Example 1
Find eigenvalues and eigenfunctions of the following equation
\[ y'' + \lambda y = 0 \]
with boundary values \( y(0) = 0 \) and \( y(L) = 0 \).

Solution
The general solution is
\[ y(x) = a \cos(\sqrt{\lambda}x) + b \sin(\sqrt{\lambda}x). \]
Then, apply the boundary values,
from \( y(0) = 0 \),
\[ a = 0 \]
from \( y(L) = 0 \) and \( a = 0 \),
\[ 0 = b \sin(\sqrt{\lambda}L). \]
We assume that \( b \) is nonzero so that \( 0 = \sin(\sqrt{\lambda}L) \). Then, we have
\[ \sqrt{\lambda}L = \pi n \]
\[ \lambda = \left( \frac{\pi n}{L} \right)^2. \]
Let’s write \( \lambda_n \) instead of \( \lambda \), which gives
\[ \lambda_n = \left( \frac{\pi n}{L} \right)^2. \]
Since we have \( a = 0 \), only sine term remains, so eigenfunctions are
\[ y_n = \sin(\sqrt{\lambda_n}x) \]
with eigenvalues
\[ \lambda_n = \left( \frac{\pi n}{L} \right)^2, \quad n = 1, 2, 3... \]
Why do we not include \( \lambda_n \) at \( n = 0 \)? The answer is simple. When \( n = 0 \), we have \( \lambda_0 = 0 \), and then an eigenfunction will be \( y_0 = \sin(0) = 0 \). However, an eigenfunction is defined to be nonzero, so this cannot be an eigenfunction.

Example 2
Find eigenvalues and eigenfunctions of the following equation
\[ y'' + \lambda y = 0 \]
with boundary values \( y'(0) = 0 \) and \( y'(L) = 0 \).
Solution

The general solution is
\[ y(x) = a \cos(\sqrt{\lambda} x) + b \sin(\sqrt{\lambda} x). \]

Then, apply the boundary values,
from \( y'(0) = 0 \),
\[ b = 0 \]
from \( y'(L) = 0 \) and \( b = 0 \),
\[ 0 = a \sqrt{\lambda} \sin(\sqrt{\lambda} L). \]
Then,
\[ \sqrt{\lambda} L = \pi n \]
\[ \lambda_n = \left( \frac{\pi n}{L} \right)^2. \]
Since we have \( b = 0 \), only cosine term remains, so eigenfunctions are
\[ y_n = \cos(\sqrt{\lambda_n} x) \]
with eigenvalues
\[ \lambda_n = \left( \frac{\pi n}{L} \right)^2, \quad n = 0, 1, 2, 3... \]

Here, we include \( \lambda_0 = 0 \) at which an eigenfunction is \( y_0 = \cos(0) = 1 \).

Example 3

Find eigenvalues and eigenfunctions of the following equation
\[ y'' + \lambda y = 0 \]
with boundary values \( y(0) = 0 \) and \( y'(L) = 0 \).

Solution

The general solution is
\[ y(x) = a \cos(\sqrt{\lambda} x) + b \sin(\sqrt{\lambda} x). \]
Then, apply the boundary values,
from \( y(0) = 0 \),
\[ a = 0 \]
from \( y'(L) = 0 \) and \( a = 0 \),
\[ 0 = b \sqrt{\lambda} \cos(\sqrt{\lambda} L). \]
Since \( \cos x = 0 \) at \( x = \frac{2n+1}{2} \pi \), we have
\[ \sqrt{\lambda} L = \frac{2n+1}{2} \pi \]
\[ \lambda_n = \left( \frac{2n+1}{2L} \pi \right)^2. \]
Then, eigenfunctions are

\[ y_n = \sin(\sqrt{\lambda_n}x) \]

with eigenvalues

\[ \lambda_n = \left(\frac{2n + 1}{2L}\right)^2, \quad n = 0, 1, 2, 3... \]

or we can also write

\[ \lambda_n = \left(\frac{2n - 1}{2L}\right)^2, \quad n = 1, 2, 3... \]

**Example 4**

Find eigenvalues and eigenfunctions of the following equation

\[ y'' + \lambda y = 0 \]

with boundary values \( y(L_1) = 0 \) and \( y(L_2) = 0 \), where \( L_2 > L_1 > 0 \).

**Solution**

The general solution is \( y(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x) \). Apply boundary values to the general equation.

From \( y(L_1) = 0 \), we get

(1) \[ 0 = a\cos(\sqrt{\lambda}L_1) + b\sin(\sqrt{\lambda}L_1) \]

from \( y(L_2) = 0 \)

(2) \[ 0 = a\cos(\sqrt{\lambda}L_2) + b\sin(\sqrt{\lambda}L_2) \]

From equations (1-2), we want to find values of \( \lambda \). Rewrite equations in a matrix form

\[ \begin{bmatrix} \cos(\sqrt{\lambda}L_1) & \sin(\sqrt{\lambda}L_1) \\ \cos(\sqrt{\lambda}L_2) & \sin(\sqrt{\lambda}L_2) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0. \]

Setting the determinant of the matrix to zero gives the following equation

\[ \sin(\sqrt{\lambda}(L_2 - L_1)) = \sin(\sqrt{\lambda}L_2) \cos(\sqrt{\lambda}L_1) - \cos(\sqrt{\lambda}L_2) \sin(\sqrt{\lambda}L_1) = 0. \]

Recall trigonometric identities, \( \sin(x - y) = \sin x \cos y - \cos x \sin y \). Then, we can write

\[ \sin[\sqrt{\lambda}(L_2 - L_1)] = \sin(\sqrt{\lambda}L_2) \cos(\sqrt{\lambda}L_1) - \cos(\sqrt{\lambda}L_2) \sin(\sqrt{\lambda}L_1) = 0. \]

We have

\[ \sin[\sqrt{\lambda}(L_2 - L_1)] = 0. \]

Then,

\[ \sqrt{\lambda}(L_2 - L_1) = n\pi \]
\[ \lambda_n = \left( \frac{n}{(L_2 - L_1)\pi} \right)^2 \]

Then, eigenfunctions are
\[ y_n = \sin(\sqrt{\lambda_n}x) \]
with eigenvalues
\[ \lambda_n = \left( \frac{n}{(L_2 - L_1)\pi} \right)^2, \quad n = 1, 2, 3... \]

Note that if we set \( L_1 = 0 \), we will have exactly same eigenfunctions as in the first example.

**Example 5**
Find the eigenvalues and eigenfunctions of the equation
\[ y'' + \lambda y = 0 \]
with given boundary values \( y(0) = y(L), y'(0) = y'(L) \).

**Solution**
I show how to solve this problem in my another note (denoted as Note 1).
Eigenfunctions are
\[ y_n = 1, \cos(\sqrt{\lambda_n}x), \sin(\sqrt{\lambda_n}x) \]
with eigenvalues
\[ \lambda_n = \left( \frac{2\pi n}{L} \right)^2, \quad n = 1, 2, ... \]

**Example 6**
Find the eigenvalues and eigenfunctions of
\[ y'' + \lambda y = 0 \]
with boundary values \( y(0) = 0 \) and \( y(L) + y'(L) = 0 \).

**Solution**
I show how to solve this problem in my another note (denoted as Note 1).
Eigenfunctions are
\[ y_n(x) = \sin(\sqrt{\lambda_n}x) \]
where eigenvalues \( \lambda_n, n = 1, 2, ..., \) are determined from
\[ \sqrt{\lambda} = -\tan(\sqrt{\lambda}L). \]
Example 7
Find the eigenvalues and eigenfunctions of
\[ y'' - 2x^{-1}y' + (\lambda + 2x^{-2})y = 0 \]
with boundary values \( y(1) = 0 \) and \( y(2) = 0 \).
(Hint: let \( y = xu(x) \))

Solution
This differential equation is also in the WileyPlus homework.
Let \( y = xu(x) \). Its derivatives are
\[
\begin{align*}
  y' & = xu' + u \\
  y'' & = xu'' + 2u'.
\end{align*}
\]
Substituting them into the equation yields
\[ xu' + 2u' - 2x^{-1}(xu' + u) + (\lambda + 2x^{-2})xu = 0, \]
which is reduced to
\[
\begin{align*}
  xu'' + \lambda xu & = 0 \\
  u'' + \lambda u & = 0.
\end{align*}
\]
Then, the general solution to the above equation is \( u(x) = a \cos(\sqrt{\lambda}x) + b \sin(\sqrt{\lambda}x) \).
Since, in the beginning, we let \( y = xu \), we obtain
\[
y(x) = ax \cos(\sqrt{\lambda}x) + bx \sin(\sqrt{\lambda}x)
\]
that is the general solution to the original equation.
Apply boundary values,
from \( y(1) = 0 \),
\[
0 = a \cos(\sqrt{\lambda}) + b \sin(\sqrt{\lambda}),
\]
from \( y(2) = 0 \),
\[
0 = 2a \cos(\sqrt{\lambda}2) + 2b2 \sin(\sqrt{\lambda}2),
\]
2 is cancelled,
\[
0 = a \cos(\sqrt{\lambda}2) + b \sin(\sqrt{\lambda}2).
\]
Write them in a matrix form,
\[
\begin{bmatrix}
  \cos(\sqrt{\lambda}) & \sin(\sqrt{\lambda}) \\
  \cos(\sqrt{\lambda}2) & \sin(\sqrt{\lambda}2)
\end{bmatrix}
\begin{bmatrix}
  a \\
  b
\end{bmatrix} = 0.
\]
Setting the determinant of the matrix to zero gives the following equation
\[
\begin{vmatrix}
  \cos(\sqrt{\lambda}) & \sin(\sqrt{\lambda}) \\
  \cos(\sqrt{\lambda}2) & \sin(\sqrt{\lambda}2)
\end{vmatrix} = \sin(\sqrt{\lambda}2) \cos(\sqrt{\lambda}) - \cos(\sqrt{\lambda}2) \sin(\sqrt{\lambda}) = 0.
\]
By using \( \sin(x + y) = \sin x \cos y - \cos x \sin y \), we get
\[
\sin[\sqrt{\lambda}(2 - 1)] = \sin(\sqrt{\lambda}2) \cos(\sqrt{\lambda}) - \cos(\sqrt{\lambda}2) \sin(\sqrt{\lambda}) = 0.
\]
We have

$$\sin(\sqrt{\lambda}) = 0.$$  

Then,

$$\sqrt{\lambda} = n\pi$$

$$\lambda_n = (n\pi)^2$$

Substitute $$\lambda_n = (n\pi)^2$$ into $$0 = a \cos(\sqrt{\lambda}) + b \sin(\sqrt{\lambda})$$, and then find that

$$0 = a \cos(n\pi) + b \sin(n\pi)$$

$$0 = a.$$

A coefficient $$a$$ is zero.

Thus, eigenfunctions are

$$y_n = x \sin(\sqrt{\lambda_n}x)$$

with eigenvalues

$$\lambda_n = (n\pi)^2, \quad n = 1, 2, ...$$