times bounded terms consisting of polynomials in $\nabla w$, so that we can use $\nabla^3 w \in L^2$ with control. Terms of $\nabla^2 w$ can appear to higher powers (actually, at most squared, which means we need only $q = 4$), but they can be handled since $\nabla^2 w \in L^q$ with control for $q < \infty$. The result here is that $\nabla^3 w \in L^q$ with control for $q$ large depending on $R_1$, and so also $\nabla^2 w \in L^\infty$ with control, by the Sobolev embedding theorem. We can now apply the methods of the subsection “Reverse Sobolev embedding” as we did just
above, and the result is that $\nabla^4 w \in L^2$ with control. Finally, computing $\mathcal{L}(\nabla^4 w)$, we obtain
\[
\nabla w, \nabla^2 w, \nabla^3 w, \nabla^4 w \in L^2 \text{ with control,}
\]
\[
\mathcal{L}(\nabla^4 w) = P(\nabla w)(\nabla^4 w) + Q(\nabla w, \nabla^2 w)(\nabla^3 w) + Q_0(\nabla w, \nabla^2 w)
+ T^4 [R(\nabla w)(\nabla^4 w) + S(\nabla w, \nabla^2 w)(\nabla^3 w) + S_0(\nabla w, \nabla^2 w)],
\]
where again by the Sobolev embedding theorem, $\nabla^3 w \in L^q$, $q < \infty$ and $\nabla^2 w \in L^\infty$ with control. Note that this time, components of both $\nabla^4 w$ and $\nabla^3 w$ appear only to the first power, multiplied by polynomials in the components of $\nabla^2 w$ and $\nabla w$, which are bounded. This is the sense in which the equations for higher order derivatives become progressively less nonlinear.

We can now iterate this process to obtain
\[
(4.25) \quad \nabla^j w \in L^2 \text{ with control, } 1 \leq j \leq \ell + 1,
\]
\[
\mathcal{L}(\nabla^{\ell+1} w) = P(\nabla w)(\nabla^{\ell+1} w) + Q(\nabla w, \nabla^2 w)(\nabla^\ell w)
+ Q_0(\nabla w, ..., \nabla^{\ell-1} w) + T^4 [R(\nabla w)(\nabla^{\ell+1} w)]
+ T^4 [S(\nabla w, \nabla^2 w)(\nabla^\ell w) + S_0(\nabla w, ..., \nabla^{\ell-1} w)].
\]
for all $\ell$ by induction on $\ell$, where $P, Q, Q_0, R, S, S_0$ are polynomials with partial derivatives of $k$ as coefficients, and as before, the Sobolev embedding theorem shows that the first line in 4.25 can be improved to
\[
\nabla^j w \in L^\infty \text{ with control, } 1 \leq j \leq \ell - 1,
\]
\[
\nabla^\ell w \in L^q \text{ with control, for } q < \infty,
\]
\[
\nabla^{\ell+1} w \in L^2 \text{ with control.}
\]
We emphasize that $\nabla^{\ell+1} w$ and $\nabla^\ell w$ appear linearly in (4.25) with coefficients involving derivatives of order at most two of $w$, and that $\nabla^{\ell-1} w$ and earlier derivatives are bounded. For example, although we will not need the following information on the form of $\mathcal{L}(\partial_y^\ell v)$, it turns out that $\mathcal{L}(\partial_y^\ell v)$ is a sum of terms of the type that arise from
\[
\mathcal{P} = \sum_{j=0}^\ell (\partial_y^{\ell+2-j} k)(\partial_y^j v^{\ell+3-j})
\]
upon expanding $\partial_y^j v^{\ell+3-j}$. More specifically, we mean that the relation between derivatives of $v$ and derivatives of $k$ in the expansion of $\mathcal{L}(\partial_y^\ell v)$ is the same as in the expansion of $\mathcal{P}$. As a consequence, $\partial_y^\ell v$ appears linearly in $\mathcal{L}(\partial_y^\ell v)$ and $\mathcal{P}$ for $i > \frac{\ell}{4}$.

Returning to the induction, if (4.25) holds for a given $\ell$, then as above, the previous subsection “An $L^p$ improvement” shows that $\nabla^{\ell+1} w \in L^q$ with control, for $q$ large, and so by the Sobolev embedding theorem that $\nabla^\ell w \in L^\infty$. The subsection “Reverse Sobolev embedding” then shows that $\nabla^{\ell+2} w \in L^2$. It is in these iterations that we require $R_1$ to be successively smaller as the constants involving earlier derivatives become progressively larger. Differentiating the equation for $\mathcal{L}(\nabla^{\ell+1} w)$ yields the same form for $\mathcal{L}(\nabla^{\ell+2} w)$. This establishes (4.25) for $\ell+1$ and completes the proof of the a priori estimates (4.1).
We remark that for $\ell \geq 4$, the technique of the section "Reverse Sobolev embedding" only requires (4.25) in order to conclude $\nabla^{\ell+2} w \in L^2$ with control, rather than having to first use the Moser iteration to obtain $\nabla^{\ell+1} w \in L^q$, for $q$ large. As a result, we can inductively prove (4.25) for $\ell \geq 5$ (assuming it holds for $\ell = 4$) without resorting to the Moser iteration techniques of the section "An $L^p$ improvement". To illustrate, we estimate the analogues of (4.21) and (4.22) with $\nabla^2 w$ replaced by $\nabla^\# w + 1$:

$$\int_R \left( \partial_x \left( \zeta \partial_\eta \nabla^{\ell+1} w \right) \right)^2 + k \left( \partial_y \left( \zeta \partial_\eta \nabla^{\ell+1} w \right) \right)^2$$

(4.26)

and

$$\int_R \left( \zeta \partial_\eta (\nabla^{\ell+1} w) \right) \left( \zeta \partial_\eta \nabla^{\ell+1} w \right).$$

(4.27)

After plugging into (4.27) part of the formula for $\nabla^\# w + 1$ in (4.25), namely

$$T^\ell \left[ R(\nabla^\ell w) \left( \nabla^{\ell+1} w \right) + S(\nabla^\ell w, \nabla^2 w) \left( \nabla^\ell w \right) \right],$$

and then moving $T^\ell$ to the other side of the integral, we obtain the following analogue of term $I$ in (4.23):

$$\int_R \left( \zeta \partial_\eta R(\nabla^\ell w) \left( \nabla^{\ell+1} w \right) \right) \left( k_j \partial_\eta \zeta \partial_\eta \nabla^{\ell+1} w \right)
+ \int_R \left( \zeta \partial_\eta S(\nabla^\ell w, \nabla^2 w) \left( \nabla^\ell w \right) \right) \left( k_j \partial_\eta \zeta \partial_\eta \nabla^{\ell+1} w \right).$$

The more problematic term is the second one which can be dominated by

$$\frac{C}{\alpha} \int_R \left| \zeta \partial_\eta S(\nabla^\ell w, \nabla^2 w) \left( \nabla^\ell w \right) \right|^2 + C_\alpha B^2 \int_R \left| k \partial_\eta \zeta \partial_\eta \nabla^{\ell+1} w \right|^2.$$  

The second term here can be absorbed into (4.26), while for the first we use

$$\partial S(\nabla^\ell w, \nabla^2 w) \eta \left( \nabla^\ell w \right) = S(\nabla^\ell w, \nabla^2 w) \eta \left( \nabla^\ell w \right) + \eta \left( \nabla^\ell w \right) \partial S(\nabla^\ell w, \nabla^2 w).$$

Since $S(\nabla^\ell w, \nabla^2 w)$ is bounded for $\ell \geq 3$, we can use that $\int_R \left| \zeta \partial_\eta \left( \nabla^\ell w \right) \right|^2$ is under control by induction to handle the $L^2$ norm of the first term here. As for the second, $\partial S(\nabla^\ell w, \nabla^2 w)$ includes components of $\nabla^3 w$, which will be bounded provided $\ell \geq 4$.

The remaining terms are also handled by such techniques.

5. PROOFS OF THE MAIN THEOREMS

In this final section, we apply Theorem 4.7, our a priori estimates in terms of the gradient, to obtain the remaining theorems mentioned at the beginning of the paper.

5.1. Close to one variable curvature. We begin by proving Theorem 2.4. Recall that the desired conclusion is

$$\| \zeta D^\alpha w \|_\infty \leq C_\alpha (L),$$

(5.1)

where $w$ is a solution of the quasilinear equation (4.2) satisfying the condition (4.3), $(x, w(x, y)) \in L$ for all $(x, y) \in \text{support } (x)$. 


We will say that an expression involving derivatives of \( w \) is under special control if it is dominated by the right side of (5.1). Note that this is a stronger condition than requiring that \( w \) is under control. Since \( |u| \leq C \sqrt{v} \) by our assumption in (2.22), it is enough by the previous theorem, Theorem 2.3, to prove that
\[
\| \zeta v \|_\infty \leq C_\alpha \left( L \right).
\]
This will be accomplished by using Plancherel’s theorem in the following way:
\[
\| \zeta v \|_{L^\infty} \leq \left\| \xi \right\|_{L^1} \leq \left\{ \int \int \frac{d\sigma d\tau}{1 + |\sigma|^2 + |\tau|^2 + |\sigma|^2 |\tau|^2} \right\} \frac{1}{2} \times \\
\left\{ \int \int \left( 1 + |\sigma|^2 + |\tau|^2 + |\sigma|^2 |\tau|^2 \right) \left| \widetilde{\zeta}^2 (\sigma, \tau) \right|^2 d\sigma d\tau \right\} \frac{1}{2} \\
\leq C \left\{ \int \int \left( |\zeta v|^2 + |(\zeta v)_x|^2 + |(\zeta v)_y|^2 \right) dx dy \right\} \frac{1}{2}.
\]
This calculation reduces matters to showing that
\begin{align*}
(5.2) \quad & \| \zeta v \|_{L^2}, \\
& \| \nabla (\zeta v) \|_{L^2}, \\
& \| \partial_x \partial_y (\zeta v) \|_{L^2},
\end{align*}
are all under special control. This in turn will be accomplished by establishing in succession that the following \( k \)-gradient integrals are under special control. Here, and often in subsequent inequalities, the cutoff functions may change from instance to instance:
\begin{align*}
(5.3) \quad & \int_R \left( |u|^2 + k |\zeta v|^2 \right) \quad \text{is under special control}, \\
& \int_R \left( |\zeta \partial_x v|^2 + k |\zeta \partial_y v|^2 \right) \quad \text{is under special control}, \\
& \int_R \left( |\zeta \partial_x \eta v|^2 + k |\zeta \partial_y \eta v|^2 \right) \quad \text{is under special control}.
\end{align*}
Indeed, Poincaré’s inequality in one variable shows that the first term \( \| \zeta v \|_{L^2} \) in (5.2) is controlled by \( \| \zeta \partial_x v \|_{L^2} + \| \zeta v \|_{L^2} \). Now the term \( \| \zeta \partial_x v \|_{L^2} \) is included in the second line of (5.3) while the other term \( \| \zeta v \|_{L^2} \) is controlled using (2.22) and the first line of (5.3) since \( k \geq c > 0 \) on the support of \( \zeta_x \).

The second term in (5.2) can be controlled, allowing for a change in cutoff functions as announced above in Cautionary Note 3.1.1, and taking into account terms already estimated, by \( \| \zeta_y v \|_{L^2} + \| \eta v \|_{L^2} \). The first of these is controlled by \( \| \zeta v \|_{L^2} \) by Poincaré’s inequality, and both of these are controlled as above. Poincaré’s inequality and earlier estimates again show that the second term, \( \| \eta v \|_{L^2} \), is controlled by \( \| \zeta \partial_x \eta \eta v \|_{L^2} + \| \eta v \|_{L^2} \). The term \( \| \zeta \partial_x \eta v \|_{L^2} \) is included in the third line of (5.3), while the term \( \| \eta v \|_{L^2} \) is controlled by the second line of (5.3) since \( k \geq c > 0 \) on the support of \( \eta_x \).

The third term in (5.2) is controlled by \( \| \zeta \partial_x \partial_y v \|_{L^2} + \| \partial_x \partial_y, \zeta v \|_{L^2} \). Now assuming here, as we may, that \( \eta = 1 \) on the support of \( \zeta \) (again see Cautionary Note 3.1.1), the first of these terms squared is included in the third line of (5.3). The second is controlled in terms of \( \| \nabla (\zeta v) \|_{L^2} \) (for a cutoff function \( \zeta \) with an enlarged
support), which is the second term in (5.2) and has already been controlled in the previous paragraph.

We have from (3.5) with \( k = k(x, w(x, y)) \) and \( L = \partial_x^2 + \partial_y k(x, w(x, y)) \partial_y \) that
\[
\int_{\mathcal{R}} \left( |\partial_x w|^2 + |\sqrt{k} \partial_y w|^2 \right) \leq -2 \int_{\mathcal{R}} (\zeta L w)(\zeta w) + 4A^2 \int_{\mathcal{R}} |\partial_1 w|^2 + 4A^2 \int_{\mathcal{R}} k |\partial_2 w|^2.
\]

Since \( Lw = 0 \), we have \( \|\zeta u\|_{L^2} + \|\sqrt{k} \partial v\|_{L^2} \leq C \|\zeta w\|_{\infty} \), which by (4.3) proves the first assertion in (5.3). Now replacing \( w \) by \( v \) we obtain
\[
\int_{\mathcal{R}} \left( |\partial_x v|^2 + k |\zeta \partial_y v|^2 \right) \leq -2 \int_{\mathcal{R}} (\zeta L v)(\zeta v) + 4A^2 \int_{\mathcal{R}} |\partial_1 v|^2 + 4A^2 \int_{\mathcal{R}} k |\partial_2 v|^2.
\]

Now \( \int_{\mathcal{R}} |\partial_1 v|^2 \leq C \int_{\mathcal{R}} k |\partial_1 v|^2 \) (since \( k \geq c \) on the support of \( \partial_1 \)) and \( \int_{\mathcal{R}} k |\partial_2 v|^2 \leq \|\sqrt{k} v\|_{L^2} \leq C \|\zeta v\|_{\infty} \) by the previous inequality. Next, by (4.4),
\[
\int_{\mathcal{R}} (\zeta L v)(\zeta v) = \int_{\mathcal{R}} (k_2 v^2) \partial_y (\zeta^2 v) \leq \int_{\mathcal{R}} (k_2 v^2) (\zeta^2 v) + \int_{\mathcal{R}} (k_2 v^2) (2\zeta \partial_y v).
\]

Now by our hypothesis (2.3), the second term is dominated by
\[
C \int_{\mathcal{R}} \zeta |\zeta_y| k^{\frac{1}{2}} |v| \leq C \int_{\mathcal{R}} \zeta |\zeta_y| k^{\frac{1}{2}} |v|^2
\]

since \( |kv| \leq C \) by (2.22). Continuing, we bound the above by
\[
C \int_{\mathcal{R}} |\zeta|^2 k |v|^2 + C \int_{\mathcal{R}} |\zeta v|^2 \leq \ C A^2 \int_{\mathcal{R}} k |\partial_2 v|^2 + C R_1^2 \int_{\mathcal{R}} |\partial_x v|^2
\]
\[
\leq \ C A^2 \int_{\mathcal{R}} k |\partial_2 v|^2 + C R_1^2 \int_{\mathcal{R}} |\zeta \partial_x v|^2 + C R_1^2 \int_{\mathcal{R}} |\zeta v|^2.
\]

The first of these terms is dominated by \( CA^2 \|\zeta v\|_{\infty}^2 \) by the first inequality in (5.3). The second term on the right can be absorbed into the left side of (5.4) for \( R_1 \) sufficiently small, and the third is controlled since \( \zeta_x \) is supported where \( k \geq c > 0 \). Indeed, we then have \( \int_{\mathcal{R}} |\zeta_x v|^2 \leq \frac{4\zeta^2}{\varepsilon} \int_{\mathcal{R}} k |\zeta v|^2 \), which is under special control by the first line in (5.3).

The first term in (5.5) satisfies
\[
\int_{\mathcal{R}} (k_2 v^2) (\zeta^2 v_y) \leq C \int_{\mathcal{R}} \left( k^{\frac{3}{2}} \zeta v_y \right) (\zeta k v^2) \leq C \varepsilon \int_{\mathcal{R}} k |\zeta v|^2 + \frac{C}{\varepsilon} \int_{\mathcal{R}} (\zeta^2 k^2 v^4) \leq C \varepsilon \int_{\mathcal{R}} k |\zeta v|^2 + \frac{C}{\varepsilon} \int_{\mathcal{R}} (\zeta v)^2,
\]
by \((2.22)\), where the term \(C\varepsilon \int_R \zeta |v_\beta|^2\) can be absorbed on the left side of \((5.4)\), and the remaining term is bounded by
\[
\frac{C}{\varepsilon} R_1^2 \int_R |\partial_x \zeta v|^2 \leq \frac{C}{\varepsilon} R_1^2 \int_R |\zeta \partial_x v|^2 + \frac{C}{\varepsilon} R_1^2 \int_R |\zeta_x v|^2.
\]
This can be handled as above, absorbing the first term on the right for \(R_1\) sufficiently small, and using \(k \geq c > 0\) on the second term. This proves the second line in \((5.3)\), and hence also \(\|\zeta v\|_{L_2} \leq C \|\zeta w\|_{L_\infty}\) by the one-dimensional Poincaré inequality.

In preparation for proving the third line in \((5.3)\), we will now use the Moser iteration technique to boost the integrability of \(v\) to \(\|\zeta v\|_{L^6} \leq C \|\zeta w\|_{L_\infty}\). (Recall that in proving Theorem 2.3, we only required local \(L^4\) integrability.) The inequality in Lemma 3.9 yields
\[
\left( |\zeta \partial_x v|^2 + k |\zeta \partial_y v|^2 \right) \leq \frac{2\beta^2}{2\beta - 1} \left| \int_R (\zeta \mathcal{L}v) (\zeta v^{2\beta-1}) \right| + \left( \frac{2\beta}{2\beta - 1} \right)^2 A^2 \int_R |\vartheta_1 v|^2 + \left( \frac{2\beta}{2\beta - 1} \right)^2 A^2 \int_R |\vartheta_2 v|^2,
\]
for \(\beta > \frac{1}{2}\). Now \(k \geq c > 0\) on the support of \(\vartheta_1\), and so \(\int_R |\vartheta_1 v|^2 \leq C \int_R k |\vartheta_1 v|^2\). As a result of this together with \((2.22)\), the second and third terms on the right above are dominated by
\[
(5.6) \quad C \left( \frac{\beta}{2\beta - 1} \right)^2 A^2 \int_R \left| \zeta v^{2\beta - \frac{1}{2}} \right|^2.
\]

**Remark 5.1.** Note that the inequality \(kv \leq C\) from \((2.22)\) has permitted us to avoid using the difficult Lemma 3.12 to handle the term \(\int_R k |\vartheta_2 v|^2\).

We will now show that the first term is bounded by a similar integral. We have
\[
(5.7) \quad \left| \int_R (\zeta \mathcal{L}v) (\zeta v^{2\beta-1}) \right| = \left| \int_R (\zeta \partial_y k_2 v^2) (\zeta v^{2\beta-1}) \right| \\
\leq \left| \int_R (\zeta k_2 v^2) (\zeta \partial_y v^{2\beta-1}) \right| + 2 \left| \int_R (\zeta y k_2 v^2) (\zeta v^{2\beta-1}) \right|.
\]
Now the first integral here satisfies
\[
\left| \int_R (\zeta k_2 v^2) (\zeta \partial_y v^{2\beta-1}) \right| = \frac{2\beta - 1}{\beta} \left| \int_R (\zeta k_2 v^{2\beta+1}) (\zeta \partial_y v^\beta) \right| \\
\leq C \frac{2\beta - 1}{\beta} \int_R |\partial_y v^{2\beta+1}| \sqrt{k} \zeta v^\beta \\
\leq C \frac{2\beta - 1}{\beta} \int_R |\zeta v^\beta| \sqrt{k} \zeta \partial_y v^\beta \\
\leq C \frac{\alpha}{\alpha} \int_R |\zeta v^\beta|^2 + C \int_R k |\zeta \partial_y v^\beta|^2,
\]
since \(\frac{2\beta - 1}{\beta} \leq 2\). The second term here can be absorbed for \(\alpha\) chosen small enough, while by Poincaré’s inequality in one variable, the first is dominated by
\[
\frac{C}{\alpha} R_1^2 \int_R |\zeta \partial_x v^\beta|^2 \leq \frac{C}{\alpha} R_1^2 \int_R |\zeta \partial_x v^\beta|^2 + \frac{C}{\alpha} R_1^2 \int_R |\zeta x v^\beta|^2.
\]
The first integral on the right here can now be absorbed for \( R_1 \) small enough, and the second is at most
\[
\frac{C}{\alpha} A^2 R_1^2 \int_{\mathcal{R}} \left| \xi v^{\beta - \frac{2}{3}} \right|^2
\]
since \( v \leq \frac{C}{\alpha} \leq \frac{C}{\epsilon} \) on the support of \( \zeta_x \). The second integral on the right side of (5.7) is at most
\[
C \int_{\mathcal{R}} k^{\frac{1}{4}} \left| \zeta_y \zeta v^{2\beta + 1} \right| \leq C \int_{\mathcal{R}} \left| \zeta_y \zeta v^{2\beta - \frac{2}{3}} \right| \leq C A \int_{\mathcal{R}} \left| \xi v^{\beta - \frac{4}{3}} \right|^2,
\]
and together with the previous estimate, (5.6) and the fact that \( v \) is bounded below by (2.22), this shows that
\[
\int_{\mathcal{R}} \left( |\zeta \partial_x v^\beta|^2 + k |\zeta \partial_y v^\beta|^2 \right) \leq C \left( \frac{\beta}{2\beta - 1} \right)^2 A^2 \int_{\mathcal{R}} \left| \xi v^{\beta - \frac{4}{3}} \right|^2.
\]
Using the one-dimensional Poincaré inequality again along with the inequality \( \sqrt{v} \leq \frac{C}{\epsilon} \leq \sqrt{\frac{\epsilon}{C}} \) on the support of \( \zeta_x \), we conclude that
\[
\int_{\mathcal{R}} |\zeta v^\beta|^2 \leq CR_1^2 \int_{\mathcal{R}} |\partial_x \zeta v^\beta|^2 \leq CR_1^2 \int_{\mathcal{R}} |\zeta \partial_x v^\beta|^2 + CR_1^2 \int_{\mathcal{R}} |\zeta_x v^\beta|^2 \leq C \left( \frac{\beta}{2\beta - 1} \right)^2 R_1^2 A^2 \int_{\mathcal{R}} \left| \xi v^{\beta - \frac{4}{3}} \right|^2.
\]
Applying this with successively \( \beta = \frac{5}{4}, \frac{6}{4}, \ldots, \frac{12}{4} \), we obtain that
\[
\int_{\mathcal{R}} |\zeta v^\beta|^2 \leq C \int_{\mathcal{R}} |\xi v|^2 \leq C \|xw\|_{\infty}^2.
\]

We now wish to show the third line in (5.3) is under special control by establishing Lemma 4.2 without assuming that \( \|v\|_{\infty} \) is under special control, rather using only that \( \|v\|_{L^4} \) is under special control along with our hypothesis (2.3), and of course (2.22). The first step in proving Lemma 4.2 is the application of Corollary 3.8 with \( k(x, y) \) replaced by \( k(x, w(x, y)) \), yielding an estimate for
\[
\int_{\mathcal{R}} \left( |\partial_x (\zeta \partial yu)| + k |\partial_y (\zeta \partial yu)|^2 \right)
\]
\[
+ \int_{\mathcal{R}} \left( |\partial_x (\zeta \partial yv)|^2 + k |\partial_y (\zeta \partial yv)|^2 \right).
\]
Of course we really only need to estimate the integrals involving \( v \). We remind the reader that \( k \) refers to \( k(x, w(x, y)) \) here. It is crucial to note that our hypothesis \( |k_2| \leq C k^{\frac{4}{3}} \) implies that \( k(x, w(x, y)) \) satisfies (2.19) - see (4.6). To illustrate the remaining argument, consider the revised estimate for the following term which arises in the proof just following (4.7):
\[
\int_{\mathcal{R}} (\zeta \partial y \mathcal{L} v) (\zeta \partial yv) = \int_{\mathcal{R}} (\zeta \partial yv^2) (k_2 \partial_y \zeta \partial yv)
\]
\[
- \int_{\mathcal{R}} (v^2) ([k_2 \partial_y, \eta \partial \zeta^2] \partial yv)
\]
\[
- \int_{\mathcal{R}} (\zeta \partial yv^2) (k_2 \zeta_y \partial yv)
\]
\[
= I + II + III.
\]
For term I we use $|k^2| \leq C k^{\frac{3}{2}}$ to get

$$|I| \leq \frac{C}{\varepsilon} \int_{\mathcal{R}} k^2 |\partial_{\eta^2}|^2 + \varepsilon \int_{\mathcal{R}} k |\partial_{\eta} \zeta \partial_{\eta^2}|^2,$$

and absorb the second term into (5.9) as usual. The first term now satisfies

$$\int_{\mathcal{R}} k^2 |\partial_{\eta^2}|^2 \leq C \int_{\mathcal{R}} k^2 |\eta \partial_{\eta^2}|^2 + C \int_{\mathcal{R}} k^2 |(\partial_{\eta}) v^2|^2$$

$$\leq C \|\eta v\|_1^2 + C \|\partial_{\eta} v\|_{L^2}^2$$

$$\leq C \|\eta v\|_1^2 + C A^2 \|\xi w\|_{\infty},$$

since $|kv| \leq C$ by (2.22) again, and since $\|\zeta\|_2$ is under special control. The first term here is now absorbed into (5.9) by Poincaré’s inequality as in Theorem 4.4. Term III is handled in similar fashion, using that $\sqrt{k^2} \partial_{\eta^2}$ already has $L^2$ norm under special control.

For term II we use

$$[k^2 \partial_{\eta^2}, \eta^2 \partial_{\zeta^2^2}] = k^2 \eta^2 \partial_{\zeta^2} + k^2 \eta^2 \partial_{\zeta^2^2 \zeta^2} - \eta^2 (\partial_{k^2}) \partial_{\eta^2},$$

to obtain

$$|II| \leq \left| \int_{\mathcal{R}} (v^2) (k^2 \eta^2 \partial_{\zeta^2^2} \partial_{\eta^2}) + \int_{\mathcal{R}} (v^2) (k^2 \eta^2 \partial_{\zeta^2^2} \zeta^2 \partial_{\eta^2}) \right|$$

$$+ \left| \int_{\mathcal{R}} (v^2) (\eta^2 (\partial_{k^2}) \partial_{\eta^2} \partial_{\eta^2}) \right|.$$

Now using $|k^2| \leq C k^{\frac{3}{2}}$, the first of the terms here satisfies

$$\left| \int_{\mathcal{R}} (v^2) (k^2 \eta^2 \partial_{\zeta^2^2} \partial_{\eta^2}) \right| = \left| \int_{\mathcal{R}} (v^2) (k^2 \eta^2 \partial_{\zeta^2^2} \partial_{\eta^2}) + \int_{\mathcal{R}} (v^2) (k^2 \eta^2 (\partial_{\zeta^2^2}) \zeta^2 \partial_{\eta^2}) \right|$$

$$\leq C \varepsilon \int_{\mathcal{R}} k |\partial_{\zeta^2^2} \partial_{\eta^2}|^2 + C \varepsilon \int_{\mathcal{R}} k^2 |\eta^2 v^2|^2$$

$$+ C \int_{\mathcal{R}} k^2 |\eta^2 (\partial_{\zeta^2^2}) v^2|^2 + C \int_{\mathcal{R}} |\partial_{\eta^2}|^2$$

$$\leq C \varepsilon \int_{\mathcal{R}} k |\partial_{\zeta^2^2} \partial_{\eta^2}|^2 + C \varepsilon A^4 \|\xi v\|_{L^2}^2 + C \|\eta v\|_1^2,$$
\[ \partial = \partial_x , \] commuting one factor of \( \zeta \) with \( \partial_x \) we obtain

\begin{align*}
(5.11) \quad \left| \int_{\mathcal{R}} (\nu^2) (\eta \zeta^2 (\partial_x k_2) \partial_y \partial_x \eta v) \right| \\
= \left| \int_{\mathcal{R}} (\nu^2) (\eta \zeta^2 (k_{21} + k_{22} u) \partial_y \partial_x \eta v) \right| \\
\leq \left| \int_{\mathcal{R}} (\eta \zeta^2) (k_{21} \partial_x \zeta \partial_y \eta v) \right| \\
+ \left| \int_{\mathcal{R}} (\eta \zeta^2) (k_{22} u \partial_x \zeta \partial_y \eta v) \right| \\
+ \left| \int_{\mathcal{R}} (\nu^2) (\eta \zeta \zeta \zeta (k_{21} + k_{22} u) \partial_y \eta v) \right|,
\end{align*}

where the final term here is under special control since \( \zeta \) is supported where \( k \) is bounded away from zero, and so by (2.22), where \( u \) and \( v \) are bounded. Indeed, the final term is at most

\[ C \int_{\mathcal{R}} |\zeta \nu (\partial_y \eta v)| \leq C \int_{\mathcal{R}} \frac{k}{c} |\zeta \nu (\partial_y \eta v)| \leq C \int_{\mathcal{R}} k |\partial_y \eta v|^2 + C \int_{\mathcal{R}} k |\zeta \nu|^2, \]

which is under special control by the first two lines of (5.3). The first two integrals on the right side of (5.11) are dominated by

\[ \frac{C}{\alpha} \int_{\mathcal{R}} |\eta \zeta \nu^2|^2 + \alpha \int_{\mathcal{R}} |\partial_x \zeta \partial_y \eta v|^2, \]

since \( |u| \leq C \sqrt{v} \) and \( 0 < c \leq v \). The second term here can be absorbed into (5.9) and the first is under special control.

Now we turn our attention to the case \( \partial = \partial_y \). For the moment we will consider \( k \) to mean \( k (x, y) \) and write \( k_2 \) for \( (\partial_y k) (x, y) \), etc. We will need the fact that \( |k_2| \leq C k \) implies \( |\nabla k_2| \leq C \sqrt{k} \). Indeed, \( k - ck_2 \geq 0 \) and so by (2.20),

\[ |\nabla (k - ck_2)| \leq C \sqrt{k - ck_2} \leq C \sqrt{k} \]

which implies by (2.20) again,

\[ c |\nabla k_2| \leq |\nabla k| + C \sqrt{k} \leq C \sqrt{k}. \]

This inequality holds for \( (x, y) \) in a compact subset of \( \Omega \), and so \( |\nabla k_2 (x, w (x, y))| \leq C k (x, w (x, y)) \) holds for \( (x, y) \) in a compact subset of the interior of \( T \Omega \). With this we now have

\[ \left| \int_{\mathcal{R}} (\nu^2) (\eta \zeta^2 \partial_y k_2 \partial_y \partial_y \eta v) \right| \leq \left| \int_{\mathcal{R}} (\nu^2) (\eta \zeta^2 (k_{22} \nu v) \partial_y \partial_y \eta v) \right| \leq \left| \int_{\mathcal{R}} (\eta \zeta \nu^3) \left( \sqrt{k} \partial_y \zeta \partial_y \eta v \right) \right|, \]

plus a term \( \int_{\mathcal{R}} |\eta \zeta \nu^3| \left| \sqrt{k} \partial_y \zeta \partial_y \eta v \right| \) that is under special control by the Cauchy-Schwartz inequality, (5.8) and the second line of (5.3). We continue with

\[ \left| \int_{\mathcal{R}} (\eta \zeta \nu^3) \left( \sqrt{k} \partial_y \zeta \partial_y \eta v \right) \right| \leq \frac{C}{\alpha} \int_{\mathcal{R}} |\eta \zeta \nu^3|^2 + \alpha \int_{\mathcal{R}} k |\partial_y \zeta \partial_y \eta v|^2 \]
where the first term is under special control and the second can be absorbed into (5.9). Similar arguments handle the term

\[ \int_{\mathbb{R}} (\zeta \partial \eta \mathcal{L} u) (\zeta \partial \eta u) \]

in (4.7), except that this time we have \( \zeta \partial \eta u = \zeta \partial_x \eta \partial u \), which is either \( \zeta \partial_x \eta u \) or \( \zeta \partial_x \eta v \), modulo terms whose \( L^2 \) norm is under special control. No factor of \( \sqrt{k} \) is needed to absorb this term and so we only require the boundedness of the second order partial derivatives of \( k \) that arise here. The remaining terms in (4.7) are easily handled using (2.3), (2.22) and the terms already proven to be under special control, thereby establishing that (5.9) is under special control, and completing the proof of the third assertion in (5.3).

As indicated at the beginning, this completes the proof of Theorem 2.4.

5.1.1. Monge-Ampère equation.

Remark 5.2. In our application to the Monge-Ampère equation, we will need (2.22), i.e. the inequalities\( w_y \geq c > 0, \ k w_y \leq C, \ |w_x| \leq C \sqrt{w_y} \) and \( \sqrt{k} |w_x| \leq C \) for smooth solutions \( w \) of (2.17) arising from the partial Legendre transform. In this case, reverting to the original variables \( (s, t) \), the inequalities follow immediately from the a priori estimates \( u_{xx} \leq C, \ u_{yy} \leq C \) in [12] since

\[
\max * k, u^2_{xy} + u^2_{y} \leq k + u^2_{xy} = u_{xx} u_{yy};
\]

\[
y_j = \frac{1}{u_{yy}} \geq \frac{1}{C},
\]

\[
k(s, y(s, t)) y_1(s, t) = k(x, y) \frac{1}{u_{yy}(x, y)} \leq u_{xx}(x, y) \leq C,
\]

\[
y_j(s, t)^2 = \frac{u_{xg}(x, y)^2}{u_{yy}(x, y)^2} \leq \frac{u_{xx}(x, y)}{u_{yy}(x, y)} \leq Cy_j(s, t),
\]

\[
k(s, y(s, t)) y_2(s, t)^2 = k(x, y) \frac{u_{xg}(x, y)^2}{u_{yy}(x, y)^2} \leq u_{xx}(x, y)^2 \leq C^2.
\]

Note that the fourth inequality also follows by combining the second and third inequalities.

Proof. (of Theorem 2.1) For \( \delta \geq 0 \), let \( u^\delta \) be the convex solution of the Monge-Ampère boundary problem

\[
\begin{cases}
  u^\delta_{xx} u^\delta_{yy} - (u^\delta_{xy})^2 = k(x, y) + \delta, & (x, y) \in \Omega \\
  u^\delta = \phi(x, y), & (x, y) \in \partial \Omega
\end{cases}
\]

For \( \delta > 0 \), \( u^\delta \) is smooth in \( \bar{\Omega} \) by [4] and for \( \delta = 0 \), \( u^0 = u \in C^{1,1}(\bar{\Omega}) \) by [12]. A simple and well known calculation shows that \( U = u^{\delta_1} - u^{\delta_2} \) solves the linear equation

\[
LU = \delta_1 - \delta_2, \quad (x, y) \in \Omega,
\]

\[
U = 0, \quad (x, y) \in \partial \Omega,
\]

where

\[
(5.12) \quad LU = \left( \frac{u^{\delta_1}_{yy} + u^{\delta_2}_{yy}}{2} \right) U_{xx} + \left( \frac{u^{\delta_1}_{xx} + u^{\delta_2}_{xx}}{2} \right) U_{yy} - (u^{\delta_1}_{xy} + u^{\delta_2}_{xy}) U_{xy}.
\]
When \( \delta_1 \geq \delta_2 \geq 0 \), \( L \) is elliptic (since it is the average of two nonnegative operators, one of which is positive) with \( C^{1,1} \) coefficients and \( U \in C^{1,1} \subset W^{2,2} = H^2 \) (the Sobolev space of functions with distributional derivatives in \( L^2 \)). The maximum principle for such equations (see e.g. Alexandrov’s maximum principle, Theorem 9.1 in [9]) shows that \( U \leq 0 \) or \( u^{\delta_1} \leq u^{\delta_2} \) in \( \overline{\Omega} \). It follows that if we write
\[
\partial \Omega = \{(x, \alpha(x)) : x \in P\Omega \} \cup \{(x, \beta(x)) : x \in P\Omega \}
\]
where \( \alpha(x) < \beta(x) \) and both \( \alpha \) and \( -\beta \) are strictly convex functions on the projection \( P\Omega \) of \( \Omega \) onto the \( x \)-axis, then
\[
(5.13) \quad u^{\delta_1}_y (x, \beta(x)) = \lim_{y \to \beta(x)^-} \frac{\phi(x, \beta(x)) - u^{\delta_1}(x, y)}{\beta(x) - y} 
\]
\[
\geq \lim_{y \to \beta(x)^-} \frac{\phi(x, \beta(x)) - u^{\delta_2}(x, y)}{\beta(x) - y} = u^{\delta_2}_y (x, \beta(x)) ,
\]
and similarly,
\[
(5.14) \quad u^{\delta_1}_y (x, \alpha(x)) \leq u^{\delta_2}_y (x, \alpha(x)) ,
\]
for \( \delta_1 > \delta_2 \).

Now apply the partial Legendre transform \( T^\delta \) as in (2.14) with \( u^\delta \) in place of \( u \),
\[
\begin{align*}
\begin{cases}
  s &= x \\
  t &= u^\delta_y (x, y)
\end{cases}
\]
and use (5.13) and (5.14) to obtain that the transformed regions
\[
T^\delta \Omega = \{(x, u^\delta_y (x, y)) : (x, y) \in \Omega \}
\]
satisfy \( T^\delta \Omega \subset T^{\delta_1} \Omega \) for \( \delta_1 > \delta_2 \). Note also that (2.8) implies that the interior of \( T^\delta \Omega \) is the domain
\[
T^\delta \Omega^* = \{(s, t) : u_y (s, \alpha(s)) < t < u_y (s, \beta(s)), s \in P\Omega \},
\]
where \( \Omega^* \) is as in (2.6). Now \( u^\delta_{yy} > 0 \) for \( \delta > 0 \), and we let \( w^\delta(s, t) \) denote the inverse function \( u^\delta_y (s, \cdot)^{-1} (t) \). We claim that Theorem 2.4 applies to the transformed functions \( w^\delta \) to show that all their derivatives are uniformly bounded on compact subsets of \( T^\delta \Omega^* \) (which is contained in \( T^\delta \Omega \)). Here we are taking \( \Omega’ = T^0 \Omega^* \). To apply Theorem 2.4 we must show that for every compact subset \( K \) of \( T^0 \Omega^* \), there is a compact subset \( L \) of \( \Omega^* \) and \( c > 0 \) such that
\[
(5.15) \quad (s, w^\delta(s, t)) \in L, \quad \text{for all} \quad (s, t) \in K, 0 < \delta < c .
\]

To see (5.15) we first prove that \( u^\delta \to u_y \) uniformly on \( \overline{\Omega} \) as \( \delta \to 0 \). By the \( C^2 \) a priori estimates in [12], the functions \( u^\delta \) and their gradients are uniformly Lipschitz, and hence equicontinuous, on the compact set \( \overline{\Omega} \). Thus given a sequence \( \{u^{\delta_n}\} \) with \( \delta_n \to 0 \), there is a subsequence converging in \( C^{0,1} (\overline{\Omega}) \) to a \( C^{1,1} \) function, say \( v \). By a result of Alexandrov (see [5] or Theorem 7.1 in the appendix), the absolutely continuous measures with density \( k + \delta_n = \det \nabla^2 u^{\delta_n} \) converge weakly to \( \det \nabla^2 v \), the representing measure (see the appendix for the definition) for the convex function \( v \). Since \( k + \delta_n \) also converges weakly to \( k \), we conclude that \( \det \nabla^2 v = k \), and thus that \( v \) is a generalized solution of (2.7). By the uniqueness of generalized solutions (see Proposition 3 in [6]), \( v = u \). Thus we’ve proved that every sequence \( \{u^{\delta_n}\} \) with \( \delta_n \to 0 \) has a subsequence converging to \( u \) in \( C^{0,1} (\overline{\Omega}) \),

\( \text{MONGE-AMPE`RE} \quad 59 \)
and it follows that \( u^\delta \to u \) in \( C^{0,1}(\Omega) \) as \( \delta \to 0 \). In particular, \( u^\delta_y \to u_y \) uniformly on \( \Omega \). Now since \( K \) is compact, there is \( \varepsilon > 0 \) such that
\[
|u_y(s, \alpha(s)) + \varepsilon - u_y(s, \beta(s))| < \varepsilon
\]
for \((s, t) \in K\), and by the uniform convergence of \( u^\varepsilon \) to \( u \), there is \( c > 0 \) such that
\[
u^\delta_y(s, \alpha(s)) + \frac{\varepsilon}{2} < t < u^\delta_y(s, \beta(s)) - \frac{\varepsilon}{2}
\]
for \((s, t) \in K\) and \( 0 < \delta < c \). Now \( u^\delta_{yy} \leq C \) by the \( C^2 \) \textit{a priori} estimates in [12], and it follows that
\[
u^\delta_y(s, \alpha(s) + \frac{\varepsilon}{2}) \leq u^\delta_y(s, \alpha(s)) + \frac{\varepsilon}{2},
\]
which implies upon taking inverses in the second variable that
\[
a(s) + \frac{\varepsilon}{2} \leq w^\delta(s, u^\delta_y(s, \alpha(s)) + \frac{\varepsilon}{2}).
\]
Similarly, we obtain
\[
w^\delta(s, u^\delta_y(s, \beta(s)) - \frac{\varepsilon}{2}) \leq \beta(s) - \frac{\varepsilon}{2C}.
\]
Combining (5.16), (5.17) and (5.18) we obtain that
\[
a(s) + \frac{\varepsilon}{2C} \leq w^\delta(s, u^\delta_y(s, \alpha(s)) + \frac{\varepsilon}{2}) < w^\delta(s, t) < w^\delta(s, u^\delta_y(s, \beta(s)) - \frac{\varepsilon}{2}) \leq \beta(s) - \frac{\varepsilon}{2C}
\]
for \((s, t) \in K\) and \( 0 < \delta < c \), which proves (5.15) with
\[
L = \{(x, y) \in \Omega : a(x) + \frac{\varepsilon}{2C} \leq w^\delta(x, y) \leq \beta(x) - \frac{\varepsilon}{2C}\}.
\]
The only other hypothesis of Theorem 2.4 that we need to verify is (2.22), which holds by Remark 5.2 and the fact that the \( C^2 \) \textit{a priori} estimates in [12] are independent of \( 0 < \delta \leq 1 \). Thus for each \( \alpha \), the derivatives \( D^\alpha w^\delta \) are bounded uniformly in \( 0 < \delta \leq 1 \) on compact subsets of \( T^\delta \Omega^* \).

Next, we apply the inverse function theorem to the inverse partial Legendre transform \((T^\delta)^{-1}(s, t) = (s, w^\delta(s, t))\), whose Jacobian is \( w^\delta_{yy} = \frac{1}{1 + \cdots} \). We claim that this implies the functions \( u^\delta_y \), together with all their derivatives, are uniformly bounded, and hence equicontinuous, on compact subsets of \( \Omega^* \). Indeed, if we denote by \( S^\delta \) the inverse transform \((T^\delta)^{-1}\), then differentiating the equation \( S^\delta \circ T^\delta(x, y) = (x, y) \) yields \( (D^\delta \circ T^\delta)(DT^\delta) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \), and so
\[
DT^\delta = (DS^\delta \circ T^\delta)^{-1} = \frac{(\text{cofactor} DS^\delta) \circ T^\delta}{(\det DS^\delta) \circ T^\delta}.
\]
Using the quotient and chain rules, induction now shows that every partial derivative of a component of \( DT^\delta \) is a sum of products of derivatives of \( S^\delta \) and nonnegative powers of \( \frac{1}{\det DS^\delta} = \frac{1}{\partial^2} = u^\delta_{yy} \). Since the functions \( u^\delta_{yy} \) are uniformly bounded in \( \delta \) by the \textit{a priori} estimates in [12], and since \( T^\delta(x, y) = (x, u^\delta_y(x, y)) \), our claim is justified. There is however a point left open in this argument. If \( L \) is a compact subset of \( \Omega^* \), we must show that \( T^\delta L \) lies in a fixed compact subset \( K \) of \( T^\delta \Omega^* \) for \( \delta \) small. However, by compactness there is \( \varepsilon > 0 \) such that
\[
T^\delta L \subset \{(s, t) : u_y(s, \alpha(s)) + \varepsilon - t < u_y(s, \beta(s)) - \varepsilon, s \in P\Omega\}.\]
We showed above that \( u_\delta^{\delta y} \rightarrow u_y \) uniformly on \( \overline{\Omega} \), and hence it follows that
\[
T^\delta L \subset \left\{(s,t) : u_y(s,\alpha(s)) + \frac{\varepsilon}{2} < t < u_y(s,\beta(s)) - \frac{\varepsilon}{2}, s \in P\Omega\right\},
\]
a precompact subset of \( T^0\Omega^* \), for sufficiently small \( \delta > 0 \).

At this point we have that \( u_\delta^{\delta y} \) and all its derivatives are uniformly bounded, and hence equicontinuous, on compact subsets of \( \Omega^* \). Using the equation \( u_\delta^{\delta xx} u_\delta^{\delta yy} - (u_\delta^{\delta xy})^2 = k + \delta \) together with
\[
u_\delta^{\delta yy} = \frac{1}{u_\delta^y} \geq c > 0
\]
on compact subsets of \( \Omega^* \), it now follows that the functions \( u_\delta^{\delta xx} \), together with all their derivatives, are uniformly bounded, and hence equicontinuous, on compact subsets of \( \Omega^* \). Recall that \( u_\delta \rightarrow u \) in \( C^0,1 \) \( \Omega^* \). We can now extract a sequence \( \{u_\delta^{\delta_n}\}_{n=1}^\infty \) with \( \delta_n \rightarrow 0 \) such that all derivatives of \( u_\delta^{\delta_n} \), of order at least two, converge uniformly on compact subsets of \( \Omega^* \). It follows that \( u \) is smooth in \( \Omega^* \).

Finally, note that (5.20) shows that \( u_\delta^{\delta yy} > 0 \) in \( \Omega^* \).

5.2. The generalized equation.

Proof. (of Theorem 2.5) The proof is very similar to that of Theorem 2.3. The main points are

- Inequality (2.19) persists in the form
  \[
  |k_j(x,y,v,p,q)| \leq C\sqrt{k(x,y,v,p,q)}, \quad 1 \leq j \leq 5,
  \]
  for \( (x,y,v,p,q) \) in a compact subset \( K \) of \( \Omega \times \mathbb{R}^3 \), and can be applied to the quasilinear equation (2.25) since
  \[
  (x,w(x,y),r(x,y),z(x,y),y)
  \]
  lies in a compact subset of \( \Omega \times \mathbb{R}^3 \) for \( (x,y) \in \mathcal{R} \), a compact subset of \( \Omega' \), by the \( C^1 \) a priori estimates in say [4] (the proofs in this reference use only \( k \geq 0 \) for these estimates).

- The gradients of the auxiliary functions \( r \) and \( z \) are expressible in terms of \( z \) and the gradient of \( w \) times smooth functions, namely from (2.26),

  \[
  \begin{align*}
  r_x &= z + yw_x, \\
  r_y &= yw_y, \\
  z_x &= kw_y, \\
  z_y &= -w_x,
  \end{align*}
  \]

  where \( k \) is evaluated at \( (x,w(x,y),r(x,y),z(x,y),y) \). Thus \( \nabla z \) satisfies the same estimates as does \( \nabla w \) at any point in the argument, and then likewise for \( \nabla r \) (recall that the sup norm bounds of both \( z \) and \( r \) appear on the right side of the conclusion of Theorem 2.5).

To illustrate, we consider the extension of Lemma 4.1 to the present setting. If we set
\[
\mathcal{L} = \partial^2_x + \partial_y k(x,y) \partial_y
\]
where \( \tilde{k}(x, y) = k(x, w(x, y), r(x, y), z(x, y), y) \), and differentiate the equation \( \mathcal{L}(w) = 0 \) with respect to \( y \), we obtain

\[
\mathcal{L}(\partial_y w) = -\partial_y \left( \partial_y \tilde{k} \right) \partial_y w.
\]

By using (5.21), we have

\[
\partial_y \tilde{k} = k_2 w_y + k_3 y w_y - k_4 w_x + k_5
\]

where the partial derivatives \( k_j \) are evaluated at the point \( (x, w(x, y), r(x, y), z(x, y), y) \).

The key step in extending Lemma 4.1 is to estimate (with \( k \) partial derivatives of (2.24). Thus for example, (5.22)

\[
\text{By using (5.21), we have}
\]

\[
\partial_y \tilde{k} = k_2 w_y + k_3 y w_y - k_4 w_x + k_5
\]

where the partial derivatives \( k_j \) are evaluated at the point \( (x, w(x, y), r(x, y), z(x, y), y) \).

The first term on the right is the only term appearing in the proof of Lemma 4.1, and it is evident that the same techniques apply to the remaining three terms. This completes our discussion of the proof of Theorem 2.5.

We now extend the argument in the previous section to prove Theorems 2.6 and 2.2 concerning the generalized Monge-Ampère equation. Additional considerations arise due to the interplay of partial derivatives of \( k \) and the derivatives of \( r \) and \( z \) in (5.21).

**Proof.** (of Theorem 2.6) We suppose that \( w \) is a smooth solution of (2.25), which in terms of \( x, y \) variables reads

\[
\mathcal{L}w = \partial_x^2 w + \partial_y k(x, w(x, y), r(x, y), z(x, y), y) \partial_y w = 0,
\]

and also suppose that \( w, r, z \) satisfy the compatibility conditions (5.21), i.e.,

\[
\begin{align*}
    r_x &= z + y w_x \\
    r_y &= y w_y \\
    z_x &= k w_y \\
    z_y &= -w_x
\end{align*}
\]

Recall that if \( \tilde{k} \) denotes the function \( k(x, w(x, y), r(x, y), z(x, y), y) \), then (2.27) holds;

\[
\begin{align*}
    \partial_x \tilde{k} &= k_1 + k_2 w_x + k_3 (z + y w_x) + k_4 k w_y, \\
    \partial_y \tilde{k} &= k_2 w_y + k_3 y w_y - k_4 w_x + k_5.
\end{align*}
\]

Just as in the section on quasilinear equations, we continue to write \( k \) in place of \( \tilde{k} \) and continue to use variables \( x \) and \( y \), writing \( k_j \) with \( j = 1, 2, 3, 4, 5 \) to indicate partial derivatives of \( k \) with respect to the original 5 variables \( x, y, v, p, q \) as in (2.24). Thus for example, \( k_4 \) means (recall that \( k = k(x, y, v, p, q) \))

\[
(\partial_y k) (x, w(x, y), r(x, y), z(x, y), y).
\]
We will say that an expression is under special control if it is dominated by 

$C (L)$, when $(x, w, r, z, y) \in L$ compact $\subset \Omega \times \mathbb{R}^3$.

We now establish analogues of the three successive assertions in (5.3). We have from (3.5), namely

$$\int_{\mathcal{R}} \left( |\zeta \partial_x w|^2 + |\zeta \sqrt{k} \partial_y w|^2 \right) \leq -2 \int_{\mathcal{R}} (\zeta \mathcal{L} w) (\zeta w) + 4 A^2 \int_{\mathcal{R}} |\varrho_1 w|^2 + 4 A^2 \int_{\mathcal{R}} k |\varrho_2 w|^2,$$

and $\mathcal{L} w = 0$, that $\| |\zeta \partial_x w||^2 + \| |\zeta \sqrt{k} \partial_y w||^2 \leq C \| \zeta w \|_\infty$, the analogue of the first line of (5.3). We now wish to estimate, writing $u = \partial_x w$ and $v = \partial_y w$ as usual,

$$\int_{\mathcal{R}} \left( |\zeta \partial_x u|^2 + k |\zeta \partial_y u|^2 \right) + \int_{\mathcal{R}} \left( |\zeta \partial_x v|^2 + k |\zeta \partial_y v|^2 \right).$$

Note that it is necessary to include the $k$-energy of $u$ as well as $v$ this time because the formulas in (5.22) each involve both $u$ and $v$ on the right hand side. Replacing $w$ by $v$ in (3.5) we obtain

$$\int_{\mathcal{R}} \left( |\zeta \partial_x v|^2 + k |\zeta \partial_y v|^2 \right) \leq -2 \int_{\mathcal{R}} (\zeta \mathcal{L} v) (\zeta v) + 4 A^2 \int_{\mathcal{R}} |\varrho_1 v|^2 + 4 A^2 \int_{\mathcal{R}} k |\varrho_2 v|^2 \leq -2 \int_{\mathcal{R}} (\zeta \mathcal{L} v) (\zeta v) + \text{TUSC},$$

where TUSC stands for terms under special control. Indeed, the indicated integrals are TUSC since $\int_{\mathcal{R}} |\varrho_1 v|^2 \leq C \int_{\mathcal{R}} k |\varrho_1 v|^2$ (since $k \geq c > 0$ on the support of $\varrho_1$) and $\int_{\mathcal{R}} k |\varrho_j v|^2 \leq \| \xi \sqrt{k} v \|^2_{L^2} \leq C \| \zeta w \|^2_{L^2}$ by the earlier inequality. Next we compute

$$\int_{\mathcal{R}} (\zeta \mathcal{L} v) (\zeta v) = - \int_{\mathcal{R}} (\zeta \partial_y (\partial_y k) v) (\zeta v)$$

$$= - \int_{\mathcal{R}} (\zeta \partial_y (k_2 v + k_3 y v - k_4 u + k_5) v) (\zeta v)$$

$$= \int_{\mathcal{R}} v^2 k_2 (\partial_y \zeta^2 v) + \int_{\mathcal{R}} y v^2 k_3 (\partial_y \zeta^2 v)$$

$$- \int_{\mathcal{R}} u v k_4 (\partial_y \zeta^2 v) + \int_{\mathcal{R}} v k_5 (\partial_y \zeta^2 v).$$

Now we use the hypotheses $\sqrt{k} |u| \leq C$ and $kv \leq C$ as well as $\partial_y \zeta^2 v = 2 \zeta \zeta_y v + \zeta^2 \partial_y v$ and note that $2 \sqrt{k} \zeta_y v$ has $L^2$ norm under special control, and that the $L^2$ norm of $\alpha \zeta \sqrt{k} \partial_y v$ can be absorbed into (5.23) for $\alpha$ sufficiently small. Moreover, we claim that $\frac{1}{\alpha} \int |\zeta v|^2$ is a sum of terms that can either be absorbed into (5.23) or are under special control. Indeed, by Poincaré’s inequality in one variable,

$$\frac{1}{\alpha} \int |\zeta v|^2 \leq C \frac{1}{\alpha} R_1^2 \int |\partial_x \zeta v|^2 \leq C \frac{1}{\alpha} R_1^2 \int |\zeta_x v|^2 + C \frac{1}{\alpha} R_1^2 \int |\partial_x v|^2.$$

The first term is under special control since $k \geq c > 0$ on the support of $\zeta_x$, while the second can be absorbed for $R_1$ sufficiently small.
From these observations, we now see that in (5.24), we need $I$ fact that $U$ special control by the Poincaré inequality in one variable, together with (5.25) if multiplied by a factor $\alpha$.

This completes the proof that (5.23) is handled. Now we turn to iteration technique to show that (5.26). The corresponding terms where $\partial_y$ hits $\zeta$ are handled similarly.

Now we use $\zeta$ and note that $2\zeta u$ has $L^2$ norm under special control, and that $\alpha \zeta \partial_y u$ can be absorbed into (5.23) in $L^2$ norm for sufficiently small $\alpha$ if multiplied by a factor $\sqrt{K}$. Using $\sqrt{K} |u| \leq C$ and $kv \leq C$ together with the fact that $\int |v|^2$ can be handled, we see that all of the above integrals are TUSCA. This completes the proof that (5.23) is under special control, the analogue of the second line of (5.3).

Altogether we now have that

$$\|\xi \partial_x v\|_{L^2} + \|\xi \sqrt{K} \partial_y v\|_{L^2} + \|\xi \partial_x u\|_{L^2} = \text{TUSC}.\tag{5.25}$$

By the Poincaré inequality in one variable, together with $k \geq c > 0$ on the support of $\zeta_x$, and $|u| \leq C \sqrt{K}$, we also have that

$$\|\xi v\|_{L^2} + \|\xi u\|_{L^4} = \text{TUSC}.\tag{5.26}$$

The next step, following the proof in the previous section, is to use the Moser iteration technique to show that $\|\xi v\|_{L^6}$ is under special control. The inequality in Lemma 3.9 yields

$$\int_R \left( |\zeta \partial_x v^\beta|^2 + k |\zeta \partial_y v^\beta|^2 \right) \leq \frac{2\beta^2}{2\beta - 1} \int_R (\zeta L v) (\zeta v^2)^{\beta - 1} \right| + \left( \frac{2\beta}{2\beta - 1} \right)^2 A^2 \int_R |\zeta v^\beta|^2 + \left( \frac{2\beta}{2\beta - 1} \right)^2 A^2 \int_R k |\zeta v^\beta|^2,$$
for $\beta > \frac{1}{2}$. Now $k \geq c > 0$ on the support of $\varphi_1$, and so $\int_{\mathcal{R}} |\varphi_1 v^\beta|^2 \leq C \int_{\mathcal{R}} k |\varphi_1 v^\beta|^2$. As a result of this together with (2.22), the second and third terms on the right above are dominated by

$$C \left( \frac{\beta}{2\beta - 1} \right)^2 A^2 \int_{\mathcal{R}} \left| \xi v^\beta - \frac{1}{2} \right|^2.$$  

We will now show that the first term is bounded by a similar integral plus terms which can be absorbed. We have

$$(5.28) \quad \int_{\mathcal{R}} (\zeta L v) (\zeta v^{2\beta - 1}) = - \int_{\mathcal{R}} (\zeta \partial_y (\partial_y k) v) (\zeta v^{2\beta - 1}) = - \int_{\mathcal{R}} (\zeta \partial_y (k_2 v + k_3 yv - k_4 u + k_5) v) (\zeta v^{2\beta - 1})$$

$$= \int_{\mathcal{R}} v^2 k_2 (\partial_y \zeta v^{2\beta - 1}) + \int_{\mathcal{R}} y^2 k_3 (\partial_y \zeta v^{2\beta - 1}) - \int_{\mathcal{R}} u v k_4 (\partial_y \zeta v^{2\beta - 1}) + \int_{\mathcal{R}} v k_5 (\partial_y \zeta v^{2\beta - 1}).$$

The first integral on the right satisfies

$$\left| \int_{\mathcal{R}} v^2 k_2 (\partial_y \zeta v^{2\beta - 1}) \right| = \left| \int_{\mathcal{R}} (\zeta v^2) (\zeta \partial_y \zeta v^{2\beta - 1}) \right| + 2 \int_{\mathcal{R}} (\zeta v^2 k_2) (\zeta v^{2\beta - 1}) = \int_{\mathcal{R}} \zeta v^2 (\zeta \partial_y \zeta v^{2\beta - 1}) + 2 \int_{\mathcal{R}} (\zeta v^2 k_2) (\zeta v^{2\beta - 1})$$

and just after (5.7) in subsection 5.1, we showed that the first integral here satisfies

$$\left| \int_{\mathcal{R}} (\zeta v^2) (\zeta \partial_y \zeta v^{2\beta - 1}) \right| \leq \frac{C}{\alpha} \int_{\mathcal{R}} |\zeta v^2|^2 + C \int_{\mathcal{R}} k |\zeta \partial_y v^\beta|^2$$

$$\leq \frac{C}{\alpha} R_1^2 \int_{\mathcal{R}} |\zeta \partial_y v^\beta|^2 + C A^2 R_1^2 \int_{\mathcal{R}} \left| \xi v^{\beta - \frac{1}{4}} \right|^2 + C \int_{\mathcal{R}} k |\zeta \partial_y v^\beta|^2,$$

since $0 < c \leq k \leq C v^{-1}$ on the support of $\zeta$, while the second integral is at most

$$C \int_{\mathcal{R}} k^2 |\zeta \partial_y \zeta v^{2\beta + 1}| \leq C A \int_{\mathcal{R}} \left| \xi v^{\beta - \frac{1}{4}} \right|^2.$$  

Using the inequalities $|k_3| \leq C \frac{k_2}{4}$, $|k_4| \leq C k$ and $|k_5| \leq C k^{\frac{1}{2}}$, we can show similar estimates for the remaining terms in (5.28), and then absorbing the relevant terms into the left side of (5.27) yields

$$\int_{\mathcal{R}} \left( |\zeta \partial_x v^\beta|^2 + k |\zeta \partial_y v^\beta|^2 \right) \leq C (\beta) A^2 \int_{\mathcal{R}} \left| \xi v^{\beta - \frac{1}{4}} \right|^2.$$  

Using the one-dimensional Poincaré inequality again, we conclude that

$$\int_{\mathcal{R}} |\zeta v^\beta|^2 \leq CR_1^2 \int_{\mathcal{R}} |\partial_y \zeta v^\beta|^2 \leq CR_1^2 \int_{\mathcal{R}} |\zeta \partial_x v^\beta|^2 + CR_1^2 \int_{\mathcal{R}} |\zeta \partial_y v^\beta|^2 \leq C (\beta) R_1^2 A^2 \int_{\mathcal{R}} \left| \xi v^{\beta - \frac{1}{4}} \right|^2.$$  

Applying this with successively $\beta = \frac{2}{3}, \frac{4}{3}, \ldots, \frac{12}{3}$, we obtain that $\int_{\mathcal{R}} |\zeta v^\beta|^2 \leq C \|\varphi\|_\infty.$
We now wish to show the analogue of the third line in (5.3). As in (5.9) we estimate
\begin{equation}
\int_{\mathcal{R}} \left( |\partial_x (\zeta \nabla \eta u)|^2 + k |\partial_y (\zeta \nabla \eta u)|^2 \right) \, dx \\
+ \int_{\mathcal{R}} \left( |\partial_x (\zeta \nabla \eta v)|^2 + k |\partial_y (\zeta \nabla \eta v)|^2 \right).
\end{equation}

In order to apply Corollary 3.8, we need $|\partial_x k| + |\partial_y k| \leq Ck^{\frac{2}{3}}$. From (5.22) and $|u|^2 \leq Cv \leq Ck^{-1}$, we see that this in fact holds provided $|k_i| \leq Ck^{d(i)}$ with $d(i) = \frac{3}{2}$ for $i = 2$ and 3, 1 for $i = 4$, $\frac{1}{2}$ for $i = 5$ and 1. This allows us to complete the estimation of all terms which result from Corollary 3.8, except the main terms involving $\mathcal{L}u$ and $\mathcal{L}v$. To estimate these main terms, we begin by using

$$
\mathcal{L}u = -\partial_y \{ k_1 v + k_2 uv + k_3 (zv + yuv) + k_4 kv^2 \},
$$

$$
\mathcal{L}v = -\partial_y \{ k_2 v^2 + k_3 yv^2 - k_4 uv + k_5 v \},
$$

to obtain

$$
\int_{\mathcal{R}} (\zeta \partial \eta \mathcal{L}u) (\zeta \partial \eta u) = -\int_{\mathcal{R}} (\zeta \partial \eta \partial_y \{ k_1 v + k_2 uv + k_3 (zv + yuv) + k_4 kv^2 \}) (\zeta \partial \eta u) = I^u + II^u + III^u,
$$

and

$$
\int_{\mathcal{R}} (\zeta \partial \eta \mathcal{L}v) (\zeta \partial \eta v) = -\int_{\mathcal{R}} (\zeta \partial \eta \partial_y \{ k_2 v^2 + k_3 yv^2 - k_4 uv + k_5 v \}) (\zeta \partial \eta v) = I^v + II^v + III^v,
$$

where the decompositions into $I^u + II^u + III^u$ and $I^v + II^v + III^v$ are as in (5.10), now forming a commutator for each $\partial_y k_i$. We have

$$
I^u = \int_{\mathcal{R}} (\zeta \partial \eta v) k_1 (\partial_y \zeta \partial \eta u) + \int_{\mathcal{R}} (\zeta \partial \eta uv) k_2 (\partial_y \zeta \partial \eta u) \\
+ \int_{\mathcal{R}} (\zeta \partial \eta \{ zv + yuv \}) k_3 (\partial_y \zeta \partial \eta u) + \int_{\mathcal{R}} (\zeta \partial \eta v^2) k_4 k (\partial_y \zeta \partial \eta u),
$$

and

$$
I^v = \int_{\mathcal{R}} (\zeta \partial \eta v^2) k_2 (\partial_y \zeta \partial \eta v) + \int_{\mathcal{R}} (\zeta \partial \eta yv^2) k_3 (\partial_y \zeta \partial \eta v) \\
- \int_{\mathcal{R}} (\zeta \partial \eta uv) k_4 (\partial_y \zeta \partial \eta v) + \int_{\mathcal{R}} (\zeta \partial \eta v) k_5 (\partial_y \zeta \partial \eta v).
$$

To handle these terms, we note that $\alpha \sqrt{k} \partial_y \zeta \partial \eta u$ and $\alpha \sqrt{k} \partial_y \zeta \partial \eta v$ have $L^2$ norms that can be absorbed into (5.29) for sufficiently small $\alpha$. We can also absorb the $L^2$ norms of terms of the form $\frac{1}{4} \zeta \partial \eta u$ and $\frac{1}{4} \zeta \partial \eta v$ by using the Poincaré inequality in one variable. Finally, we see that all of the above terms can be handled with our hypotheses on $k_j$ by manipulations of the form

$$
\zeta \partial \eta v^2 = 2v (\zeta \partial \eta v) - \zeta (\partial \eta) v^2,
$$

and

$$
\partial_y \zeta \partial \eta u = [\partial_y, \zeta \partial \eta] u - [\partial_x, \zeta \partial \eta] v + \partial_x \zeta \partial \eta v,
$$
where we have used $\partial_y u = \partial_x v$. For example, the first equality renders the first integral on the right side of $I^v$ tractable as follows:

$$\left| \int_R (\zeta \partial \eta v^2) k_2 (\partial_y \zeta \partial \eta) \right| \leq \left| \int_R (2v (\zeta \partial \eta v)) k_2 (\partial_y \zeta \partial \eta) \right|$$

$$+ \left| \int_R (\zeta (\partial \eta) v^2) k_2 (\partial_y \zeta \partial \eta) \right|$$

$$\leq \int_R (2v |\zeta \partial \eta v|) k^{\frac{2}{3}} |\partial_y \zeta \partial \eta v|$$

$$+ \int_R |\zeta (\partial \eta) v^2| k^{\frac{2}{3}} |\partial_y \zeta \partial \eta v|$$

$$\leq \int_R (2|\zeta \partial \eta v|) k^{\frac{1}{3}} |\partial_y \zeta \partial \eta v|$$

$$+ \int_R |\zeta (\partial \eta) v| k^{\frac{1}{3}} |\partial_y \zeta \partial \eta v| .$$

Here we can absorb $\|k^{\frac{2}{3}} \partial_y \zeta \partial \eta v\|_{L^2}$ and then use Poincaré in one variable to absorb $\|\zeta \partial \eta v\|_{L^2}$, and finally note that $\|\zeta (\partial \eta) v\|_{L^2}$ is under special control. The second identity renders the third integral on the right side of $I^v$ tractable as follows:

$$\int_R (\zeta \partial \eta v) k_4 (\partial_y \zeta \partial \eta v) = \int_R (\zeta \partial \eta v) k_4 ([\partial_y, \zeta \partial \eta] u)$$

$$- \int_R (\zeta \partial \eta v) k_4 ([\partial_x, \zeta \partial \eta] v)$$

$$+ \int_R (\zeta \partial \eta v) k_4 (\partial_x \zeta \partial \eta v) .$$

Each of the terms

$$[\partial_y, \zeta \partial \eta] u = \zeta_x \partial \eta u + \zeta \partial \eta_x u,$$

$$[\partial_x, \zeta \partial \eta] v = \zeta_x \partial \eta v + \zeta \partial \eta_x v,$$

lies in $L^2$ under special control since $\partial u = \partial_x \partial w \in L^2$ under special control by (5.25) and (5.26), and $\zeta_x$ and $\eta_x$ are supported where $k \geq c > 0$. Moreover $\partial_x \zeta \partial \eta v$ has $L^2$ norm that can be absorbed into (5.29). Then we can use

$$\int_R |(\zeta \partial \eta u) k_4|^2 \leq \int_R |(\zeta \partial \eta u) v k|^2 + \int_R |(\zeta \partial \eta v) u k|^2 + \text{TUSC}$$

$$\leq C \int_R |(\zeta \partial \eta u)|^2 + \int_R k |(\zeta \partial \eta v)|^2 + \text{TUSC}$$

$$\leq C \int_R |(\zeta \partial_x \eta \partial w)|^2 + \int_R k |(\zeta \partial \eta v)|^2 + \text{TUSC} .$$

Thus by (5.25) and (5.26) all of the terms in $I^u$ and $I^v$ are now under special control. The type $\text{III}$ terms are given by

$$I^{\text{III}} = \int_R (\zeta \partial \eta v) k_1 (\zeta_y \partial \eta u) + \int_R (\zeta \partial \eta v) k_2 (\zeta_y \partial \eta u)$$

$$+ \int_R (\zeta \partial \eta (zv + yuv)) k_3 (\zeta_y \partial \eta u) + \int_R (\zeta \partial \eta v^2) k_4 k \partial_y \partial \eta u .$$
\[ III^v = \int \mathbb{R} (\zeta \partial v^2) k_2 (\zeta \partial \eta u) + \int \mathbb{R} (\zeta \partial \eta v^2) k_3 (\zeta \partial \eta u) \]

\[ - \int \mathbb{R} (\zeta \partial \eta w) k_4 (\zeta \partial \eta u) + \int \mathbb{R} (\zeta \partial \eta v) k_5 (\zeta \partial \eta u), \]

and are handled in similar fashion to the type I terms.

We now turn to the type II terms, which require the hypotheses \(|k_{ij}| \leq Ck^{\frac{j}{2}}\) for \(2 \leq i, j \leq 5\). Since by the hypothesis (2.28) we already have \(|k_j| \leq Ck\) for \(2 \leq j \leq 4\), it follows from (2.20) as before that if \(ck_j \leq k\), then

\[ c |\nabla k_j| = |\nabla k - \nabla (k - ck_j)| \leq |\nabla k| + |\nabla (k - ck_j)| \leq C \sqrt{k} + C \sqrt{k - ck_j} \leq Ck^{\frac{j}{2}} \]

for \(2 \leq j \leq 4\). Thus \(|k_{55}| \leq Ck^{\frac{7}{2}}\) is the only second derivative estimate that must be assumed in the hypotheses. We begin with the identities

\[ [k_j \partial_y, \eta \partial \zeta^2] = k_j \eta \zeta^2 + k_j \eta \partial_z \zeta_y - \eta \zeta^2 (\partial k_j) \partial_y = A_j + B_j - C_j, \]

for \(1 \leq i, j \leq 5\). We will write \(II^v = II_A^v + II_B^v + II_C^v\). \(II_A^v = \sum_{j=1}^4 II_{A_j}^v\) and \(II_C^v = \sum_{j=2}^5 II_{C_j}^v\) etc. with the obvious meanings. We have

\[ II_A^v = \int \mathbb{R} vA_1 \partial \eta u + \int \mathbb{R} uvA_2 \partial \eta u + \int \mathbb{R} (zw + yw) A_3 \partial \eta u + \int \mathbb{R} kv^2 A_4 \partial \eta u \]

\[ = \int \mathbb{R} vk_1 \eta \partial \zeta^2 \partial \eta u + \int \mathbb{R} uvk_2 \eta \partial \zeta^2 \partial \eta u \]

\[ + \int \mathbb{R} (zw + yw) k_3 \eta \partial \zeta^2 \partial \eta u + \int \mathbb{R} kv^2 k_4 \eta \partial \zeta^2 \partial \eta u \]

and these terms can be decomposed into terms that are either under special control, or can be absorbed into (5.29) after applying the Poincaré inequality in one variable. Indeed, since \(\partial u = \partial_x \partial \eta\), we have \(\partial \zeta \partial \eta u = \partial_x \zeta \partial \eta (\partial w)\) modulo terms of the form \(\partial_x u, \partial_y u, \partial_x v\) and \(\partial_y v\) with appropriate cutoff functions included, and terms involving only \(u, v\) or \(w\) with cutoff functions. Now the terms of the form \(\partial_x u\) and \(\partial_y u = \partial_z v\) have \(L^2\) norm under special control since the integrals in (5.23) are under special control. The remaining term \(\partial_y v\) can be absorbed into (5.29) after applying the Poincaré inequality in one variable so as to obtain the \(L^2\) norm of \(\partial_x \partial_y v\) (with cutoff functions) multiplied by \(R^2\). Finally the main term \(\partial_x \zeta \partial \eta (\partial w)\) has \(L^2\) norm that can be absorbed into (5.29) if multiplied by a sufficiently small \(\alpha\). Similarly the term \(II_B^v\) is under special control.

Turning to the term \(II_C^v\) we have

\[ II_C^v = \int \mathbb{R} vC_1 \partial \eta u + \int \mathbb{R} uvC_2 \partial \eta u + \int \mathbb{R} (zw + yw) C_3 \partial \eta u + \int \mathbb{R} kv^2 C_4 \partial \eta u \]

\[ = \int \mathbb{R} v\zeta^2 (\partial k_1) \partial_y \partial \eta u + \int \mathbb{R} uv \zeta^2 (\partial k_2) \partial_y \partial \eta u \]

\[ + \int \mathbb{R} (zw + yw) \zeta^2 (\partial k_3) \partial_y \partial \eta u + \int \mathbb{R} kv^2 \zeta^2 (\partial k_4) \partial_y \partial \eta u. \]
We now need the second derivatives,
\[
\partial_x k_j = \partial_x k_j (x, w, r, z, y) = k_{j1} + k_{j2}u + k_{j3} (z + yu) + k_{j4}kv,
\]
\[
\partial_y k_j = \partial_y k_j (x, w, r, z, y) = k_{j2}v + k_{j3}vy - k_{j4}u + k_{j5}.
\]
Considering first the case \( \partial = \partial_x \), we compute
\[
II_C^\alpha = \int_R vC_1 \partial \eta \mu = \int_R v\eta^2 (\partial_x k_1) \partial_y \partial \eta \mu
\]
\[
= \int_R v\eta^2 (k_{11}) \partial_y \partial \eta \mu + \int_R v\eta^2 (k_{12}u) \partial_y \partial \eta \mu
\]
\[
+ \int_R v\eta^2 (k_{13} (z + yu)) \partial_y \partial \eta \mu + \int_R v\eta^2 (k_{14}kv) \partial_y \partial \eta \mu.
\]
Now since \( \partial_y u = \partial_x v \), we can write as above \( \zeta \partial_y \partial \eta \mu = \partial_x \zeta \partial \eta \nu \) modulo terms either with \( L^2 \) norm under special control, or that can be absorbed into (5.29) after applying the Poincaré inequality in one variable. Since the term \( \partial_x \zeta \partial \eta \nu \) has \( L^2 \) norm that can be absorbed into (5.29) if multiplied by a sufficiently small \( \alpha \), and \( ||v||_{L^\alpha} \) is under special control, we see that we need \( |k_{ij}| \leq C \) to obtain that \( II_C^\alpha \) is under special control. Similarly, we have \( II_C^\alpha \) is under special control for all \( \ell \) provided \( |k_{ij}| \leq C \). This completes the proof that \( II_C^\alpha \) is under special control in the case \( \partial = \partial_x \).

Turning now to the case \( \partial = \partial_y \), we compute
\[
II_C^\alpha = \int_R vC_1 \partial \eta \mu = \int_R v\eta^2 (\partial_y k_1) \partial_y \partial \eta \mu
\]
\[
= \int_R v\eta^2 (k_{12}v) \partial_y \partial \eta \mu + \int_R v\eta^2 (k_{13}vy) \partial_y \partial \eta \mu
\]
\[
- \int_R v\eta^2 (k_{14}u) \partial_y \partial \eta \mu + \int_R v\eta^2 (k_{15}) \partial_y \partial \eta \mu.
\]
Again we can write \( \zeta \partial_y \partial_y \eta \mu = \partial_x \zeta \partial \eta \nu \) modulo terms either with \( L^2 \) norm under special control, or that can be absorbed into (5.29) after applying the Poincaré inequality in one variable. Since the term \( \partial_x \zeta \partial \eta \nu \) has \( L^2 \) norm that can be absorbed into (5.29) if multiplied by a sufficiently small \( \alpha \), we see that \( II_C^\alpha \) is under special control since \( ||v||_{L^\alpha} \) is. Similarly, we have \( II_C^\alpha \) is under special control for all \( \ell \) provided \( |k_{ij}| \leq C \). This completes the proof that \( II_C^\alpha \) is under special control in the case \( \partial = \partial_y \).

We now investigate the corresponding estimates for \( II_A^\alpha \), \( II_B^\alpha \) and \( II_C^\alpha \). We have
\[
II_A^\alpha = \int_R v^2 A_2 \partial \eta \mu + \int_R yv^2 A_3 \partial \eta \mu - \int_R uvA_4 \partial \eta \mu + \int_R vA_5 \partial \eta \mu
\]
\[
= \int_R v^2 k_2 \eta \partial \zeta \partial \eta \mu + \int_R yv^2 k_3 \eta \partial \zeta \partial \eta \mu
\]
\[
- \int_R uv k_4 \eta \partial \zeta \partial \eta \mu + \int_R v k_5 \eta \partial \zeta \partial \eta \mu.
\]
Now \( \sqrt{k} \partial \zeta \partial \eta \mu \) can be absorbed into (5.29) if multiplied by a sufficiently small \( \alpha \), \( kv \leq C \), \( \xi v \) lies in \( L^2 \) under special control, and the \( L^2 \) norm of \( \zeta \partial \eta \nu \) can be absorbed using Poincaré’s inequality in one variable. Thus we see that we need \( |k_2| \leq C k^2, |k_3| \leq C k^2, |k_4| \leq C k^2 \) and \( |k_5| \leq C k^2 \) in order to have \( II_A^\alpha \) under special control. Similarly the term \( II_B^\alpha \) is under special control.
Turning to the term \( II_C \), we have
\[
II_C = \int_\mathcal{R} v^2 C_2 \partial \eta v + \int_\mathcal{R} y v^2 C_3 \partial \eta v - \int_\mathcal{R} u v C_4 \partial \eta v + \int_\mathcal{R} v C_5 \partial \eta v
\]
\[
= \int_\mathcal{R} v^2 \eta^2 \zeta_1 (\partial k_2) \partial y \partial \eta v + \int_\mathcal{R} y v^2 \eta^2 \zeta_2 (\partial k_3) \partial y \partial \eta v
\]
\[
- \int_\mathcal{R} w v \eta^2 \zeta_3 (\partial k_4) \partial y \partial \eta v + \int_\mathcal{R} v \eta^2 \zeta_4 (\partial k_5) \partial y \partial \eta v.
\]

We recall the second derivatives,
\[
\partial_x k_j = \partial_x k_j (x, w, r, z, y) = k_{j1} + k_{j2} u + k_{j3} (z + y) + k_{j4} v,
\]
\[
\partial_y k_j = \partial_y k_j (x, u, r, z, y) = k_{j2} v + k_{j3} y - k_{j4} u + k_{j5}.
\]

Considering first the case \( \partial = \partial_x \), we compute
\[
II_C^n = \int_\mathcal{R} v^2 C_2 \partial \eta v = \int_\mathcal{R} v^2 \eta^2 \zeta_1 (\partial_x k_2) \partial y \partial \eta v
\]
\[
= \int_\mathcal{R} v^2 \eta^2 \zeta_2 (k_{21}) \partial y \partial \eta v + \int_\mathcal{R} v^2 \eta^2 \zeta_4 (k_{22} u) \partial y \partial \eta v
\]
\[
+ \int_\mathcal{R} v^2 \eta^2 \zeta_5 (k_{23} (z + y)) \partial y \partial \eta v + \int_\mathcal{R} v^2 \eta^2 \zeta_6 (k_{24} v) \partial y \partial \eta v.
\]

Now \( \zeta_\partial \partial_x \eta v = \partial_x \zeta \partial \eta v \) modulo a term that is under special control by \( (5.25) \) since it is supported where \( k \geq c > 0 \). Since \( \partial_x \zeta \partial_y \eta v \) has \( L^2 \) norm that can be absorbed into \( (5.29) \) if multiplied by a sufficiently small \( \alpha \), and since \( \| \xi v \|_{L^6} \) is under special control, we see that \( II_C^n \) is under special control. Similarly, we have \( II_C^i \) is under special control for all \( \ell \) provided \( |k_{ij}| \leq C \). This completes the proof that \( II_C \) is under special control in the case that \( \partial = \partial_x \).

Turning now to the final case \( \partial = \partial_y \), we compute
\[
II_C^\eta = \int_\mathcal{R} v^2 C_2 \partial \eta v = \int_\mathcal{R} v^2 \eta^2 \zeta_1 (\partial_y k_2) \partial y \partial \eta v
\]
\[
= \int_\mathcal{R} v^2 \eta^2 \zeta_2 (k_{22} v) \partial y \partial \eta v + \int_\mathcal{R} v^2 \eta^2 \zeta_4 (k_{23} y v) \partial y \partial \eta v
\]
\[
- \int_\mathcal{R} v^2 \eta^2 \zeta_5 (k_{24} u) \partial y \partial \eta v + \int_\mathcal{R} v^2 \eta^2 \zeta_6 (k_{25}) \partial y \partial \eta v.
\]

This time we need an additional factor of \( \sqrt{k} \) to go with \( \partial_x \zeta \partial_y \eta v \) so that \( \sqrt{k} \partial_x \zeta \partial_y \eta v \) has \( L^2 \) norm that can be absorbed into \( (5.29) \) if multiplied by a sufficiently small \( \alpha \). Since \( \| \xi v \|_{L^6} \) is under special control, we see that we only need \( |k_{ij}| \leq C k^\frac{6}{5} \) for \( 2 \leq j \leq 5 \) in order to have \( II_C^\eta \) under special control. As mentioned above, these follow from our assumption that \( |k_{ij}| \leq C k^\frac{6}{5} \). Similarly, we have \( II_C^\eta \) under special control for \( i = 3, 4, 5 \) if \( |k_{ij}| \leq C k^\frac{6}{5} \) for \( 2 \leq i, j \leq 5 \). Again, this follows from our assumptions on \( k_2, k_3, \) and \( k_4 \), with the exception of \( |k_{55}| \leq C k^\frac{6}{5} \), which is part of the hypotheses. This completes the proof that \( II_C \) is under special control in the case \( \partial = \partial_y \), and with this, the proof that the main terms in the application of Corollary 3.8 to \( (5.29) \) are under special control. The remaining terms are easier to handle, and then just as in the previous section, we conclude that \( \| \xi u \|_{L^\infty} \) and \( \| \xi v \|_{L^\infty} \) are under special control. Theorem 2.5 now completes the proof of Theorem 2.6.
Remark 5.3. Instead of using Theorem 2.5 above, we could have continued by establishing the special control of the higher order derivatives of $w$ directly. The special conditions (2.22) on the solution $w$ permit some simplification of the corresponding arguments in the proof of Theorem 2.5, especially in circumventing the use of Lemma 3.12 by invoking instead the argument used above to prove that $\zeta u \in L^6$ with special control.

5.2.1. Prescribed Gaussian curvature. Here we prove Theorem 2.2 on prescribed Gaussian curvature.

Proof. (of Theorem 2.2) For $\delta \geq 0$ such that $K(x, y) + \delta < \lambda$ for $(x, y) \in \overline{\Omega}$ (where $\lambda$ is as in (2.10)), let $u^\delta$ be the solution in [15] of the prescribed Gaussian curvature boundary problem with homogeneous boundary data,

$$
\begin{cases}
  u_{x x}^\delta u_{y y}^\delta - (u_{x y}^\delta)^2 &= (K(x, y) + \delta) \left(1 + |\nabla u^\delta(x, y)|^2\right)^2, \quad (x, y) \in \Omega \\
  u^\delta &= 0, \quad (x, y) \in \partial \Omega.
\end{cases}
$$

Thus $u^0 = u$, and just as in the case of the Monge-Ampère equation, with $L$ as in (5.12), $U = u^{\delta_1} - u^{\delta_2} \in C^{1,1}$ satisfies an elliptic equation with $C^{1,1}$ coefficients,

$$
LU = (\delta_1 - \delta_2) \left(1 + |\nabla u^{\delta_1}(x, y)|^2\right)^2
+ (K(x, y) + \delta_2) \left\{ \left(1 + |\nabla u^{\delta_1}(x, y)|^2\right)^2 - \left(1 + |\nabla u^{\delta_2}(x, y)|^2\right)^2 \right\}
= (\delta_1 - \delta_2) \left(1 + |\nabla u^{\delta_1}(x, y)|^2\right)^2
+ (K(x, y) + \delta_2) \int_0^1 \frac{d}{d\theta} F(\nabla u^{\delta_2} + \theta \nabla U) \, d\theta,
$$

where $F(p, q) = (1 + p^2 + q^2)^2$. When $\delta_1 > \delta_2 \geq 0$, we can rewrite this equation as

$$
LU - (K(x, y) + \delta_2) \left\{ \int_0^1 (\nabla F)(\nabla u^{\delta_2} + \theta \nabla U) \, d\theta \right\} \cdot \nabla U \geq 0
$$

where the second term on the left is linear in $\nabla U$ with coefficients satisfying the hypotheses of the maximum principle, Theorem 9.6 in [9]. We conclude that $U \leq 0$ or $u^{\delta_1} \leq u^{\delta_2}$ in $\overline{\Omega}$. It follows that with $y = \alpha(x)$ and $y = \beta(x)$ parameterizing the bottom and top boundary curves of $\partial \Omega$,

$$
u_{y}^{\delta_1}(x, \beta(x)) = \lim_{y \to \beta(x)^-} \frac{0 - u_{y}^{\delta_1}(x, y)}{\beta(x) - y}
\geq \lim_{y \to \beta(x)^-} \frac{0 - u_{y}^{\delta_2}(x, y)}{\beta(x) - y} = u_{y}^{\delta_2}(x, \beta(x)),
$$

and similarly,

$$u_{y}^{\delta_1}(x, \alpha(x)) \leq u_{y}^{\delta_2}(x, \alpha(x)),$$

for $\delta_1 > \delta_2$. Note also that

$$u_{y}(x, \alpha(x)) \leq c_1 < 0 < c_2 \leq u_{y}(x, \beta(x))$$

for $x$ near 0, since $u$ is a nonconstant convex function with zero boundary values.
Now apply the partial Legendre transform $T^\delta$ as in (2.14) with $u^\delta$ in place of $u$,
\[
\begin{align*}
  s &= x \\
  t &= u^\delta_y (x, y)
\end{align*}
\]
and use the above inequalities to obtain that the transformed regions
\[
T^\delta \Omega = \{ (x, u^\delta_y (x, y)) : (x, y) \in \Omega \}
\]
satisfy $T^{\delta_2} \Omega \subset T^{\delta_1} \Omega$ for $\delta_1 > \delta_2$, as well as that the interior of $T^0 \Omega$ is the domain $T^0 \Omega^*$. Now for $\delta > 0$, let $w^\delta (s, t)$ denote the inverse function $u^\delta_y (s, t)^{-1} (t)$. Since
\[
k^\delta (x, y, v, p, q) = (\mathcal{K} (x, y) + \delta) \left( 1 + p^2 + q^2 \right)^2
\]
satisfies
\[
\left\| \frac{\left( |k_1^\delta| + |k_5^\delta| + |k_{55}^\delta| \right)}{\sqrt{k^\delta}} + \sum_{i=2}^{4} \frac{|k_i^\delta|}{(k^\delta)^{d(i)}} \right\|_{L^\infty(L)} \leq C \left( L \right), \quad L \text{ compact } \subset \Omega,
\]
equally in $1 \geq \delta > 0$, we can apply Theorem 2.6 to the transformed functions $w^\delta$ to show that they, together with all their derivatives, are uniformly bounded on compact subsets of $T^0 \Omega^*$. Indeed, just as in the proof of Theorem 2.1 for the Monge-Ampère equation (but using instead the $C^2$ a priori estimates in [11] for the equation of prescribed Gaussian curvature), we can show that $u^\delta \rightarrow u$ in $C^{0,1} (\Omega)$ as $\delta \rightarrow 0$, and that for every compact subset $K$ of $T^0 \Omega^*$, there is a compact subset $L$ of $\Omega^* \times \mathbb{R}^3$ and $c > 0$ such that
\[
(x, w^\delta (x, y), r^\delta (x, y), z^\delta (x, y), y) \in L, \quad \text{for all} \ (x, y) \in K, 0 < \delta < c.
\]
Moreover, the inequalities in (2.22) also follow from the a priori estimates in [11]. Finally, as mentioned earlier, in the case of the prescribed Gaussian curvature equation, all of the inequalities in (2.28) hold automatically, save for $|k_2| \leq C k^\delta$, which is part of the hypotheses of Theorem 2.2.

It now follows from the inverse function theorem, just as in the proof for the Monge-Ampere equation, that the functions $u^\delta_y$, together with all their derivatives, are uniformly bounded, and hence equicontinuous, on compact subsets of $\Omega^*$. Indeed, for $K$ a compact subset of $\Omega^*$, the sets $T^\delta K$ lie in a fixed compact subset $L$ of $T^0 \Omega^*$ for $\delta$ small by (5.19), whose proof follows using the $C^2$ a priori estimates in [11] for the equation of prescribed Gaussian curvature. Using the equation
\[
u^\delta_x u^\delta_y - \left( u^\delta_{xy} \right)^2 = (\mathcal{K} (x, y) + \delta) \left( 1 + |\nabla u^\delta (x, y)|^2 \right),
\]
it follows that the functions $u^\delta_{xx}$, together with all their derivatives, are uniformly bounded, and hence equicontinuous, on compact subsets of $\Omega^*$. Just as in the proof for the Monge-Ampere equation, it now follows that $u$ is smooth and $u_{yy} > 0$ in $\Omega^*$.

6. Appendix A: Pogorelov Segments

Let $\Omega$ be an open set in the plane and let $u$ be a continuous function defined in either $\Omega$ or $\overline{\Omega}$. A segment $L$ in $\overline{\Omega}$ is said to be a Pogorelov segment for $u$ if both of the following hold:

(a) $L$ is the projection onto $\overline{\Omega}$ of a maximal line segment in the graph of $u$,
(b) one endpoint of $L$ lies in $\partial \Omega$ while the other endpoint lies in $\Omega$. 

Note that the endpoint of $L$ in $\partial \Omega$ will or will not belong to $L$ according as $u$ is defined in $\Omega$ or $\Omega$. We prove here that if $u$ is a nontrivial convex solution to the homogeneous Monge-Ampère boundary value problem

$$
\begin{aligned}
\det D^2 u &= k & \text{in } \Omega \\
u &= 0 & \text{on } \partial \Omega,
\end{aligned}
$$

(6.1)

with $k = 0$ on the $y$-axis and $\partial \Omega$ positively curved, then $u \notin C^2(\overline{\Omega}) \cap C^3(\Omega)$ if there is a Pogorelov segment in the $y$-axis. Moreover, if in addition $\Omega$ and $k$ are symmetric about the $y$-axis, we obtain the stronger conclusion that $u \notin C^2(\overline{\Omega})$ whether or not it has a Pogorelov segment, rendering $u$ as irregular as it can possibly be when $k \in C^{1,1}(\overline{\Omega})$ and $\partial \Omega$ is $C^{3,1}$ - indeed, in this case $u \in C^{1,1}(\overline{\Omega})$ by the results in Guan [11]. A result for the nonhomogeneous problem is given in the second section below.

Before stating and proving our results in detail, we sketch our arguments in the simple case when $u \in C^4(\overline{\Omega})$; the extra differentiability permits the simplifying use of differential inequalities.

**Theorem 6.1.** Suppose $u \in C^4(\overline{\Omega})$ is a nontrivial convex solution to (6.1) where $k$ vanishes when $x = 0$, and $\partial \Omega$ is positively curved. Then there are no Pogorelov segments in the $y$-axis.

**Proof.** Suppose, in order to derive a contradiction, that $L$ is a Pogorelov segment for $u$ lying in the $y$-axis. We compute

$$
k_{xx} = u_{xxxx}u_{yy} + 2u_{xxx}u_{xxy} + u_{xxx}u_{xyy} - 2(u_{xxy})^2 - 2u_{xxy}u_{xxx}. $$

Now $k$ achieves its minimum value of 0 on the $y$-axis, so the second derivative test yields $k_{xx} \geq 0$ on $L$. Also $u_{yy} = 0$ on $L$ since $u$ restricted to $L$ is affine, and it follows that $u_{xy}^2 = u_{xx}u_{yy} - k = 0$ on $L$ as well. Finally then, $u_{xxy} = \partial_y(u_{xy}) = 0$ on $L$ and we have

$$0 \leq k_{xx} = u_{xx}\partial_y^2 u_{xx} - 2(\partial_y u_{xx})^2, \quad \text{on } L. $$

Now by the homogeneous boundary condition and the positive curvature of $\partial \Omega$, the chain rule yields that $u_{xx}$ is positive at the endpoint of $L$ in $\partial \Omega$ (see the proof of Corollary 6.4 below). The above inequality then shows that $\partial_y^2 \left( \frac{1}{u_{xx}} \right) \leq 0$ where $u_{xx} > 0$ on $L$, and it follows easily that $\frac{1}{u_{xx}}$ is concave, and that $u_{xx}$ is bounded below by a positive constant, on $L$ (see the proof of Theorem 6.2 below).

Next we compute

$$
\partial_y k = u_{xx}\partial_y u_{yy} + u_{xxy}u_{yy} - 2u_{xxy}\partial_x u_{yy},
$$

which when restricted to the $y$-axis yields

$$u_{xx} |\partial_y u_{yy}| \leq C u_{yy},$$

since both $|u_{xy}|$ and $|\partial_x u_{yy}|$ are dominated by $C \sqrt{u_{yy}}$ on the $y$-axis. Indeed, $u_{xy}^2 = u_{xx}u_{yy} - k$ yields the former, while the latter holds because $u_{yy}$ is $C^2$ and nonnegative on $\Omega$, and thus satisfies

$$|\partial_x u_{yy}| \leq C \|\partial_y^2 u_{yy}\|_\infty \sqrt{u_{yy}},$$

by (2.20) with $u_{yy}$ in place of $k$. In the previous paragraph we proved that $u_{xx}$ is bounded below by a positive constant on $L$, and hence on an extension of $L$ in $\Omega$. Then unique continuation, or Gronwall’s inequality, shows that $u_{yy}$ vanishes on a
segment of the $y$-axis strictly larger than $L$, contradicting the maximality of the Pogorelov segment $L$, and completing the proof of the theorem.

Our main technical results on Pogorelov segments do not involve boundary behaviour.

**Theorem 6.2.** Suppose $u \in C^2(\Omega)$ is convex and $L = \{(0, y) : y \in I\}$ is a Pogorelov segment in the $y$-axis. Let

$$f(y) = u_{xx}(0, y), \quad y \in I.$$ 

Then either $f$ is identically zero on $I$, or $f$ is positive on $I$ and $\frac{1}{f}$ is concave on $I$.

**Theorem 6.3.** Suppose $u$ is convex on $\Omega$ and that either $u \in C^2(\Omega)$ and is symmetric about the $y$-axis, or $u \in C^3(\Omega)$. Moreover, we assume that $\det D^2u = 0$ on the $y$-axis. Then if $L$ is a Pogorelov segment in the $y$-axis, $D^2u = 0$ on $L$, i.e. the Hessian of $u$ has rank zero on $L$.

If we assume that $u$ is defined on $\overline{\Omega}$ and vanishes on the boundary of $\Omega$, then we are able to obtain some negative regularity results.

**Corollary 6.4.** If $k$ vanishes on the $y$-axis, and $\partial \Omega$ is positively curved (at the intersection of the $y$-axis and $\Omega$), then a nontrivial convex solution $u$ to (6.1) has no Pogorelov segments on the $y$-axis if $u \in C^2(\overline{\Omega}) \cap C^3(\Omega)$. If in addition both $\Omega$ and $k$ are symmetric about the $y$-axis, then there is no nontrivial convex $C^2(\overline{\Omega})$ solution to (6.1).

**Proof.** (of Theorem 6.2) We have that $u(0, y)$ is affine for $y$ belonging to some nontrivial maximal interval $I = (\rho(0), a]$ where the lower half of $\partial \Omega$ has graph $(x, \rho(x))$. Then $u(0, y) = \beta(y - \rho(0))$ for $\rho(0) < y \leq a$ for some $\beta < 0$ (since $u$ is nontrivial). Let $k = \det D^2u$. Now $u_{yy} = 0$ on $L$ together with $k = u_{xx}u_{yy} - u_{xy}^2 \geq 0$ on the $y$-axis, yields $u_{xy} = 0$ on $L$, and so $u_x$ is a constant, say $\gamma$, on $L$. By Taylor’s formula in the $x$-variable with $y$ fixed, and with $h(y) = \beta(y - \rho(0))$ and $f(y) = \frac{\partial^2}{\partial y^2}u(0, y)$, we then have

$$u(x, y) = h(y) + \gamma x + \frac{1}{2}f(y)x^2 + \eta(x, y)x^2,$$

for $\rho(0) < y \leq a$, where $\eta(x, y)$ is continuous in $x$ for each fixed $y$ and satisfies

$$\lim_{x \to 0} \eta(x, y) = 0, \quad \rho(0) < y \leq a.$$  \hspace{1cm} (6.2)

We will now establish that either $f$ vanishes identically on $(\rho(0), a]$, or $f$ is positive on $(\rho(0), a]$ and $\frac{1}{f}$ is concave on $(\rho(0), a]$. Fix $\theta \in (0, 1)$, $\rho(0) < y_1 < y_2 \leq a$, and for $x_1, x_2$ nonnegative, small and not both zero, we set

$$x_\theta = (1 - \theta)x_1 + \theta x_2,$$

$$y_\theta = (1 - \theta)y_1 + \theta y_2.$$
The convexity of $u$ implies

$$h(y_0) + \gamma x_0 + \left\{ \frac{1}{2} f(y_0) + \eta(x_0, y_0) \right\} (x_0)^2 = u(x_0, y_0) \leq (1 - \theta) u(x_1, y_1) + \theta u(x_2, y_2) = (1 - \theta) h(y_1) + \theta h(y_2) + (1 - \theta) \gamma x_1 + \theta \gamma x_2 + (1 - \theta) \left\{ \frac{1}{2} f(y_1) + \eta(x_1, y_1) \right\} x_1^2 + \theta \left\{ \frac{1}{2} f(y_2) + \eta(x_2, y_2) \right\} x_2^2.$$

Since $h$ is affine on $(\rho(0), a]$, then $h(y_0) = (1 - \theta) h(y_1) + \theta h(y_2)$, and we obtain

$$f(y_0)(x_0)^2 \leq f(y_1)(1 - \theta) x_1^2 + f(y_2) \theta x_2^2 + o(x_1^2 + x_2^2),$$

since

$$(1 - \theta) \eta(x_1, y_1) x_1^2 + \theta \eta(x_2, y_2) x_2^2 \leq \eta(x_0, y_0) (x_0)^2 = o(x_1^2 + x_2^2),$$

by (6.2) with $y = y_1, y_2$ and $y_0$. Now using that

$$(x_{\theta})^2 = ((1 - \theta) x_1 + \theta x_2)^2 \approx x_1^2 + x_2^2,$$

with constants depending on $\theta$ (but $\theta$ is fixed in $(0, 1)$), we obtain upon dividing (6.4) throughout by $(x_{\theta})^2$,

$$f(y_0) \leq f(y_1) \frac{(1 - \theta) x_1}{x_{\theta}}^2 + f(y_2) \frac{\theta x_2}{x_{\theta}}^2 + o(1)$$

as $x_1^2 + x_2^2 \to 0$.

Now if $f(y_1)$ (respectively $f(y_2)$) is zero, we may choose $x_2$ (respectively $x_1$) to be zero, and obtain in the limit from (6.5) that $f(y_0) = 0$. Thus if $f$ vanishes at any point of $(\rho(0), a]$, then $f$ vanishes identically on $(\rho(0), a]$, and by continuity on $(\rho(0), a]$. So we now assume that $f$ is positive on $(\rho(0), a]$ and prove that $\frac{1}{f}$ is concave on $(\rho(0), a]$.

For $A, B > 0$ the function

$$A(1 - \alpha)^2 + B\alpha^2$$

has minimum value equal to the harmonic mean $\frac{1}{\frac{1}{A} + \frac{1}{B}}$, and is minimized by choosing $\alpha = \frac{A}{A + B}$. Thus if we let the pair $(x_1, x_2) \to (0, 0)$ through values satisfying the linear relation

$$\frac{\theta x_2}{(1 - \theta) x_1 + \theta x_2} = \alpha = \frac{A}{A + B} = \frac{f(y_1)}{1 - \theta} = \frac{f(y_2)}{\theta},$$

then we conclude from (6.5) that

$$f(y_0) \leq \frac{1}{\frac{1}{A} + \frac{1}{B}} = \frac{1 - \theta}{f(y_1) + \theta f(y_2)},$$

or $\frac{1}{f(y_0)} \geq \frac{1 - \theta}{f(y_1) + \theta f(y_2)}$, and this establishes the concavity of $\frac{1}{f}$ on $(\rho(0), a]$.

**Remark 6.1.** If we replace the regularity assumption $u \in C^2(\Omega)$ by $u \in C^1(\Omega)$, then the above argument demonstrates that the function

$$f^+(y) = \lim_{x \to 0^+} \frac{u(x, y)}{x^2}$$
either vanishes identically on \((\rho(0), a)\), or \(f^+\) is positive on \((\rho(0), a)\) and \(\frac{1}{u}\) is concave on \((\rho(0), a)\) (with a similar result for \(f^-\) defined with \(x \to 0^+\)). Indeed, simply use \(u(x, y) \leq h(y) + \{f^+(y) + o(1)\} x^2\) to estimate both \(u(x_1, y_1)\) and \(u(x_2, y_2)\) in (6.3), and then choose a sequence of pairs \((x_1^n, x_2^n) \to (0, 0)\) satisfying (6.6) such that the corresponding sequence of points \(x^n_0 = (1 - \theta) x^n_1 + \theta x^n_2\) satisfy \(f^+(y) = \lim_{n \to \infty} \frac{u(x^n_1, y_0)}{(x^n_0)^2}\).

Proof. (of Theorem 6.3) Suppose \(L\) is a Pogorelov segment as in Theorem 6.2 above. With the notation of the previous proof, we now claim that

\[
(6.7) \quad f(a) = u_{xx}(0, a) = 0,
\]

By Theorem 6.2 this shows that \(u_{xx}\) vanishes on \(L\), and since \(u_{yy}\) vanishes on \(L\) by definition, we conclude that \(D^2u\) vanishes on \(L\) as well, completing the proof of the theorem. There are two cases to consider in proving (6.7).

If \(u\) is symmetric about the \(y\)-axis, then \(u_x = 0\) on the \(y\)-axis. Thus \(u_{xy} = 0\) on the \(y\)-axis, and along with \(k = 0\) there, we conclude that \(u_{xx} u_{yy} = 0\) on the \(y\)-axis. Thus \(u_{xx}\) vanishes on the set \(E = \{(0, y) : u_{yy}(0, y) > 0\}\), and since \((0, a)\) is a limit point of \(E\) (by maximality of the Pogorelov segment \(L\)), we have (6.7) as \(u \in C^2(\Omega)\).

In the other case, there is no symmetry, but there is additional regularity instead, namely \(u \in C^3(\Omega)\). We compute as before

\[
\partial_y k = u_{xx} \partial_y u_{yy} + u_{xx y} u_{yy} - 2u_{xy} \partial_x u_{yy},
\]

We assume, in order to derive a contradiction, that \(u_{xx}(0, a) > 0\), and show that \(u_{yy}(0, y)\) satisfies a differential inequality of Gronwall type for \(y\) in a neighbourhood of \(a\). Again we use \(|u_{xy}| \leq \sqrt{u_{xx} u_{yy}} \leq C \sqrt{u_{yy}}\) on the \(y\)-axis, but since we are no longer assuming \(u \in C^4(\Omega)\), we must calculate \(\partial_x u_{yy}\) more carefully. Now for some \(\varepsilon > 0\), \(u_{xx}(0, y) > 0\) for \(\rho(0) < y < a + \varepsilon\), and in this range we have \(u_{yy} = \frac{k + u_{xx}^2}{u_{xx}}\), and so

\[
\partial_x u_{yy} = \partial_x \left( \frac{k + u_{xx}^2}{u_{xx}} \right) = \frac{k_x + 2u_{xx} u_{xy} u_{xx} - k + u_{xx}^2}{u_{xx}^2} u_{xxx}.
\]

Now \(|k_x| \leq C \sqrt{k} \leq C \sqrt{u_{yy}}\) near \((0, a)\) by (2.20) since \(k\) is smooth and nonnegative, and \(|u_{xy}| \leq C \sqrt{u_{yy}}\) was established above. Since \(u \in C^3(\Omega)\), we thus obtain from (6.9) that

\[
|\partial_x u_{yy}(0, y)| \leq C \sqrt{u_{yy}(0, y)}, \quad \rho(0) < y < a + \varepsilon.
\]

Altogether, we have from (6.8) and the fact that \(k = 0\) on the \(y\)-axis,

\[
|\partial_y u_{yy}(0, y)| \leq \frac{|u_{xxy}| u_{yy}}{u_{xx}} + 2 |u_{xy}| |\partial_x u_{yy}| u_{xx} \leq C u_{yy}(0, y),
\]

at least for \(y\) near \(a\). With \(g(y) = u_{yy}(0, y)\), we thus have

\[
\begin{align*}
|g'(y)| & \leq C g(y) \\
g(a) & = 0,
\end{align*}
\]

which yields \(g \equiv 0\) on \((\rho(0), a + \varepsilon)\) by Gronwall’s inequality, which contradicts the maximality of the Pogorelov segment \(L\), and provides the desired contradiction. Thus we have established (6.7) and as indicated above, this completes the proof of the theorem.
Remark 6.2. In our section on statements of results above, we mentioned that $u$ has no Pogorelov segments in the $y$-axis if $u$ is symmetric about the $y$-axis and if the rank of $D^2u$ is at least one everywhere. This is of course immediate from Theorem 6.3, but actually follows from just the first two paragraphs of the proof of Theorem 6.3, and does not need Theorem 6.2 at all.

Proof. (of Corollary) We first claim that if $u \in C^2(\overline{\Omega})$ is a nontrivial convex solution to (6.10), then

$$u_{xx}(0, \rho(0)) > 0,$$

provided that either the data are symmetric about the $y$-axis, or that $u$ has a Pogorelov segment with an endpoint at $(0, \rho(0))$. To see this, we first differentiate $0 = u(x, \rho(x))$ twice with respect to $x$ to get

$$0 = u_{xx} + 2u_{xy}\rho' + u_{yy}(\rho')^2 + u_{y}\rho''.$$

Thus we obtain

$$u_{xx} + 2u_{xy}\rho' + u_{yy}(\rho')^2 = -u_{y}\rho'' > 0$$

at the boundary point $(0, \rho(0))$ since $u_y(0, \rho(0)) < 0$ for $u$ a nontrivial convex function with zero boundary data, and since $\rho''(0) > 0$ as $\partial\Omega$ is positively curved. Now if the data are symmetric about the $y$-axis, then $\rho'(0) = 0$ and (6.10) follows from (6.11). If $u$ has a Pogorelov segment ending at $(0, \rho(0))$, then $u_{yy}(0, \rho(0)) = 0$, and so also $u_{xy}(0, \rho(0)) = 0$. Thus (6.10) again follows from (6.11).

The first part of the corollary now follows from (6.10) since $u \in C^2(\overline{\Omega})$ then implies $u_{xx}(0, y) > 0$ for $y$ near $\rho(0)$ and this contradicts Theorem 6.3. To prove the second part of the corollary, we note that the symmetry of the data, together with uniqueness, shows that $u$ is symmetric about the $y$-axis. Thus $u_x = 0$ on the $y$-axis, and so then does $u_{xy}$. Since $u \in C^2(\overline{\Omega})$, we have $u_{xx} > 0$ in a neighbourhood of $(0, \rho(0))$, and so $0 = k = u_{xx}u_{yy} - u_{xy}^2$ on the $y$-axis implies that $u_{yy}(0, y) = 0$ for $y$ near $\rho(0)$, thus establishing the existence of a Pogorelov segment in the $y$-axis. This contradicts Theorem 6.3 and completes the proof of the corollary.

6.1. More general boundary values. The condition of homogeneous boundary data in the corollary can be relaxed somewhat. For convenience we consider only the case where all the data are symmetric about the $y$-axis. In order to state the result, we begin with a quantitative version of the Hopf boundary point lemma. We denote by $\nu u(P)$ the inward normal derivative at $P \in \partial\Omega$ of a differentiable function $u$ defined on a domain $\Omega$ with smooth boundary $\partial\Omega$. We let $\rho_{\max}(\Omega) = \max_{P \in \partial\Omega} \frac{1}{\Gamma(P)}$ and $\rho_{\min}(\Omega) = \min_{P \in \partial\Omega} \frac{1}{\Gamma(P)}$ where $\Gamma(P)$ is the curvature of $\partial\Omega$ at $P \in \partial\Omega$ (i.e. $\rho_{\max}$ and $\rho_{\min}$ are the maximum and minimum radii of curvature). For $\varepsilon > 0$, we define $\Omega_\varepsilon = \{(x, y) \in \Omega : \text{dist}((x, y), \partial\Omega) \geq \varepsilon\}$. The following estimate of Hopf type is essentially sharp - see Remark 6.5 below.

Lemma 6.5. Let $\Omega$ be convex with $\partial\Omega$ smooth and positively curved. Then for all nonpositive convex functions $u \in C^{1,1}(\overline{\Omega})$, and all $P \in \partial\Omega$ with $u(P) = 0$,

$$\nu u(P) \leq -\frac{1}{2\sqrt{\pi\rho_{\max}}} \sup_{\varepsilon > 0} \varepsilon \left( \int_{\Omega_\varepsilon} \det D^2u \right)^{\frac{1}{2}},$$

(6.12)
and with \( c_0 = \frac{1}{4\sqrt{\pi} \rho_{\text{max}} |\Omega|} \), we also have

\[
(6.13) \quad \mathbf{n} u(P) \leq -c_0 \left( \frac{\int_\Omega \det D^2 u}{\| \det D^2 u \|_{L^\infty(\Omega)}} \right)^{\frac{3}{2}}.
\]

**Proof.** Given \( P \in \partial \Omega \) with \( u(P) = 0 \), let \( Q \) be the point in \( \Omega \) of distance \( \rho_{\text{min}} \) from \( P \) such that \( PQ \) is normal to \( \partial \Omega \) at \( P \). Let \( A \in \Omega_\varepsilon \), \( \varepsilon > 0 \), and let \( m \) be the line through \( A \) and \( Q \). If \( m \cap \Omega \) has endpoints \( S \) and \( T \), chosen so that \( SQAT \) are consecutive in \( m \), then \(|SA| \leq 2\rho_{\text{max}} \) and \( |SQ| \geq \rho_{\text{min}} \) since the balls \( B_{\rho_{\text{min}}} \) and \( B_{\rho_{\text{max}}} \) tangent to \( \partial \Omega \) at \( P \) of radius \( \rho_{\text{min}} \) and \( \rho_{\text{max}} \) respectively, satisfy \( B_{\rho_{\text{min}}} \subset \Omega \subset B_{\rho_{\text{max}}} \). Since \( u \) is convex and nonpositive, we conclude that

\[
u(Q) = \frac{|QA|}{|SA|} u(S) + \frac{|SQ|}{|SA|} u(A) \leq \frac{|SQ|}{|SA|} u(A) \leq \frac{\rho_{\text{min}}}{2\rho_{\text{max}}} u(A).
\]

Now the modulus of the slope \((p, q)\) of the tangent plane at \((A, u(A))\) is at most \( \frac{|\nabla u(A)|}{\varepsilon} \) since \( u \) is convex and nonpositive on the ball of radius \( \varepsilon \) centered at \( A \). From the previous inequality we thus obtain

\[
\sqrt{p^2 + q^2} \leq \frac{|u(A)|}{\varepsilon} \leq \frac{2\rho_{\text{max}}}{\varepsilon \rho_{\text{min}}} |u(Q)|.
\]

If

\[
(p, q) = \chi_u(x, y) = (u_x(x, y), u_y(x, y))
\]

is the gradient map of \( u \), then the Jacobian of \( \chi_u \) is \( \det D^2 u \), and we have

\[
\int_{\Omega_\varepsilon} \det D^2 u = \int_{\Omega_\varepsilon} \frac{\partial (p, q)}{\partial (x, y)} dx dy = \int_{\chi_u(\Omega_\varepsilon)} dp dq = |\chi_u(\Omega_\varepsilon)|
\]

\[
\leq \pi \left( \frac{2\rho_{\text{max}}}{\varepsilon^{\rho_{\text{min}}}} u(Q) \right)^2 = 4\pi \rho_{\text{max}}^2 \left( \frac{u(Q)}{\rho_{\text{min}}} \right)^2,
\]

where the second equality follows from the change of variable formula applied to \( \chi_u + \delta I \) and then letting \( \delta \to 0 \). Since \( u(P) = 0 \), we have

\[
\mathbf{n} u(P) \leq \frac{u(Q) - u(P)}{|QP|} = \frac{u(Q)}{\rho_{\text{min}}} \leq -\frac{\varepsilon}{2\sqrt{\pi} \rho_{\text{max}} \left( \int_{\Omega_\varepsilon} \det D^2 u \right)^{\frac{3}{2}}},
\]

which proves (6.12).

To obtain (6.13), we note that

\[
\int_{\Omega_\varepsilon} \det D^2 u = \int_{\Omega} \det D^2 u - \int_{\Omega - \Omega_\varepsilon} \det D^2 u
\]

\[
\geq \int_{\Omega} \det D^2 u - \varepsilon |\partial \Omega| \| \det D^2 u \|_{L^\infty(\Omega)}
\]

\[
\geq \frac{1}{2} \int_{\Omega} \det D^2 u
\]

if we choose \( \varepsilon = \frac{1}{2\sqrt{\pi} \rho_{\text{max}} |\partial \Omega|} \int_{\Omega} \det D^2 u \). Plugging this choice of \( \varepsilon \) into (6.12) yields (6.13) with \( c_0 = \frac{1}{4\sqrt{\pi} \rho_{\text{max}} |\partial \Omega|} \).
Remark 6.3. This lower bound for $|\mathbf{n}u(P)|$ should be compared with the formula in Lemma 17.1 of [18] for the $L^2$ average of $|\mathbf{n}u(P)|$ with respect to the probability measure $\frac{\Gamma_{\partial \Omega}}{2\pi}$, valid for those $u \in C^{1,1}(\Omega)$ that vanish on $\partial \Omega$:

$$\left( \int_{\partial \Omega} |\mathbf{n}u(P)|^2 \frac{\Gamma_{ds}}{2\pi} \right)^{\frac{1}{2}} = \left( \frac{1}{\pi} \int_{\Omega} \det D^2u \right)^{\frac{1}{2}}.$$  

We now have the following generalization of Theorem 6.3 and its corollary for the general boundary value problem

$$(6.14) \quad \begin{cases} \det D^2u = k & \text{in } \Omega \\ u = \phi & \text{on } \partial \Omega. \end{cases}$$

Let the lower half of $\partial \Omega$ be given by the graph $(x, \rho(x))$.

**Theorem 6.6.** Suppose $\Omega$, $k$ and $\phi$ are symmetric about the $y$-axis, $k \in C^{1,1}(\Omega)$ vanishes on the $y$-axis, $k \geq 0$ and nontrivial on $\Omega$, $\phi \in C^{3,1}(\partial \Omega)$, and $\partial \Omega$ is smooth and positively curved. Suppose moreover that $\Omega$, $k$ and $\phi$ are related at the boundary point $(0, \rho(0))$ by the condition

$$(6.15) \quad \frac{d^2}{dx^2} [\phi(x, \rho(x))] |_{x=0} = \rho''(0) \left\{ \sup_{(x, y) \in \partial \Omega \setminus \{(0, \rho(0))\}} \frac{\phi(x, y) - \phi(0, \rho(0))}{y - \rho(0)} - c_0 \left( \int_{\Omega} k \right)^{\frac{2}{3}} \right\},$$

where $c_0$ is as in Lemma 6.5. If $u \in C^{1,1}(\Omega)$ is the convex solution to the boundary value problem (6.14), then $u \notin C^2(\Omega)$.

**Remark 6.4.** A useful special case of the theorem occurs when $\phi(0, \rho(0)) = 0$ and $\phi \leq 0$ on $\partial \Omega$, and we assume the following inequality stronger than (6.15):

$$(6.16) \quad \frac{d^2}{dx^2} [\phi(x, \rho(x))] |_{x=0} = \rho''(0) > -c_0 \left( \int_{\Omega} k \right)^{\frac{2}{3}}.$$  

**Proof.** We begin by proving the special case in Remark 6.4 above. So assume, in order to derive a contradiction, that $u \in C^2(\Omega)$. Differentiating $\phi(x, \rho(x)) = u(x, \rho(x))$ twice with respect to $x$, setting $x = 0$, and using $\rho'(0) = 0$ yields

$$\frac{d^2}{dx^2} [\phi(x, \rho(x))] |_{x=0} = u_{xx}(0, \rho(0)) + u_y(0, \rho(0)) \rho''(0).$$

Thus by (6.16) and Lemma 6.5, we have

$$u_{xx}(0, \rho(0)) > -c_0 \left( \int_{\Omega} k \right)^{\frac{2}{3}} \rho''(0) + u_y(0, \rho(0)) \rho''(0) \geq 0,$$

since $\mathbf{n}u(0, \rho(0)) = u_y(0, \rho(0))$. With this inequality, the proof of Theorem 6.3 and its corollary carry over without change.

Now let $\theta = \sup_{(x, y) \in \partial \Omega} \frac{\phi(x, y) - \phi(0, \rho(0))}{y - \rho(0)}$, and define

$$\tilde{u}(x, y) = u(x, y) - \phi(0, \rho(0)) - \theta (y - \rho(0))$$

and

$$\tilde{\phi}(x, y) = \phi(x, y) - \phi(0, \rho(0)) - \theta (y - \rho(0)).$$
so that \( \tilde{u}(0, \rho(0)) = 0 \) and \( \tilde{\phi} \leq 0 \) on \( \partial \Omega \), and (6.14) holds with \( \tilde{u} \) and \( \tilde{\phi} \) in place of \( u \) and \( \phi \). Then (6.15) implies that \( \tilde{\phi} \) satisfies (6.16), and applying Remark 6.4, proved in the previous paragraph, completes the proof of the theorem.

**Remark 6.5.** The exponent \( \frac{3}{2} \) in conclusion (6.13) of Lemma 6.5 is sharp. This is most easily seen by first noting that Lemma 6.5 carries over to arbitrary dimension \( n \) in the form

\[
|\{|(u_x(x, y), u_y(x, y)) : (x, y) \in E\}| = \int \int_E \left( u_{xx} u_{yy} - (u_{xy})^2 \right) dxdy
\]

where \( \|\det D^2 u\|_{C^0(\Omega)} = 1 \) and \( \int_\Omega \det D^2 u \) can be arbitrarily small.

7. **Appendix B**

We collect some standard material here for the reader’s convenience.

7.1. **Generalized solutions.** Let \( u \) be a nonnegative Borel measure on \( \Omega \). A. D. Alexandrov introduced in [1] the concept of a generalized convex solution \( u \) to the Monge-Ampère equation in \( \Omega \),

\[
u_{xx} u_{yy} - (u_{xy})^2 = \mu,
\]

based on the change of variable formula

\[
\int \int f(s, t) dsdt = \int \int f(u_x, u_y) \left\{ u_{xx} u_{yy} - (u_{xy})^2 \right\} dxdy.
\]

In particular, if \( u \) is smooth and convex, and the change of variables is one-to-one, then

\[
|\{(u_x(x, y), u_y(x, y)) ; (x, y) \in E\}| = \int \int_E \left( u_{xx} u_{yy} - (u_{xy})^2 \right) dxdy
\]
for all Borel subsets $E$ of $\Omega$. Given any convex function $u$ on $\Omega$ and a point $(x, y) \in \Omega$, define as in [5] the set

$$B(x, y) = \{(a, b) \in \mathbb{R}^2 : u(s, t) \geq u(x, y) + a(s - x) + b(t - y), (s, t) \in \Omega\}$$

of slopes of all supporting planes of the graph of $u$ at $(x, y)$. To each Borel set $E$ in $\Omega$, let

$$B(E) = \bigcup_{(x, y) \in E} B(x, y)$$

and define $\mu_u(E)$ to be the Lebesgue measure of the set $B(E)$. In [1] (see also [5]) it is proved that $B(E)$ is Lebesgue measurable if $E$ is Borel, and that $\mu_u$ is a nonnegative Borel measure on $\Omega$, referred to as the representing measure of $u$ (and to be thought of as the generalized determinant of the Hessian of $u$). One says that $u$ is a generalized convex solution of (7.1) if its representing measure is $\mu$. In particular, $u$ is a generalized convex solution of $u_{xx}u_{yy} - (u_{xy})^2 = k$ if $u$ is convex and $d\mu_u = k dx dy$.

In the case that $u$ is $C^{1,1}$, it turns out that (7.2) implies that $\mu_u$ is absolutely continuous with density $u_{xx}u_{yy} - (u_{xy})^2$. Thus the above notion of solution generalizes the classical notion of solution for the Monge-Ampère equation. One of the key properties of generalized solutions is the following weak convergence theorem.

**Theorem 7.1.** If a sequence of convex functions $\{u_n\}_{n=1}^{\infty}$ converges uniformly on compact subsets of $\Omega$ to a convex function $u$, then the associated measures $\mu_{u_n}$ converge weakly to the measure $\mu_u$ in the sense that

$$\lim_{n \to \infty} \int f d\mu_{u_n} = \int f d\mu_u$$

for all continuous functions $f$ with compact support in $\Omega$.


7.2. **Radial solutions.** Radial solutions to the Monge-Ampère equation,

\[
\begin{align*}
    u_{xx}u_{yy} - (u_{xy})^2 &= k(x, y), \quad (x, y) \in D \\
    u &= C, \quad (x, y) \in \partial D
\end{align*}
\]

in the unit disk $D$ are easily characterized. Since the determinant of the Hessian is rotation invariant, $u$ is radial in (2.1) if and only if $k$ is radial and $\phi$ is a constant $C$ on the unit circle $\partial D$. Let

$$k(x, y) = f\left(\frac{r^2}{2}\right),$$

$$u(x, y) = g\left(\frac{r^2}{2}\right),$$

where $r = \sqrt{x^2 + y^2}$. Then one easily computes

$$u_{xx}u_{yy} - u_{xy}^2 = g''\left(\frac{r^2}{2}\right)g'\left(\frac{r^2}{2}\right)r^2 + g'\left(\frac{r^2}{2}\right)^2$$

and so

$$f(t) = g''(t)g'(t)2t + g'(t)^2 = \left[tg'(t)^2\right]'.$$
Thus
\[ g(t) = C + \int_0^t \sqrt{\frac{1}{s} \int_0^s f \, ds} \]
and \( u(x, y) = g\left(\frac{x^2 + y^2}{2}\right) \) solves (7.3). In particular, if \( f(t) = t^N \) so that \( k(x, y) = cr^{2N} \), then \( u - C \approx r^{N+2} \), which fails to be smooth for \( N \) a positive odd integer. Note that if we add a positive constant of integration \( K \) to \( I_s \), then \( u(r) = Cr^2 + 2r^N \), which fails to be smooth for \( N \) a positive odd integer.

We can just as easily compute radial solutions \( u \) to the prescribed Gaussian curvature equation in the unit disk,
\[
\begin{cases}
  u_{xx}u_{yy} - (u_{xy})^2 &= K(x, y) \left( 1 + u_x^2 + u_y^2 \right)^2, & (x, y) \in D \\
  u &= C, & (x, y) \in \partial D,
\end{cases}
\]
where again, since the Hessian and the gradient are rotation invariant (the curvature is actually an isometry invariant), the data must be rotation invariant if the solution \( u \) is. With
\[
K(x, y) = f\left(\frac{r^2}{2}\right),
\]
\[
u(x, y) = g\left(\frac{r^2}{2}\right),
\]
as above, we have
\[
(1 + u_x^2 + u_y^2)^2 = \left(1 + g'\left(\frac{r^2}{2}\right)^2 r^2\right)^2
\]
and
\[
f(t) = g''(t)g'(t) \frac{2t + g'(t)^2}{1 + 2tg'(t)^2} = \frac{tg'(t)^2}{1 + 2tg'(t)^2} = -\frac{1}{2} \left[ \frac{1}{\left(1 + 2tg'(t)^2\right)} \right].
\]
Integrating, we obtain
\[
\frac{1}{\left(1 + 2tg'(t)^2\right)} - 1 = -2 \int_0^t f
\]
and
\[
g(t) = C_1 + \int_0^t \sqrt{\frac{1}{1 - 2\int_0^s f}} \, ds.
\]
Thus \( u(x, y) = g\left(\frac{r^2}{2}\right) \) solves (7.4) provided \( 2\int_0^t f < 1 \). This is of course the familiar condition
\[
\int_D K(x, y) \, dx dy = \int_0^{2\pi} \int_0^1 f\left(\frac{r^2}{2}\right) r \, dr \, d\theta = 2\pi \int_0^1 f(t) \, dt < \pi,
\]
which is necessary if $K$ is the curvature of a convex $C^{1,1}$ function $u$ on $\overline{D}$. The necessity of (7.5) follows from the change of variable $(s, t) = T(x, y)$ given by

\[
\begin{align*}
s &= u_x(x, y) \\
t &= u_y(x, y)
\end{align*}
\]

in the computation

\[
\pi = \int_0^{2\pi} \int_0^\infty (1 + r^2)^{-2} r dr d\theta = \int_{\mathbb{R}^2} (1 + s^2 + t^2)^{-2} ds dt > \int_{T(D)} (1 + s^2 + t^2)^{-2} ds dt = \int_D K(x, y) dxdy,
\]

where we have used that $K$ is the curvature of $u$, the Jacobian of $T$ is $u_{xx}u_{yy} - (u_{xy})^2$, and that $T(D)$ is bounded if $u$ has bounded derivatives on $D$. See for example [29] for a discussion of prescribed Gaussian curvature and condition (7.5).

### 7.3. Almost one-variable.

Our condition (2.3),

\[
|k_2(x, y)| \leq C k(x, y)^{3/2},
\]

says that $k$ is close to being a function of $x$ alone when $k$ is small. In fact, the zero set of such a function $k$ is a union of vertical lines by Gronwall’s inequality. Thus for a given $x$, we may suppose that $k(x, y) \neq 0$ for all $y$ and rewrite (7.6) for such $(x, y)$ as

\[
\left| \frac{1}{\sqrt{k(x, y)}} - \frac{1}{\sqrt{k(x, 0)}} \right| \leq C |y|,
\]

which implies

\[
\left| \frac{1}{\sqrt{k(x, y)}} \right| \leq C |y|.
\]

Since $k$ is bounded, we conclude that $k(x, y) \approx k(x, 0)$, and so

\[
k(x, y) = f(x) \left[ 1 + \sqrt{f(x)} h(x, y) \right]
\]

where $f(x) = k(x, 0)$ is nonnegative and both $h(x, y) = \frac{k(x, y) - k(x, 0)}{k(x, 0)^{3/2}}$ and $\partial_y h$ are bounded:

\[
|h(x, y)| = \left| \frac{\sqrt{k(x, y)} - \sqrt{k(x, 0)}}{k(x, 0)^{3/2}} \right| (\sqrt{k(x, y)} + \sqrt{k(x, 0)}) \leq C |y| \left( \sqrt{k(x, y)} + \sqrt{k(x, 0)} \right) \leq 2C |y|;
\]

\[
|\partial_y h(x, y)| = \left| \frac{\partial_y k(x, y)}{k(x, 0)^{3/2}} \right| \leq C.
\]

Conversely, functions $k$ of the form (7.7) with $f, h$ and $\partial_y h$ bounded, are themselves bounded and satisfy (7.6). This observation led to the refinement of the Sibony example in section 2.5 above. We thank C. Rios for a suggestion to combine this with the example of Sibony.
7.4. Interpolation inequalities. We begin with the classical interpolation inequality for a smooth nonnegative function \( k \) defined on a domain \( \Omega \),

\[
|\nabla k(x,y)| \leq B \sqrt{k(x,y)}, \quad (x,y) \in \mathcal{R},
\]
valid for a compact subset \( \mathcal{R} \) of \( \Omega \). We first note that (7.8) holds for \( k \) nonnegative and smooth on all of \( \mathbb{R}^2 \) with bounded second derivatives. Indeed, the inequality follows from the one-dimensional version, which in turn follows from Taylor’s formula,

\[
0 \leq k(y) = k(x) + k'(x)(y-x) + \frac{1}{2} k''(c)(y-x)^2,
\]
upon choosing \( y = x - \frac{k'(x)}{\|k''\|_{\infty}} \) (if \( \|k''\|_{\infty} = 0 \), the result is trivial). The inequality

\[
|\nabla k(x,y)| \leq \sqrt{2} \|\nabla^2 k\|_{\infty}^{\frac{1}{2}} \sqrt{k(x,y)}, \quad (x,y) \in \Omega,
\]
valid for \( k \) nonnegative with bounded second derivatives on \( \mathbb{R}^2 \), remains true in the form

\[
|\nabla k(x,y)| \leq C \left\{ \|\nabla^2 k\|_{\infty}^{\frac{1}{2}} + \left( \text{dist} \left( (x,y), \partial \Omega \right) \right)^{-\frac{3}{2}} \right\} \sqrt{k(x,y)}, \quad (x,y) \in \Omega,
\]
if \( k \) is merely nonnegative with bounded first and second derivatives on a domain \( \Omega \). To see this recall that Taylor’s formula in one dimension yields

\[
0 \leq k(y) = k(x) + k'(x)(y-x) + \frac{1}{2} k''(c)(y-x)^2.
\]

We obtain \( |k'(x)| \leq 2 \|k''\|_{\infty} \sqrt{k(x)} \) upon choosing \( y = x - \frac{k'(x)}{\|k''\|_{\infty}} \), but if \( \frac{|k'(x)|}{\|k''\|_{\infty}} \) exceeds the distance \( d \) to the boundary (or if \( k \) is linear), then with \( y = x - d \text{ sign}(k'(x)) \) we can only achieve

\[
0 \leq k(x) - |k'(x)| d + \frac{1}{2} \|k''\|_{\infty} d^2
\]
\[
\leq k(x) - \frac{1}{2} |k'(x)| d,
\]
which yields \( |k'(x)| \leq 2 \frac{k(x)}{d} \). Since \( k' \) is bounded, we may write \( |k'(x)| \leq C \sqrt{\frac{k(x)}{d}} \) as claimed.

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