Introduction to o-minimal structures

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Chapter 1

Ordered structures

Throughout these notes[1], we fix an ordered structure $\mathcal{M} = (M, <, \dots)$ such that $(M, <)$ is a linear order. Unless otherwise specified, "definable" means "definable with parameters". For the purposes of these notes, an **interval** is always a *nonempty* interval with endpoints in $M \cup \{-\infty, +\infty\}$ (and hence definable). We consider $\mathcal{M}$ with its order topology on M and the corresponding product topologies on M^k, for $k \in \mathbb{N}$.

For $1 \leq m \leq n$ and a strictly increasing $\iota : \{1, \dots, m\} \longrightarrow \{1, \dots, n\}$, we denote by $\Pi^n_\iota : M^n \longrightarrow M^m$ (or Π_ι if n is clear from context) the projection on the coordinates $(x_{\iota(1)}, \dots, x_{\iota(m)})$. If $\iota(i) = i$ for all i, we also write Π^n_m or simply Π_m in place of Π^n_ι. Note that Π^n_ι is a definable, continuous and open map.

Example 1.1. Let P be the set of all **Puiseux series**, that is, series of the form $G(X) = X^{p/d} \cdot F(X^{1/d})$, where $F(X) \in \mathbb{R}[\![X]\!]$ is a formal power series with real coefficients, $d \in \mathbb{N}$ is nonzero and $p \in \mathbb{Z}$. For such a series $G(X)$, the number $(\mathrm{ord}(F) + p)/d$ is called its **order** and denoted by $\mathrm{ord}(G)$, and the **leading coefficient** of $G(X)$ is the coefficient $\mathrm{lc}(G)$ of $G(X)$ for the monomial $X^{\mathrm{ord}(G)}$. We set $G(X) < 0$ if and only if $\mathrm{lc}(G) < 0$, and we set $G(X) < H(X)$ if and only if $G(X) - H(X) < 0$. Then $(P, <)$ is a linearly ordered structure, and if $+$ and $\cdot$ denote the usual addition and multiplication of such series, the structure $\mathcal{P} := (P, <, +, \cdot)$ is a real closed ordered field.

Exercise 1.2. Prove that $\mathcal{P}$ is totally disconnected.

A definable set $S \subseteq M^n$ is **definably connected** if there are no definable open sets $U, V \subseteq M^n$ such that $S \subseteq U \cup V$, $S \cap U \cap V = \emptyset$ and both $S \cap U$ and $S \cap V$ are nonempty.

Exercise 1.3. (1) Prove that the image of a definably connected, definable set under a definable, continuous map is definably connected.

(2) Let $S, T \subseteq M^n$ be definable and definably connected, and assume that $\mathrm{cl}\, S \cap T \neq \emptyset$. Prove that $S \cup T$ is definably connected.

A **box** is a set of the form $B = I_1 \times \cdots \times I_k$ with each I_i a definable interval. We call B **open** if each I_i is open, and we call B **closed** if each I_i is closed.

Remark. The open boxes form a basis for the product topology on M^n induced by the order topology on M; in particular, they are open sets in this topology.

[1] Partially based on van den Dries [12], Peterzil [8] and Starchenko [11]

Definably complete structures

The structure $\mathcal{M}$ is **definably complete** if every definable subset of M has an infimum and a supremum in $M \cup \{-\infty, +\infty\}$.

Exercise 1.4. Assume that $\mathcal{M}$ is definably complete.

(1) Prove that every interval is definably connected.

(2) **Intermediate Value Theorem:** Let $f, g : I \longrightarrow M$ be definable and continuous, with $I \subseteq M$ and interval, and assume that $f(x) \neq g(x)$ for $x \in I$. Prove that either $f(x) > g(x)$ for all $x \in I$, or $f(x) < g(x)$ for all $x \in I$.

Let $S \subseteq M^{n+m}$. For $x \in M^n$, we denote by $S_x := \{y \in M^m : (x, y) \in S\}$ the **fiber of** S **over** x. The projection $\Pi_n|_S$: $S \longrightarrow \Pi_n(S)$ is a **local homeomorphism** if, for every $(x, y) \in S$, there are an open box $B \subseteq M^{n+m}$ containing (x, y) and a continuous function $f : \Pi_n(B) \longrightarrow M^m$ such that $S \cap B = \mathrm{gr}(f)$. Note that, if $\Pi_n|_S$ is a local homeomorphism, then $\Pi_n(S)$ is open.

For $S \subseteq M^{n+m}$ and $x \in M^n$, we say that S is **locally bounded at** x if there exists an open box $B \subseteq M^n$ containing x such that $S \cap (B \times M^m)$ is bounded.

Exercise 1.5. Assume that $\mathcal{M}$ is definably complete, and let $S \subseteq M^{n+1}$ be definable such that S_x is finite for all $x \in \Pi_n(S)$. Assume in addition that:

(i) $\Pi_n(S)$ is definably connected,

(ii) S is locally bounded at every $x \in \Pi_n(S)$,

(iii) S is closed in $\Pi_n(S) \times M$, and

(iv) $\Pi_n|_S$: $S \longrightarrow \Pi_n(S)$ is a local homeomorphism.

Prove that $|S_x|$ is constant as x ranges over $\Pi_n(S)$. Also, for any three of the conditions (i)–(iv), find an example satisfying these three conditions where $|S_x|$ is not constant.

O-minimal structures

We call $\mathcal{M}$ **o-minimal** if every definable subset of M is a finite union of points and intervals.

Examples 1.6. (1) By quantifier elimination, every dense linear order without endpoints is o-minimal.

(2) Let $\mathcal{V} = (V, <, +, (\lambda_k)_{k \in K})$ be an ordered vector space over an ordered field K. By quantifier elimination, $\mathcal{V}$ is o-minimal.

(3) By Tarski's Theorem, every real closed field is o-minimal.

(4) By Wilkie's [13] and Khovanskii's [2] Theorems, the real expenential field is o-minimal.

We will discuss examples of o-minimal structures later. We assume from now on that $\mathcal{M}$ is o-minimal.

Exercise 1.7. Assume that $\mathcal{M}$ is o-minimal.

(1) Let $A \subseteq M^{n+1}$ be definable. Prove that the set $\{x \in M^n : \ A_x \text{ is finite}\}$ is definable.

(2) Prove that every infinite definable subset of M contains an interval.

(3) Prove that $\mathcal{M}$ is definably complete.

Lemma 1.8. *Let $S \subseteq M$ be definable and $a \in M$. Then there exists $\epsilon > a$ such that either $(a, \epsilon) \subseteq S$ or $(a, \epsilon) \subseteq M \setminus S$.*

Proof. If a is not in the boundary $\operatorname{bd} S$ of S, then either a is in the interior of S or a is in the interior of $M \setminus S$; the lemma follows in both cases. So we assume that $a \in \operatorname{bd}(S)$. By o-minimality, $\operatorname{bd}(S)$ is finite, so we are in one of the following cases: if a is an isolated point of S or the right endpoint of an interval contained in S, then $(a, \epsilon) \subseteq M \setminus S$ for some $\epsilon > 0$; if a is the left endpoint of some interval contained in S, then $(a, \epsilon) \subseteq M$ for some $\epsilon > 0$. $\square$

The first big question about o-minimality is the following: is o-minimality an elementary property, that is, given $\mathcal{N} \equiv \mathcal{M}$, is $\mathcal{N}$ necessarily o-minimal?

Exercise 1.9. Prove that the following are equivalent:

(i) every $\mathcal{N} \equiv \mathcal{M}$ is o-minimal;

(ii) for every definable $A \subseteq M^{n+m}$, there exists $k \in \mathbb{N}$ such that, for all $x \in M^n$, the fiber A_x is finite if and only if $|A_x| \leq k$.

Condition (ii), called the **uniform finiteness property (UFP)**, will be a direct consequence of the cell decomposition theorem (CDT), probably the most fundamental theorem of o-minimality. Indeed, special cases of (UFP) need to be proved inductively along with the proof of (CDT).

Exercise 1.10. Assume that $\mathcal{M}$ is o-minimal and $\aleph_1$-saturated. Prove that $\mathcal{M}$ has (UFP); in particular, every $\mathcal{N} \equiv \mathcal{M}$ is o-minimal.

Chapter 2

Monotonicity

We start by studying definable one-variable functions: let $f : I \longrightarrow M$ be definable, with $I = (a,b)$ an interval. We call f **strictly monotone** if either f is constant, or f is strictly increasing, or f is strictly decreasing. Also, f is strictly monotone at $a \in I$ if there exist $c_1 < a < c_2$ such that the restriction of f to (c_1, c_2) is strictly monotone.

Exercise 2.1. (1) Assume that f is strictly monotone at every $a \in I$. Prove that f is strictly monotone.

(2) Assume that f is strictly monotone. Then there is an interval $J \subseteq I$ such that $f|_J$ is continuous. [Hint: if f is not constant, then $f(I)$ contains an interval J, and $f^{-1}(J)$ is an interval on which f is either an order-preserving or an order-reversing bijection.]

Lemma 2.2. *Assume that $f(x) > x$ for all $x \in I$. Then there exist an open interval $J \subseteq I$ and $c > J$ such that $f(x) > c$ for all $x \in J$.*

Proof. Let $B := \{y \in I : \ f(y) \geq f(x) \text{ for all } x \in (a,y)\}$; we distinguish two cases based on Lemma 1.8.

Case 1: $(a,\epsilon) \subseteq B$ for some $\epsilon > a$; so f is increasing on (a,ϵ). Choose $a < \alpha < \beta < \epsilon$ such that $\beta < f(\alpha)$, and put $J := (\alpha,\beta)$ and $c := f(\alpha)$.

Case 2: $(a,\epsilon) \subseteq I \setminus B$ for some $\epsilon > a$. Choose $c \in (a,\epsilon)$; so there exists $x_1 \in (a,c)$ such that $f(x_1) > f(c)$. Iterating this, we find $x_1 > x_2 > \cdots > x_i > \cdots > a$, for $i \in \mathbb{N}$, such that $f(x_{i+1}) > f(x_i) > f(c)$ for $i \geq 1$. So by Lemma 1.8, there exists $\delta \in (a,c)$ such that $f(x) > f(c)$ for all $x \in (a,\delta)$, so we take $J := (a,\delta)$. $\square$

Proposition 2.3. *Let $S \subseteq I^2$ be definable. There exists an open interval $J \subseteq I$ such that the set*

$$\Delta^>(J) := \left\{(x,y) \in J^2 : \ y > x\right\}$$

is a subset either of S or of $I^2 \setminus S$.

Proof. Let $B := \{x \in I : \ \exists\epsilon > x \text{ such that } (x,\epsilon) \subseteq S_x\}$. By Lemma 1.8, after replacing S by $I^2 \setminus S$ if necessary, we may assume that B contains an interval I'. Now define $f : I' \longrightarrow I$ by $f(x) := \sup\{\epsilon \in (x,b] : \ (x,\epsilon) \subseteq S_x\}$; then f is definable and $f(x) > x$ for all $x \in I'$. By Lemma 2.2, there are an open interval $J \subseteq I'$ and $c > J$ such that $f(x) > c$ for all $x \in J$. The proposition follows. $\square$

Corollary 2.4. *Let $S_1, \dots, S_k \subseteq M^2$ be definable, and assume that $I^2 \subseteq S_1 \cup \dots \cup S_k$. Then there exist $l \in \{1, \dots, k\}$ and an open interval $J \subseteq I$ such that $\Delta^>(J) \subseteq S_l$.* □

Theorem 2.5 (Monotonicity [4]). *There are $k \in \mathbb{N}$ and definable $a_1, \dots, a_k \in M$ such that $a_0 := a < a_1 < \dots < a_k < a_{k+1} := b$ and, for $i = 0, \dots, k$, the restriction $f|_{(a_i, a_{i+1})}$ of f to (a_i, a_{i+1}) is strictly monotone and continuous.*

Proof. By Exercises 2.1 and o-minimality, it suffices to show that the set

$$A := \{x \in I : \ f \text{ is strictly monotone at } a\}$$

is contains an open interval. Note that I^2 is covered by the definable sets

$$X_* := \left\{(x, y) \in I^2 : \ f(x) * f(y)\right\},$$

where $* \in \{<, =, >\}$. So, by Corollary 2.4, there exist $* \in \{<, +, >\}$ and an open interval $J \subseteq I$ such that $\Delta^>(J) \subseteq X_*$. But this means that the restriction of f to J is strictly monotone, as required. □

Corollary 2.6. (1) *The limits* $\lim_{x \to a^+} f(x)$, $\lim_{x \to b^-} f(x)$ *and, for $c \in (a, b)$, the limits* $\lim_{x \to c^-} f(x)$ *and* $\lim_{x \to c^+} f(x)$ *exist in $M \cup \{-\infty, +\infty\}$.*

(2) *If $c, d \in M$ and $g : [c, d] \longrightarrow M$ is definable and continuous, then g attains a maximum and a minimum in $[c, d]$.*

Chapter 3

Definable closure

For $A \subseteq M$, the **definable closure** of A is defined by

$$\mathrm{dcl}(A) := \{b \in M : \ \{b\} \text{ is } A\text{-definable}\}\,.$$

For $A, B \subseteq M$, $a = (a_1, \ldots, a_k) \in M^k$ and $b = (b_1, \ldots, b_l) \in M^l$, we shall write $\mathrm{dcl}(AB)$, $\mathrm{dcl}(Aa)$ and $\mathrm{dcl}(ab)$ in place of $\mathrm{dcl}(A \cup B)$, $\mathrm{dcl}(A \cup \{a_1, \ldots, a_k\})$ and $\mathrm{dcl}(\{a_1, \ldots, a_k, b_1, \ldots, b_l\})$, respectively.

Exercise 3.1. (1) Let $A \subseteq M$. Prove that $\mathrm{dcl}(A) = \mathrm{acl}(A)$.

(2) Let $A \subseteq M$ and $\phi(x)$ a formula with parameters in A. Prove that $\phi(M)$ is infinite if and only if there exist an elementary extension $\mathcal{M}^*$ of $\mathcal{M}$ and $b \in \phi(M^*)$ such that $b \notin \mathrm{dcl}(A)$.

Since the boundary of any definable subset of M is finite, we obtain:

Corollary 3.2. *Let $S \subseteq M$ be A-definable. Then* $\mathrm{bd}(B) \subseteq \mathrm{dcl}(A)$. □

Lemma 3.3. *Let $A \subseteq M$ and $a, b \in M$. Then $b \in \mathrm{dcl}(Aa)$ if and only if there is an A-definable function $g : I \longrightarrow M$ such that $a \in I$ and $b = g(a)$.*

Proof. Assume first that $b \in \mathrm{dcl}(Aa)$, and let $\phi(x, y)$ be a formula with parameters from A such that $\{b\} = \phi(a, M)$. So $\mathcal{M} \models \exists! y \phi(a, y)$, where "$\exists! y$" abbreviates "there is a unique y". Thus, the set

$$I := \{x \in M : \ \exists! y \phi(x, y)\}$$

is definable over A and contains a, and $\phi(M^2) \cap (I \times M)$ is the graph of a function $g : I \longrightarrow M$ definable over A such that $g(a) = b$.

Conversely, assume that $b = g(a)$ for some function $g : I \longrightarrow M$ definable over A. Let $\phi(x, y)$ be a formula with parameters from A such that $\phi(M^2)$ is the graph of g. Then $\{b\} = \phi(a, M)$, so $b \in \mathrm{dcl}(Aa)$. □

Exercise 3.4. Let $A \subseteq M$, and let $f : I \longrightarrow M$ be A-definable, with $I \subseteq M$. Prove that the a_i obtained by the Monotonicity Theorem for f can be chosen to lie in $\mathrm{dcl}(A)$.

Proposition 3.5. *The pair (M, dcl) is a pregeometry.*

Proof. Since dcl = acl, it suffices to establish the exchange property. Let $A \subseteq M$ and $a, b \in M$ be such that $a \in \mathrm{dcl}(Ab) \setminus \mathrm{dcl}(A)$. By Lemma 3.3, it suffices to show that there exists $g : J \longrightarrow M$ definable over A, with $J \subseteq M$, such that $a \in J$ and $b = g(a)$.

Also by Lemma 3.3, there is an $f : I \longrightarrow M$ definable over A such that $I \subseteq M$, $b \in I$ and $a = f(b)$. By o-minimality and because $\mathrm{bd}(I) \subseteq \mathrm{dcl}(A)$, we may assume that I is a singleton or an interval. If I is a singleton, then $I = \{b\} = \mathrm{bd}(I) \subseteq \mathrm{dcl}(A)$, so $a \in \mathrm{dcl}(A)$, a contradiction. We therefore may assume that $I = (c, d)$ for some $c, d \in M \cup \{-\infty, +\infty\}$ with $c < b < d$.

By the Monotonicity Theorem and Exercise 3.4, there are $a_1, \ldots, a_k \in \mathrm{dcl}(A)$ such that $a_0 := c < a_1 < \cdots < a_k < a_{k+1} := d$ and, for $i = 0, \ldots, k$, the restriction of f to (a_i, a_{i+1}) is strictly monotone. As in the previous paragraph, we must have $b \neq a_i$ for each i. Thus, replacing I by (a_i, a_{i+1}) for some i if necessary, we may assume that f is strictly monotone.

If f is constant, then $f(x) = a$ for all $x \in I$, so $\lim_{x \to c^+} f(x) = a$ as well. Hence $\{a\} = \psi(M)$, where $\psi(y)$ is the $\mathcal{L}(A)$-formula

$$\forall y_1 y_2 x_1 \left(y_1 < y < y_2 \wedge c < x_1 \rightarrow \exists x (c < x < x_1 \wedge y_1 < f(x) < y_2)\right),$$

that is, $a \in \mathrm{dcl}(A)$, a contradiction. Therefore, f must be injective; let $g : J \longrightarrow M$ be the compositional inverse of f. Then g is definable over A, $b \in J$ and $a = g(b)$, as desired. $\square$

It follows that every set $A \subseteq M$ has a well-defined dimension, denoted here by $\mathrm{pdim}\, A$. More generally, for $A, B \subseteq M$, the set B has a well-defined **dimension over** A, denoted here by $\mathrm{pdim}(B/A)$. It is well known that in this situation, we have

$$\mathrm{pdim}(AB) = \mathrm{pdim}\, A + \mathrm{pdim}(B/A). \tag{3.1}$$

For $a = (a_1, \ldots, a_k) \in M^k$, we set

$$\dim(a/A) := \mathrm{pdim}(\{a_1, \ldots, a_k\}/A) \in \{0, \ldots, k\}.$$

This dimension is not very usefull, as long as we do not know whether it is defined in elementary extensions of $\mathcal{M}$, as the following example shows:

Example 3.6. Let R be the set of all real algebraic numbers, together with the usual ordering, addition and multiplication. Then $\mathbb{R}_{\text{alg}} := (R, <, +, \cdot)$ is a real closed field, hence is o-minimal. However, for any $a \in R^n$, we have $\dim a = 0$; and this remains so for any o-minimal expansion $\mathcal{R}$ of $\mathbb{R}_{\text{alg}}$ (for which we do not yet know whether any elementary extension is o-minimal).

Chapter 4

Sparse subsets of M^2

The next step towards proving that o-minimality is an elementary property is to show that subsets of M^2 have the uniform finiteness property. It suffices to prove this for the following sets: a set $S \subseteq M^n$ is called **sparse** if S has empty interior.

Lemma 4.1. *Let $S \subseteq M^2$ be definable. The following are equivalent:*

(1) *S is sparse;*

(2) *the set S' of all $x \in M$ such that S_x is infinite is finite;*

(3) *S is nowhere dense in M^2.*

Proof. (1) $\Rightarrow$ (2): assume that S' is infinite. Then there is an open interval $I \subseteq M$ such that S_x contains an interval, for each $x \in I$. For each x, let I_x be the first open interval contained in S_x (with respect to $<$), and consider the definable functions $i : I \longrightarrow M \cup \{-\infty\}$ and $s : I \longrightarrow M \cup \{+\infty\}$ defined by

$$i(x) := \inf I_x \quad \text{and} \quad s(x) := \sup I_x.$$

By the Monotonicity Theorem, there exists an open interval $J \subseteq I$ such that $i|_J$ and $s|_J$ are continuous. Since $i(x) < s(x)$ for all x, the set $\{(x, y) : \ x \in J, \ i(x) < y < s(x)\}$ is open and contained in S.

(2) $\Rightarrow$ (3): assume that S' is finite, and let $U \subseteq M^2$ be open and definable. Then $(S \cap U)'$ is finite, so $(U \setminus S)'$ is infinite. From the previous implication, it follows that $U \setminus S$ has nonempty interior, that is, S is not dense in U.

(3) $\Rightarrow$ (2) is obvious. $\square$

Corollary 4.2. *If $S \subseteq M^2$ is definable and sparse, then so is* $\mathrm{cl}(S)$. $\square$

Lemma 4.3. *Let $S \subseteq M^2$ be definable and sparse.*

(1) *If $\Pi_1(S)$ is infinite, there is a definable, continuous $f : I \longrightarrow M$, with I an open interval, such that $\mathrm{gr}(f) \subseteq S$.*

(2) *If there exists a definable, continuous $f : I \longrightarrow M$, with I an open interval, such that $\mathrm{gr}(f) \subseteq S$, then there exist $x_0 \in I$ and an open box B containing $(x_0, f(x_0))$ such that $B \cap S = \mathrm{gr}(f)$.*

Proof. (1) Assume that $\Pi_1(S)$ is infinite; then it contains an open interval J. Since S is sparse, after shrinking J if necessary, we may assume that S_x is finite for all $x \in J$. So we define $f : J \longrightarrow M$ by $f(x) := \min S_x$; this f is definable and, by the Monotonicity Theorem, contains an open interval I such that $f|_I$ is continuous.

(2) Let I be an open interval and $f : I \longrightarrow M$ be definable and continuous such that $\mathrm{gr}(f) \subseteq S$. Again shrinking I if necessary, we may assume that S_x is finite for every $x \in I$. Define $g, h : I \longrightarrow M \cup \{-\infty, +\infty\}$ by

$$g(x) := \sup \{y \in S_x : \ y < f(x)\} \quad \text{and} \quad h(x) := \inf \{y \in S_x : \ y > f(x)\}.$$

By the Monotonicity Theorem and because the sets $\{x \in I : \ g(x) = -\infty\}$ and $\{x \in I : h(x) = +\infty\}$ are definable, there exists an open interval $J \subseteq I$ such that both $g|_J$ and $h|_J$ are continuous. Since $g(x) < f(x) < h(x)$ for all $x \in I$, part (2) follows. $\square$

For $S \subseteq M^2$, we let $G(S)$ be the definable set of all $(x, y) \in S$ for which there exists an open box $B \subseteq M^2$ containing (x, y) and a definable, continuous $f : \Pi_1(B) \longrightarrow M$ such that $B \cap S = \mathrm{gr}(f)$. We also let $B(S)$ be the set of all $x \in M$ at which S is locally bounded.

Exercise 4.4. Let $S \subseteq M^2$ be definable and sparse. Prove that $M \setminus B(S)$ is finite.

Finally, we let $\mathrm{fr}(S) := \mathrm{cl}(S) \setminus S$ be the **frontier** of S.

Corollary 4.5. *Let $S \subseteq M^2$ be definable and sparse.*

(1) *Let $T \subseteq M^2$ be such that $\Pi_1(T)$ is infinite. Then there exist an open box $B \subseteq M^2$ and a definable, continuous $f : \Pi_1(B) \longrightarrow M$ such that $B \cap S = B \cap T = \mathrm{gr}(f)$.*

(2) *The set $\Pi_1(\mathrm{fr}(S))$ is finite.*

(3) *The set $\Pi_1(S \setminus G(S))$ is finite.*

Proof. (1) First, apply Lemma 4.3(1) with T in place of S to obtain a corresponding function f, then apply Lemma 4.3(2) with this f.

(2) Assume for a contradiction that $\Pi_1(\mathrm{fr}(S))$ is infinite. Applying part(1) with $\mathrm{cl}(S)$ and $\mathrm{fr}(S)$ in place of S and T yields an open box B such that $B \cap \mathrm{cl}(S) = B \cap \mathrm{fr}(S) = \mathrm{gr}(f)$ for some continuous $f : \Pi_1(B) \longrightarrow M$; in particular, $B \cap S = \emptyset$. But the latter implies that $B \cap \mathrm{cl}(S) = \emptyset$, because B is open; contradiction.

(3) follows from part (1) with $T = S \setminus G(S)$. $\square$

Theorem 4.6 (Uniform finiteness). *Let $S \subseteq M^2$ be definable and sparse. Then there exist $k \in \mathbb{N}$, $-\infty = a_0 < a_1 < \cdots < a_k < a_{k+1} = +\infty$ in $M \cup \{-\infty, +\infty\}$ and $i_j \in \mathbb{N}$, for $j = 0, \ldots, k$, such that $|S_x| = i_j$ for $j \in \{0, \ldots, k\}$ and $x \in (a_j, a_{j+1})$.*

Proof. By o-minimality, it suffices to show that there exist $k \in \mathbb{N}$ and $-\infty = a_0 < a_1 < \cdots < a_k < a_{k+1} = +\infty$ such that, for $j = 0, \ldots, k$, the set S_j satisfies conditions (i)–(iv) of Exercise 1.5, where $S_j := S \cap \big((a_j, a_{j+1}) \times M\big)$. By Lemma 4.1, Exercise 4.4 and Corollary 4.5, it suffices to choose finitely many a_j-s such that each $x \in M$ with S_x infinite, each $x \in \Pi_1(\mathrm{fr}(S))$, each $x \in \Pi_1(S \setminus G(S))$ and each $x \in M \setminus B(S)$ is listed among them. $\square$

Let X be a set and $Y_1, \dots, Y_l \subseteq X$, and put $\mathcal{Y} := \{Y_1, \dots, Y_l\}$. We say that the $\mathcal{Y}$ **partitions** X if $X = Y_1 \cup \dots \cup Y_l$ and the Y_j-s are pairwise disjoint. Given $Z \subseteq X$, we say that $\mathcal{Y}$ is **compatible with** Z if, for every j, either $Y_j \subseteq Z$ or $Y_j \cap Z = \emptyset$.

Exercise 4.7. Let $Z_1, \dots, Z_k \subseteq X$, and let $\mathcal{B}$ be the finite boolean algebra of subsets of X generated by $Z_1, \dots, Z_k$. Prove that $\mathcal{Y}$ is compatible with each Z_i if and only if $\mathcal{Y}$ is compatible with each atom of $\mathcal{B}$.

Our proof of Theorem 4.6 actually shows the following: for an open interval I and continuous functions $f, g : I \longrightarrow M \cup \{-\infty, \infty\}$ satisfying $f(x) < g(x)$ for all $x \in I$, we set

$$(f, g)_I := \left\{(x, y) \in M^2 : \ f(x) < y < g(x)\right\}.$$

Theorem 4.8 (Cell decomposition in M^2). *Let $S_1, \dots, S_k \subseteq M^2$ be definable. Then there exist $a_0 = -\infty < a_1 < \dots < a_l < a_{l+1} = +\infty$ and, for each $j = 1, \dots, l$, there are definable continuous functions $f_{j,1}, \dots, f_{j,p(j)} : (a_j, a_{j+1}) \longrightarrow M$ such that, with $f_{j,0} := -\infty|_{(a_j,a_{j+1})}$ and $f_{j,p(j)+1} := +\infty|_{(a_j,a_{j+1})}$, we have for each j:*

(1) *$f_{j,1}(x) < \dots < f_{j,p(j)}(x)$ for all $x \in (a_j, a_{j+1})$;*

(2) *for each $i \in \{1, \dots, k\}$, the collection $\mathcal{C}_j$ of all sets $\mathrm{gr}(f_{j,q})$ and $(f_{j,q}, f_{j,q+1})$, with $q \in \{0, \dots, p(j)\}$, is compatible with S_i.*

Proof. For each $i = 1, \dots, k$, let $T_i := \{(x, y) \in M^2 : \ y \in \mathrm{bd}((S_i)_x)\}$. Note that each T_i is definable and sparse and that it suffices to prove the theorem with the T_i-s in place of the S_i-s (exercise). By Exercise 4.7, we may also assume that the sets T_i are pairwise disjoint. Now choose the a_j-s such that the ones chosen in the proof of Theorem 4.6 with each T_i in place of S are all listed among the a_j-s. $\square$

Exercise 4.9. In the situation of Theorem 4.8, what can you say about the sets $S_i \cap \big(\{a_j\} \times M\big)$?

Chapter 5

Cells and Cell Decomposition

Inspired by the previous chapter, we now make the following definition: let $\sigma \in \{0,1\}^n$ and set $\sigma' := \sigma|_{\{0,1\}^{n-1}}$. We say that a definable set $C \subseteq M^n$ is a **σ-cell** whenever the following holds:

(i) if $n = 1$ and $\sigma(1) = 0$, then $C = \{a\}$ for some $a \in M$;

(ii) if $n = 1$ and $\sigma(1) = 1$, then C is a nonempty, open, definable interval;

(iii) if $n > 1$ and $\sigma(n) = 0$, then $C' := \Pi_{n-1}(C)$ is a σ'-cell and C is the graph of a definable, continuous $f : C' \longrightarrow M$;

(iv) if $n > 1$ and $\sigma(n) = 1$, then $C' := \Pi_{n-1}(C)$ is a σ'-cell and C is the set

$$(f,g)_{C'} := \left\{(x,y) \in M^{n-1} \times M : \ x \in C' \text{ and } f(x) < y < g(x)\right\},$$

where either $f : C' \longrightarrow \{-\infty\}$ or $f : C' \longrightarrow M$ is continuous and definable, and either $g : C' \longrightarrow \{+\infty\}$ or $g : C' \longrightarrow M$ is continuous and definable, and where $f(x) < g(x)$ for all $x \in C'$.

Note that, if $C \subseteq M^n$ is a cell and $m \leq n$, then $\Pi_m(C)$ is a cell and, for each $a \in M^m$, the fiber C_a is a cell. For the next exercises, let $n \in \mathbb{N}$, $\sigma \in \{0,1\}^n$ and $C \subseteq M^n$ be a σ-cell. We call C **open** if $\sigma(i) = 1$ for all i. For σ, we define $\sum \sigma := \sum_{i=1}^n \sigma(i)$. We also associate to σ the unique strictly increasing map $\iota = \iota_\sigma : \{1, \ldots, \sum \sigma\} \longrightarrow \{1, \ldots, n\}$ such that $\sigma(i) = 1$ if and only if $i = \iota(j)$ for some j, and we set $C^\sigma := \Pi_{\iota_\sigma}(C)$.

Exercise 5.1. Prove that C^σ is an open cell and that the map $\pi_\sigma := \Pi_{\iota_\sigma}|_C \colon C \longrightarrow C^\sigma$ is a definable homeomorphism (with respect to the subspace topology on C).

Let $\mathcal{C}$ be a finite collection of cells in M^n and $U \subseteq M^n$. We call $\mathcal{C}$ a **cell decomposition of** U if $\mathcal{C}$ is a partition of U and, if $n > 1$, the collection

$$\Pi_{n-1}\mathcal{C} := \{\Pi_{n-1}(C) : \ C \in \mathcal{C}\}$$

is a cell decomposition of $\Pi_{n-1}(U)$. If $\mathcal{C}$ and $\mathcal{D}$ are cell decompositions of U, we call $\mathcal{D}$ a **refinement** of $\mathcal{C}$ if $\mathcal{D}$ is compatible with each $C \in \mathcal{C}$.

Remark. Let $\mathcal{C}$ be a cell decomposition of $U \subseteq M^{n+m}$, and let $x \in M^n$. Then $\mathcal{C}_x := \{C_x : C \in \mathcal{C}\}$ is a cell decomposition of U_x.

Theorem 5.2 (Cell Decomposition [4]). *$(I)_n$ Let $S_1, \dots, S_k \subseteq M^n$ be definable. Then there exists a cell decomposition $\mathcal{C}$ of M^n compatible with each S_i.*

$(II)_n$ Let $f : S \longrightarrow M$ be definable, with $S \subseteq M^n$. Then there exists a cell decomposition $\mathcal{C}$ of M^n compatible with S such that, for each $C \in \mathcal{C}$, the restriction of f to C is continuous.

The proof of the cell decomposition theorem in general proceeds by induction on n, mimicking the proof in the previous chapter. $(\mathrm{I})_1$ is of course just the definition of o-minimality, and we proved $(\mathrm{II})_1$ (the Monotonicity Theorem) in Chapter 2. So we assume $n \geq 2$ and that the cell decomposition theorem holds for lower values of n.

To prove $(\mathrm{I})_n$, we first need to consider sparse sets:

Exercise 5.3. Let $S \subseteq M^n$ be definable. Prove that S is sparse if and only if the set S' of all $x \in M^{n-1}$ such that S_x is infinite is sparse.

Lemma 5.4 (Uniform finiteness). *Let $S \subseteq M^n$ be definable and sparse. Then there exist a cell decomposition $\mathcal{C}$ of M^{n-1} and, for each open $C \in \mathcal{C}$, an $i_C \in \mathbb{N}$ such that $|S_x| = i_C$ for all $x \in C$.*

Proof. The proof of Theorem 4.6 goes through with the corresponding modifications, such as using "sparse" in place of "finite", "open box in M^{n-1}" in place of "open interval" and "not sparse" in place of "infinite", and using $(\mathrm{I})_{n-1}$ in place of the o-minimality axiom and $(\mathrm{II})_{n-1}$ in place of the Monotonicity Theorem. □

Proof of $(I)_n$. As in the proof of Theorem 4.8, we may assume that each S_i is sparse and that the S_i-s are pairwise disjoint. For each i, let $\mathcal{C}_i$ be a cell decomposition of M^{n-1} obtained from Lemma 5.4 with S_i in place of S. By $(\mathrm{I})_{n-1}$, there is a cell decomposition $\mathcal{D}$ of M^{n-1} that is a refinement of each $\mathcal{C}_i$. Since the S_i-s are pairwise disjoint, for each open $C \in \mathcal{D}$, we now obtain a cell decomposition $\mathcal{D}_C$ of $C \times M$, as in the proof of Theorem 4.8, that is compatible with each S_i. On the other hand, let $C \in \mathcal{D}$ be such that C is not open, so that C is a σ-cell for some $\sigma \in \{0,1\}^{n-1}$ with $\sigma(j) = 0$ for some j. Denote by $\Pi_\sigma : M^{n-1} \times M \longrightarrow M^{\sum \sigma + 1}$ the projection $\Pi_\sigma(x, y) := (\pi_\sigma(x), y)$. By the inductive hypothesis, there is a cell decomposition $\mathcal{D}'_C$ of $C^\sigma \times M$ compatible with each $\Pi_\sigma(S_i \cap (C \times M))$. Now set

$$\mathcal{D}_C := \left\{\Pi_\sigma^{-1}(D') \cap (C \times M) : \ D' \in \mathcal{D}'_C\right\};$$

then $\mathcal{D}_C$ is a cell decomposition of $C \times M$ compatible with each S_i. Finally, we take $\mathcal{C} := \bigcup\{\mathcal{D}_C : \ C \in \mathcal{D}\}$. □

The key to proving $(\mathrm{II})_n$ is the following:

Exercise 5.5. Let $U \subseteq M^n$ be open, $I \subseteq M$ be an open interval and $f : U \times I \longrightarrow M$ be such that

(i) for each $x \in U$, the function $f_x : I \longrightarrow M$ defined by $f_x(y) := f(x, y)$ is continuous and strictly monotone;

(ii) for each $y \in I$, the function $f^y : U \longrightarrow M$ defined by $f^y(x) := f(x,y)$ is continuous.

Prove that f is continuous.

For $S \subseteq M^n$ and $y \in M$, we set $S^y := \{x \in M^{n-1} : (x,y) \in S\}$, and we denote by $\Pi^1 : M^n \longrightarrow M$ the projection on the last coordinate. For a function $f : S \longrightarrow M$, with $S \subseteq M^n$, and for $x \in \Pi_{n-1}(S)$ and $y \in \Pi^1(S)$, we define $f_x : S_x \longrightarrow M$ and $f^y : S^y \longrightarrow M$ by $f_x(y) := f(x,y)$ and $f^y(x) := f(x,y)$.

Proof of (II)$_n$. By (I)$_n$, there is a cell decomposition $\mathcal{C}$ of M^n compatible with both S and the definable set

$$T := \{(x,y) \in S : \ f_x \text{ is continuous and strictly monotone at } y, \ f^y \text{ is continuous at } x\}.$$

If $C \in \mathcal{C}$ is open, then $f|_C$ is continuous by Exercise 5.5. If $C \in \mathcal{C}$ is not open we obtain, along the lines of the proof of (I)$_n$ and using (II)$_{n-1}$, a cell decomposition $\mathcal{D}_C$ of C such that $f|_D$ is continuous for each $D \in \mathcal{D}_C$. □

Corollary 5.6. *Let $S \subseteq M^n$ be definable.*

(1) *M has finitely many definably connected components, and each of them is definable.*

(2) *If $m \leq n$, there exists $k \in \mathbb{N}$ such that M_x has at most k definably connected components, for each $x \in M^m$.*

(3) *If $m \leq n$, the set $\{x \in M^m : \ S_x \text{ is finite}\}$ is definable.*

(4) *If $\mathcal{N} \equiv \mathcal{M}$, then $\mathcal{N}$ is o-minimal.*

Proof. Exercise. □

Chapter 6

Dimension

Let $A \subseteq M$ be *finite* and $\phi(x_1, \dots, x_n)$ be an $\mathcal{L}(A)$-formula, and we set $S := \phi(M^n)$. We define

$$\dim S := \sup \{\dim(a/A) : \; \mathcal{M}^* \models \phi(a), \; \mathcal{M}^* \text{ an elementary extension of } \mathcal{M}\}.$$

Note that $\dim S \in \{-\infty, 0, \dots, n\}$ and that $\dim S = -\infty$ if and only if $S = \emptyset$.

A priori, the number $\dim S$ depends on A; we shall see below that this is not the case, as long as A contains all parameters of ϕ. Until then, we keep A fixed.

Exercise 6.1. (1) If $\mathcal{M}$ is $|\mathcal{L}|^+$-saturated, prove that $\dim S = \max \{\dim(a/A) : \; a \in S\}$.

(2) Let $S \subseteq T \subseteq M^n$ be A-definable. Prove that $\dim S \leq \dim T$.

(3) Let $S, T \subseteq M^n$ be A-definable. Prove that $\dim(S \cup T) = \max\{\dim S, \dim T\}$.

(4) Let $S \subseteq M^n$ be A-definable and $m \leq n$. Prove that $\dim \Pi_m(S) \leq \dim S$.

(5) Let $C \subseteq M^n$ be an open cell, with $\sigma \in \{0, 1\}^n$. Prove that $\dim C = n$.

(6) Let $S \subseteq M^n$ and $f : S \longrightarrow M$ be A-definable. Prove that the graph $\mathrm{gr}(f)$ of f satisfies $\dim \mathrm{gr}(f) = \dim S$, and that $\dim f(S) \leq \dim S$.

Proposition 6.2. *Let $S \subseteq M^n$ and $f : S \longrightarrow M^m$ be A-definable.*

(1) *If S is a σ-cell, then* $\dim S = \sum \sigma$.

(2) $\dim S$ *is equal to the maximal $m \leq n$ for which there exists a strictly increasing $\iota : \{1, \dots, m\} \longrightarrow \{1, \dots, n\}$ such that $\Pi_\iota(S)$ is not sparse.*

(3) *The number* $\dim S$ *is independent of the particular A, as long as the parameters of ϕ belong to A.*

(4) *If f is injective, then* $\dim f(S) = \dim S$.

Proof. Part (1) follows from Exercise 6.1(5,6). Part (2) follows from (1), the Cell Decomposition Theorem and Exercise 6.1(3). Since (2) is independent of A (as long as S is A-definable), part (3) follows. Part (4) follows from Exercise 6.1(6). □

Proposition 6.3. *Let $S \subseteq M^{n+m}$ be definable and nonempty. For $d \in \{-\infty, 0, \dots, m\}$, set $S_d := \{x \in M^n : \dim S_x = d\}$. Then each S_d is definable and*

$$\dim\big(S \cap (S_d \times M^m)\big) = \dim S_d + d.$$

Proof. This follows immediately from the Cell Decomposition Theorem and Proposition 6.2. □

A key property of nonempty definable sets is that their frontier has strictly smaller dimension than the set itself. To prove this, we need the following:

Lemma 6.4. *Let $S \subseteq M^{n+m}$ be definable and put*

$$S' := \{x \in M^n : (\operatorname{cl} S)_x \neq \operatorname{cl}(S_x)\} = \{x \in M^n : (\operatorname{fr} S)_x \neq \operatorname{fr}(S_x)\}.$$

Then the set S' is sparse.

Proof. Let $\mathcal{B}_m := \{(x_1, y_1, \dots, x_m, y_m) \in M^{2m} : x_i < y_i \text{ for each } i\}$; we think of each $z \in \mathcal{B}_m$ as parametrizing the open box $B(z) := \prod_i (x_i, y_i) \subseteq M^m$, that is, $\mathcal{B}_m$ represents the definable family of all open boxes in M^m. We let T be the set of all boxes witnessing the defining inequality for S', that is,

$$T := \{(x, z) \in M^n \times \mathcal{B}^m : (\operatorname{cl} S)_x \cap B(z) \neq \emptyset \text{ but } \operatorname{cl}(S_x) \cap B(z) = \emptyset\}.$$

Then T is definable and $\Pi_n(T) = S'$ and, by definition, for every $x \in S'$, the fiber T_x has nonempty interior. So by Proposition 6.3, we have $\dim T = \dim S' + 2m$. On the other hand, the definition of T also implies that, for every $z \in \mathcal{B}_m$, the fiber T^z is sparse. Hence by Proposition 6.2, we have $\dim T < n + 2m$, so that $\dim S' < n$, that is, S' is sparse, as required. □

Theorem 6.5. *Let $S \subseteq M^n$ be definable and nonempty. Then $\dim \operatorname{fr}(S) < \dim S$.*

Proof. By induction on n; the case $n = 1$ follows from the definition of o-minimality, so we assume $n > 1$ and the theorem holds for lower values of n. By Lemma 6.4, the set

$$S_1 := \{x \in M : (\operatorname{fr} S)_x \neq \operatorname{fr}(S_x)\}$$

is finite and, by the inductive hypothesis, we have $\dim \operatorname{fr}(S_x) < \dim S_x$ for all $x \in M$. It follows from Proposition 6.3 that the definable set

$$T_1 := \operatorname{fr} S \cap \big((M \setminus S_1) \times M^{n-1}\big)$$

has dimension less than $\dim S$ and satisfies $\Pi_1(\operatorname{fr} S \setminus T_1) \subseteq S_1$; that is, the required inequality holds outside a set whose projection on the first coordinate is finite.

Next, for $i = 1, \dots, n$, we let σ_i be the permutation of coordinates exchanging x_1 and x_i and leaving the other coordinates fixed. Arguing as above with $\sigma_i(S)$ in place of S produces a set $T_i \subseteq \operatorname{fr} S$ such that $\dim T_i < \dim S$ and the projection of $\operatorname{fr} S \setminus T_i$ on the ith coordinate is finite. Therefore, $\dim(T_1 \cup \dots \cup T_n) < \dim S$ and the set $\operatorname{fr} S \setminus (T_1 \cup \dots \cup T_n)$ is finite, so the theorem is proved. □

Exercise 6.6. (1) Let $C \subseteq M^n$ be a bounded, open cell and $f : C \longrightarrow M$ be bounded, definable and continuous. For $x \in M^n$, we say that C is **locally connected at** x if, for every open box B containing x, there is an open box $B' \subseteq B$ containing x such that $B' \cap C$ is definably connected.

(i)$_n$ Prove that there is a definable set $S \subseteq \operatorname{fr} C$ of dimension at most $n-2$ such that C is locally connected at every $x \in M^n \setminus S$.

(ii)$_n$ Prove that the set $\{x \in \operatorname{cl}(C) : \ f \text{ does not extend continuously to } x\}$ has dimension at most $n-2$.

[Hint: prove (i)$_n$ and (ii)$_n$ together by induction on n.]

(2) Find an example to show that (1) is optimal.

Exercise 6.7. Let $S \subseteq M^{n+m}$ and $f : S \longrightarrow M$ be definable.

(1) Assume that S_x is open for every $x \in M^n$. Prove that there is a cell decomposition $\mathcal{C}$ of M^n such that, for every $C \in \mathcal{C}$, the set $S \cap (C \times M^m)$ is an open subset of $C \times M^m$.

(2) Assume that f_x is continuous for every $x \in M^n$. Prove that there exists a cell decomposition $\mathcal{C}$ of M^n such that, for every $C \in \mathcal{C}$, the restriction of f to $S \cap (C \times M^m)$ is continuous.

[Hint: use Lemma 6.4.]

Chapter 7

Definable choice

We assume from now on that $\mathcal{M} = (M, <, +, 0, \dots)$ is an o-minimal expansion of an ordered group.

Exercise 7.1. Prove that $(M, <, +, 0)$ is abelian and divisible.

Note that the function $x \mapsto |x| : M \longrightarrow M$ is now definable. For $x = (x_1, \dots, x_n) \in M^n$, we set $|x| := \max\{|x_1|, \dots, |x_n|\}$. We also have a definable choice functions:

Proposition 7.2 (Definable Choice). *Let $S \subseteq M^{n+m}$ be definable. Then there is a definable function $f = f_{S,n} : \Pi_n(S) \longrightarrow M^m$ such, for $x \in \Pi_n(S)$, we have $f(x) \in S_x$ and, for $x, y \in \Pi_n(S)$, we have $f(x) = f(y)$ whenever $S_x = S_y$.*

Proof. By induction on m, simultaneously for all n. If $m = 1$ and $x \in \Pi_n(S)$, we choose $f(x) \in S_x$ as follows:

(i) if $\min \mathrm{bd}(S_x) \neq \inf S_x$, we set $f(x) := \min \mathrm{bd}(S_x) - 1$;

(ii) if $\min \mathrm{bd}(S_x) = \inf S_x \in S_x$, we set $f(x) := \min S_x$;

(iii) if $\min \mathrm{bd}(S_x) = \inf S_x \notin S_x$ and $|\mathrm{bd}(S_x)| \geq 2$, we let $f(x)$ be the midpoint between the least two points of $\mathrm{bd}(S_x)$;

(iv) otherwise, we set $f(x) := \min \mathrm{bd}(S_x) + 1$.

Assume now that $m > 1$ and the proposition holds for lower values of m, and put $S' := \Pi_{n+m-1}(S)$. Then we define $f_{S,n} : \Pi_n(S) \longrightarrow M^m$ by

$$f_{S,n}(x) := (f_{S',n}(x), f_{S,n+m-1}(x, f_{S',n}(x)))\,.$$

It is straightforward to see that this f has the required properties. $\square$

Exercise 7.3. (1) Let $E \subseteq M^{2n}$ be a definable equivalence relation on M^n. Prove that there are $k \in \mathbb{N}$ a definable function $f : M^n \longrightarrow M^k$ such that, for all $x, y \in M^n$, we have xEy if and only if $f(x) = f(y)$. (In particular, $\mathcal{M}^{\mathrm{eq}} = \mathcal{M}$.)

(2) Let $A \subseteq M$ be different from $\{0\}$. Prove that $\mathrm{dcl}(A)$ is the underlying set of an elementary substructure of $\mathcal{M}$. [Hint: use Tarski's test and definable choice.]

(3) Let $\mathcal{M}$ be an arbitrary o-minimal structure (not necessarily an expansion of an ordered group). Prove that there are definable choice functions for closed and bounded definable sets $S \subseteq M^{n+m}$.

A particular case of the models of the theory of $\mathcal{M}$ described in Exercise 7.3(2) is the following: let $\mathcal{N}$ be a saturated elementary extension of $\mathcal{M}$, and let $\tau \in N$ be such that $\tau > M$. We denote by $M\langle\tau\rangle$ the definable closure of $M \cup \{\tau\}$ in N; this is the underlying set of an elementary extension of $\mathcal{M}$ denoted by $\mathcal{M}\langle\tau\rangle$.

On the other hand, let $\mathcal{D}$ be the set of all definable functions $f : M \longrightarrow M$. For $f, g \in \mathcal{D}$, we set $f \sim g$ if there exists $a \in M$ such that $f|_{(a,\infty)} = g|_{(a,\infty)}$. Let $G := \mathcal{D}/\sim$; each element of G is called the **germ at** $+\infty$ of any of its representatives.

Exercise 7.4. (1) Prove that G is the underlying set of an elementary extension of $\mathcal{M}$ denoted by $\mathcal{G}$.

(2) Prove that, for $f, g \in \mathcal{D}$, we have $f(\tau) = g(\tau)$ if and only if $f \sim g$.

(3) Prove that the map $[f]_\sim \mapsto f(\tau) : G \longrightarrow M\langle\tau\rangle$ is a structure isomorphism.

Proposition 7.5 (Curve Selection). *Let $S \subseteq M^n$ be definable and $x \in \operatorname{fr} S$. Then there is a definable, continuous curve $f : (0, \epsilon) \longrightarrow S$ such that $\lim_{t\to 0} f(t) = x$.*

Proof. We let $\widetilde{S} \subseteq M^{1+n}$ be the definable set

$$\widetilde{S} := \{(t, y) \in (0, \infty) \times S : \ |y - x| = t\}.$$

The hypothesis and o-minimality imply that there exists $\epsilon > 0$ such that $(0, \epsilon) \subseteq \Pi_1(\widetilde{S})$. The restriction f of the choice function $f_{\widetilde{S},1}$ to $(0, \epsilon)$ is a definable curve with values in S and, by the Monotonicity Theorem and after shrinking ϵ if necessary, we may assume that f is continuous. By construction, we have $\lim_{t\to 0} f(t) = x$. □

Exercise 7.6. Let $S \subseteq M^n$ be definable. We call S **definably compact** if, for every definable curve $f : (0, \epsilon) \longrightarrow S$, we have $\lim_{t\to 0} f(t) \in S$.

(1) Prove that S is definably compact if and only if S is closed and bounded.

(2) Assume S closed and bounded, and let $f : S \longrightarrow M^k$ be definable and continuous. Prove that $f(S)$ is closed and bounded.

Chapter 8

Differentiability

We assume from now on that $\mathcal{M}$ expands a ring $(M, <, +, \cdot, 0, 1, \dots)$ with unit 1.

Exercise 8.1. Prove that $(M, <, +, \cdot, 0, 1)$ is a real closed ordered field.

Let $I \subseteq M$ be an open interval and $f : I \longrightarrow M$ be definable, and set $D := \{(x, y) \in M^2 : \; x = y\}$. We define $\Delta f : I^2 \setminus D \longrightarrow M$ by

$$\Delta f(x, y) := \frac{f(x) - f(y)}{x - y},$$

a definable function. Recall that f is differentiable at $x \in I$ if and only if $\lim_{y \to x} \Delta f(x, y)$ exists in M; in particular, the set

$$D(f) := \{x \in I : \; f \text{ is differentiable at } x\}$$

is definable. As usual, for $x \in D(f)$, we write $f'(x) := \lim_{y \to x} \Delta f(x, y)$. We call f **differentiable** if $D(f) = I$.

Exercise 8.2. (1) State and prove Rolle's Theorem and the Mean Value Theorem for f.

(2) Assume f is differentiable and that $f' = 0$. Prove that f is constant.

Lemma 8.3. *The set $I \setminus D(f)$ is finite.*

Proof. For $x \in I$, we set $f'(x^-) := \lim_{y \to x^-} \Delta f(x, y)$ and $f'(x^+) := \lim_{y \to x^+} \Delta f(x, y)$. By the Monotonicity Theorem, we have $f'(x^-), f'(x^+) \in M \cup \{-\infty, +\infty\}$ for all x. So it suffices to prove the following two claims:

Claim 1: The set $S_1 := \{x \in I : \; f'(x^-) \neq f'(x^+)\}$ is finite.

To see this claim, we assume for a contradiction that S_1 contains an interval J. By the Monotonicity Theorem, after shrinking J if necessary, we may assume that both $x \mapsto f'(x^-)$ and $x \mapsto f'(x^+)$ are continuous on J. By the Intermediate Value Theorem, it follows that either $f'(x^+) > f'(x^-)$ for all $x \in J$, or $f'(x^+) < f'(x^-)$ for all $x \in J$; we assume the former, the proof in the latter case being similar. Again shrinking J, if necessary, we may assume that there exists $c \in M$ such that $f'(x^+) > c > f'(x^-)$ for all $x \in J$. Now consider the function $g : J \longrightarrow M$ defined by $f(x) - cx$; shrinking J again, if necessary, we may assume that g is continuous and strictly monotone. But $g'(x^+) > 0$ for all x, so g must be strictly increasing; while $g'(x^-) < 0$ for all x, so g must be strictly decreasing, a contradiction.

Claim 2: The set $S_2 := \{x \in I : f'(x^-) \in \{-\infty, +\infty\}\}$ is finite.

To see this claim, we assume for a contradiction that S_2 contains an interval $J = [a, b]$; shrinking J if necessary, we may assume, by Claim 1, that f is continuous on J and $f'(x^+) = f'(x^-) = +\infty$ for all $x \in J$ (the case $f'(x^+) = f'(x^-) = -\infty$ for all $x \in J$ is handled similarly). Consider an affine function $h(x) := cx+d$ such that $h(a) = f(a)$ and $h(b) = f(b)$, and define $g : J \longrightarrow M$ by $g(x) := f(x) - h(x)$. Then $g'(x^+) = g'(x^-) = +\infty$ and $g(a) = g(b) = 0$. By Exercise 7.6(2), g attains a maximum or a minimum at some $c \in J$. But if $g(c)$ is a maximum, then $g'(x^+) \leq 0$; and if $g(c)$ is a minimum, then $g'(x^-) \leq 0$, a contradiction. □

Next, we let $U \subseteq M^n$ be definable and open and $f : U \longrightarrow M$. Recall that f is differentiable at $a = (a_1, \ldots, a_n) \in U$ if and only if each partial derivative $\partial f/\partial x_i(a)$ exists in M. Another way to say this is as follows: for each $i = 1, \ldots, n$, let

$$U_{a_i} := \{t \in M : (a_1, \ldots, a_{i-1}, t, a_{i+1}, \ldots, a_n) \in U\},$$

and let $f_{a_i} : U_{a_i} \longrightarrow M$ be given by $f_{a_i}(t) := f(a_1, \ldots, a_{i-1}, t, a_{i+1}, \ldots, a_n)$. Then f is differentiable at a if and only if each f_{a_i} is differentiable at a_i; in particular, the set $D(f) := \{x \in U : f \text{ is differentiable at } x\}$ is definable and, by the previous corollary and cell decomposition, its complement $U \setminus D(f)$ is sparse.

Finally, let $f = (f_1, \ldots, f_k) : U \longrightarrow M^k$ be definable. Recall that f is differentiable at $a \in U$ if each f_j is differentiable at a. For $p \in \mathbb{N}$, we call f of **class** C^p if the following holds:

(i) if $p = 0$, then f is continuous;

(ii) if $p = 1$, then each $D(f_j) = U$ and each partial derivative $\partial f_j/\partial x_i$ is continuous;

(iii) if $p > 1$, then each partial derivative $\partial f_j/\partial x_i$ is of class C^{p-1}.

Corollary 8.4. *Let $p \in \mathbb{N}$. Then there exists a cell decomposition $\mathcal{C}$ of U such that, for every open $C \in \mathcal{C}$, the restriction of f to C is of class C^p.* □

Assume that f is differentiable at $a \in U$. We denote by $J_a f$ the jacobian matrix of f at a. The linear map $d_a f : M^n \longrightarrow M^k$ defined by $d_a f(x) := J_a f \cdot x$ is called the **differential** of f at a.

Exercise 8.5. State and prove the Inverse and Implicit Function Theorems for definable f.

Let $p \in \mathbb{N}$, and assume now that f is of class C^p. Moreover, let $g : \mathrm{gr}(f) \longrightarrow M^l$ be definable, and assume that $g \circ f : U \longrightarrow M^l$ is of class C^p. Then the function $G : U \times M^k \longrightarrow M^l$ given by $G(x, y) := g(x, f(x))$ is of class C^p and satisfies $G|_{\mathrm{gr}\, f} = g$. (What is $d_{(x,y)}G$?)

The previous observation leads to the following definition: let $S \subseteq M^n$ and $f : S \longrightarrow M^k$ be definable. We say that f is of **class** C^p if there exists an open set $U \supseteq S$ and a definable C^P-map $F : U \longrightarrow M^k$ such that $F|_S = f$.

Correspondingly, a cell $C \subseteq M^n$ is a C^p**-cell** if all the functions used in the construction of C are of class C^p.

Theorem 8.6 (**C^p-Cell Decomposition**). *Let $p \in \mathbb{N}$.*

(I)$_n$ Let $S_1, \dots, S_k \subseteq M^n$ be definable. Then there exists a C^p-cell decomposition $\mathcal{C}$ of M^n compatible with each S_i.

(II)$_n$ Let $f : S \longrightarrow M$ be definable, with $S \subseteq M^n$. Then there exists a C^p-cell decomposition $\mathcal{C}$ of M^n compatible with S such that, for each $C \in \mathcal{C}$, the restriction of f to C is of class C^p.

Proof. As for the Cell Decomposition Theorem, we proceed by induction on n. (I)$_1$ follows from o-minimality and (II)$_1$ from Lemma 8.3; so we assume $n > 1$ and the theorem hold for lower values of n.

To prove (I)$_n$, we let $\mathcal{C}$ be a cell decomposition of M^n compatible with each S_i. Now use (II)$_{n-1}$ to obtain a refinement $\mathcal{D}$ of $\Pi_{n-1}(\mathcal{C})$ by C^p-cells such that, for every $C \in \mathcal{C}$ of the form $\operatorname{gr} f$ with $f : \Pi_{n-1}(C) \longrightarrow M$ continuous and every $D \in \mathcal{D}$ contained in $\Pi_{n-1}(C)$, the restriction $f|_D$ is of class C^p. The corresponding refinement of $\mathcal{C}$ is a C^p-cell decomposition compatible with each S_i, as required.

For the proof of (II)$_n$, we need the following:

Claim. *Let $C \subseteq M^n$ be a cell and $g : C \longrightarrow M$ be definable, and let $p \in \mathbb{N}$. Then there is a definable, open subset C' of C such that $g|_{C'}$ is of class C^p and $\dim(C \setminus C') < \dim C$.*

To prove the claim, by (I)$_n$, we may assume that C is a C^p-cell of dimension k, say. Permuting the coordinates, if necessary, this means that $C = \operatorname{gr} h$ for some definable C^p-map $h : D \longrightarrow M^{n-k}$ with $D := \Pi_k(C)$ an open C^p-cell (why can we permute these coordinates?). It follows from Corollary 8.4 that there is a definable, open subset $D' \subseteq D$ such that $g \circ h|_{D'}$ is of class C^p and $\dim(D \setminus D') < \dim D$. Hence the restriction of g to the graph C' of $h|_{D'}$ is of class C^p and satisfies $\dim(C \setminus C') < \dim C$, as required.

We now return to the proof of the C^p-cell decomposition theorem: we proceed by induction on $\dim S$. If $\dim S = 0$, there is nothing to do, so we assume $\dim S > 0$ and (II)$_n$ holds for lower values of $\dim S$. By (I)$_n$, we may assume that S is a cell C, and we let C' be obtained from the claim and a C^p-cell decomposition be obtained from the inductive hypothesis applied to $f|_{C \setminus C'}$. Now let $\mathcal{C}$ be a C^p-cell decomposition of M^n compatible with each $D \in \mathcal{C}'$ and with C'. □

Assume now that $M = \mathbb{R}$, and let $f : U \longrightarrow \mathbb{R}^k$ be definable, with $U \subseteq \mathbb{R}^n$ open. We say that f is of **class** C^∞ if f is of class C^p for every $p \in \mathbb{N}$. We call f **real analytic at** $a \in U$ if there exists a convergent power series $F(X) \in \mathbb{C}[\![X]\!]$, with $X = (X_1, \dots, X_n)$, such that $f(a + x) = F(x)$ for all x in a neighbourhood of a. Note that, if f is analytic, then it is C^∞.

Correspondingly, we define C^∞**-cells** and **analytic cells** in analogy with the C^p definition above. We say that $\mathcal{M}$ **admits** C^∞**-cell decomposition** (resp., **analytic cell decomposition**) if Theorem 8.6 holds with "C^∞" (resp, "analytic") in place of "C^p".

Until the turn of the millenium, all known examples of o-minimal expansions of the real field admitted analytic cell decomposition. More recently, examples were constructed that show C^p-cell decomposition for finite p to be optimal, even for o-minimal expansions of the real field:

Theorem 8.7. (1) *There exist o-minimal expansions of the real field that admit C^∞-cell decomposition, but not analytic cell decomposition [10].*

(2) *There exist o-minimal expansions of the real field that do not admit C^∞-cell decomposition [5].*

Grassmannians

For $k, l \in \mathbb{N}$, we identify the M-vector space $M_{k,l}(M)$ of all M-valued $(k \times l)$-matrices with M^{kl} via the map $A = (a_{ij}) \mapsto z_A = (z_1, \ldots, z_{kl})$ defined by $a_{ij} = z_{k(i-1)+j}$. As usual, we write $M_n(M)$ in place of $M_{n,n}(M)$.

Let $l \leq n$. I denote by $G_n^l(M)$ the **Grassmannian** of all l-dimensional vector subspaces of M^n. This $G_n^l(M)$ is a definable variety with a natural embedding into the vector space $M_n(M)$: each l-dimensional vector space E is identified with the unique matrix A_E (with respect to the standard basis of M^n) corresponding to the orthogonal projection on the orthogonal complement of E (see Section 3.4.2 of [1] for the case $M = \mathbb{R}$); in particular, $E = \ker(A_E)$. We identify $G_n^l(M)$ with its image in M^{n^2} under the above map. Note that, under the above identification, $G_n(M) := \bigcup_{p=0}^n G_n^p(M)$ is definable in $\mathcal{M}$ and the sets $G_n^0(M), \ldots, G_n^n(M)$ are the definably connected components of $G_n(M)$.

Example 8.8. Let $C \subseteq M^n$ be a C^1-cell of dimension k. Under the above identification, we can view the tangent bundle TC of C as the graph of the definable map $g_C : C \longrightarrow G_n^k(M)$ given by

$$g_C(x) := T_x C.$$

The map g_C is also called the **Gauss map** of C.

Chapter 9

Polynomially bounded vs. exponential, and an open question

Let $\mathcal{M}$ be an o-minimal expansion of an ordered field $(M, <, +, \cdot, 0, 1)$. We call $\mathcal{M}$ **polynomially bounded** if for every definable function $f : M \longrightarrow M$, there exist $n \in \mathbb{N}$ and $a \in M$ such that $f(x) \leq x^n$ for all $x > a$.

Example 9.1. The real field $\overline{\mathbb{R}}$ is polynomially bounded: by Tarski's Theorem, $\overline{\mathbb{R}}$ admits quantifier elimination and universal axiomatization in the language

$$\mathcal{L} = \big(<, +, \cdot, 0, 1, (\sqrt[n]{\ })_{n \in \mathbb{N}}\big) ;$$

now apply Exercise 9.2 below.

Exercise 9.2. Assume $\mathcal{M}$ admits quantifier elimination and is universally axiomatized. Prove that every definable function $f : M^n \longrightarrow M$ is piecewise given by terms, that is, for each such f there exist $k \in \mathbb{N}$ and terms $t_1, \dots, t_k$ such that

$$\mathcal{M} \models \forall x \big(f(x) = t_1(x) \vee \cdots \vee f(x) = t_k(x)\big).$$

Clearly, the exponential function is not definable in any polynomially bounded expansion of the real field.

More generally, a **power function** is a group isomorphism ϕ from $(M^{>0}, \cdot, 1)$ onto itself. For a definable power function ϕ, the definable element $\mu := \lim_{x \to 1} x \cdot \phi'(x)/\phi(x)$ of M is called the **definable exponent** of ϕ, and we usually write $x^\mu = \phi(x)$ for $x > 0$ and set $x^\mu := 0$ for $x \leq 0$. We denote by $K = K(\mathcal{M})$ the set of all definable exponents of $\mathcal{M}$; note that K is a subfield of M.

Exercise 9.3. What are $K\left(\overline{\mathbb{R}}\right)$ and $K(\mathbb{R}_{\text{exp}})$? Is $K(\mathcal{M})$ a definable subfield of M?

We call $\mathcal{M}$ **power-bounded** if, for every definable function $f : M \longrightarrow M$, there exist $\mu \in K$ and $a \in M$ such that $f(x) \leq x^\mu$ for all $x > a$. Every polynomially bounded $\mathcal{M}$ is power bounded.

In the same spirit, an **exponential function** is a group isomorphism ψ from $(M, +, 0)$ onto $(M^{>0}, \cdot, 1)$. If $\mathcal{M}$ is power bounded, no exponential function is definable in $\mathcal{M}$. A fundamental fact about o-minimal structures is the following:

Theorem 9.4 (Miller [7]). *$\mathcal{M}$ is either power bounded, or there is a unique exponential function definable in $\mathcal{M}$.*

Thus, we call $\mathcal{M}$ **exponential** if an exponential function is definable in $\mathcal{M}$. We call $\mathcal{M}$ **exponentially bounded** if either $\mathcal{M}$ is power bounded, or $\mathcal{M}$ is exponential with unique exponential function exp and, for every definable function $f : M \longrightarrow M$, there exist $n \in \mathbb{N}$ and $a \in M$ such that $f(x) < \exp(\exp(\cdots(\exp(x))\cdots))$ (n times) for all $x > a$.

Fact 9.5 (based on [6]). *Every known (as of April 2011) o-minimal expansion of the real field is exponentially bounded.*

Question 9.6. *Are there transexponential o-minimal structures, that is, o-minimal structures that are exponential but not exponentially bounded?*

One of the principal tools in studying these questions is the Hardy field: for real functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$, we set $f \sim g$ if there exists $a \in \mathbb{R}$ such that $f|_{(a,+\infty)} = g|_{(a,+\infty)}$. The corresponding equivalence classes are called **germs at** $+\infty$ of real functions; they are added and multiplied in the obvious way. Such a germ is **differentiable** if it has a representative that is differentiable on some interval $(a, +\infty)$. A **Hardy field** is a field of differentiable germs at $+\infty$ that is also closed under differentiation.

Exercise 9.7. (1) Prove that the field $\mathbb{R}(x)$ of all rational functions is a set of representatives of a Hardy field (also denoted by $\mathbb{R}(x)$).

(2) Assume that $M = \mathbb{R}$. Prove that the set G defined before Exercise 7.4 is a Hardy field.

In view of the previous exercise, we call the set G defined before Exercise 7.4 for arbitrary $\mathcal{M}$ the **Hardy field associated to** $\mathcal{M}$ and denote it by $\mathcal{H} = \mathcal{H}(\mathcal{M})$.

Assume now that $M = \mathbb{R}$. By the previous exercise, the o-minimality of $\mathcal{M}$ implies that $\mathcal{H}$ is a Hardy field. Thus, if we want to find a transexponential o-minimal structure, there must also be a transexponential Hardy field, that is, a Hardy field containing a germ that is larger than the germ of any finite compositional iterate of exp. Such Hardy fields do exist: consider the functional equation

$$f(x + 1) = \exp(f(x)). \tag{9.1}$$

Theorem 9.8 (Boshernitzan). *The functional equation* (9.1) *has a solution f that generates a Hardy field over $\mathbb{R}(x)$.*

However, generating a Hardy field is not sufficient for generating an o-minimal structure:

Theorem 9.9 (Rolin, Sanz and Schaefke [9]). *There exists a Hardy field that is not the Hardy field of any o-minimal expansion of the real field.*

Thus, Question 9.6 remains open. If there does exist a solution f of the functional equation 9.1 that generates an o-minimal structure, however, it is amusing to consider the following consequence: the function $x \mapsto f(f^{-1}(x) + y)$ would then define the yth iterate of exp, for every $y \in \mathbb{R}$.

Chapter 10

Exponential Polynomials

In this chapter, we consider the following question: given a polynomial $P(x, y) \in \mathbb{R}[x, y]$, with $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, does the zeroset (or positivity set, or negativity set) of the exponential polynomial $P(x, \exp x)$ have finitely many connected components, where $\exp x := (\exp x_1, \dots, \exp x_n)$? This question represents a first step towards proving the o-minimality of $\mathbb{R}_{\exp} := (\mathbb{R}, <, +, \cdot, 0, 1, \exp)$ but, as we shall see, its solution leads to a theory for a large class of functions called **pfaffian functions** relative to any given o-minimal expansion of the real field.

We first consider the special case $x = x_1$ and $y = y_1$. Then the zeroset of $P(x, \exp x)$ is given by the projection on the first coordinate of the intersection of the zeroset of $P(x, y)$ with the graph of exp.

Khovanskii's point of view [2]

The answer to our question in this case is based on a version of Rolle's Theorem. Rather than viewing the latter as a theorem about differentiable functions, we view it as a theorem about the line field $d_{\text{horizontal}} : \mathbb{R}^2 \longrightarrow G_2^1$ given by $d_{\text{horizontal}}(x, y) := \{y = 0\}$ and horizontal affine lines:

Rolle's Theorem. *Let $L = \{y = 0\} + a$ for some $a \in \mathbb{R}$, and let $\gamma : [0, 1] \longrightarrow \mathbb{R}^2$ be a C^1 curve such that $\gamma(0), \gamma(1) \in L$. Then there exists $t \in [0, 1]$ such that $\gamma'(t)$ is tangent to $d(\gamma(t))$, that is, $\gamma'(t) \in d(\gamma(t))$.*

Khovanskii realized that this theorem is true for other line fields: for example, for $x \in \mathbb{R}^2$ we let $d_{\exp}(x)$ be the kernel of the 1-form $\omega_{\exp} := dy_1 - y_1 dx_1$, that is, $d_{\exp}(x)$ is the orthogonal complement of the vector $(-y_1, 1)$ in $\mathbb{R}^2$. Note that this line field is definable in the real field.

Lemma 10.1 (Rolle-Khovanksii). *Let $\gamma : [0, 1] \longrightarrow \mathbb{R}^2$ be a C^1 curve such that $\gamma(0), \gamma(1) \in$ $\mathrm{gr}(\exp)$. Then there exists $t \in [0, 1]$ such that $\gamma'(t) \in d_{\exp}(\gamma(t))$.*

Proof. Write $d = d_{\exp}$ and $\omega = \omega_{\exp}$. Since exp is C^1 and total, each of the sets $C_1 := \{(x, y) \in \mathbb{R}^{n+1} : y < \exp(x)\}$ and $C_2 := \{(x, y) \in \mathbb{R}^{n+1} : y > f \exp(x)\}$ is connected and $\mathrm{gr}(\exp)$ is a closed leaf of d. Let $\gamma : [0, 1] \longrightarrow \mathbb{R}^{n+1}$ be a curve with $\gamma(0), \gamma(1) \in \mathrm{gr}(\exp)$. Note that the continuous function $t \mapsto \omega(\gamma(t))(\gamma'(t)) : [0, 1] \longrightarrow \mathbb{R}$ is a measure of the orientation of $\gamma'(t)$ with respect to $d(\gamma(t))$, with $\omega(\gamma(t))(\gamma'(t)) = 0$ if and only if $\gamma'(t) \in d(\gamma(t))$. Without loss of generality, we may assume that $\omega(\gamma(0))\big(\gamma'(0)\big)$ and $\omega(\gamma(1))\big(\gamma'(1)\big)$ are both nonzero and $\gamma\big((0,1)\big)$ is contained in either C_1 or C_2.

We now claim that $\omega(\gamma(0))\big(\gamma'(0)\big)$ and $\omega(\gamma(1))\big(\gamma'(1)\big)$ must have opposite signs. For if $\omega(\gamma(0))\big(\gamma'(0)\big) > 0$, say, there is an $\epsilon > 0$ such that $\gamma\big((0,\epsilon)\big) \subseteq C_1$, and so by the above $\gamma\big((0,1)\big) \subseteq C_1$; but if also $\omega(\gamma(1))\big(\gamma'(1)\big) > 0$, there is a $\delta > 0$ such that $\gamma\big((\delta,1)\big) \subseteq C_2$, so that $\gamma\big((0,1)\big) \subseteq C_2$, a contradiction. We obtain a similar contradiction if both $\omega(\gamma(0))\big(\gamma'(0)\big)$ and $\omega(\gamma(1))\big(\gamma'(1)\big)$ are negative, so the claim is proved.

It follows from the claim and Rolle's Theorem that there exists a $t \in (0,1)$ such that $\omega(\gamma(t))\big(\gamma'(t)\big) = 0$. This is equivalent to saying that $\gamma'(t) \in d(\gamma(t))$, so the lemma is proved. $\square$

Back to the zeroset of $P(x, \exp(x))$: let now $\mathcal{C}$ be a C^1-cell decomposition (definable in the real field) compatible with both the zeroset of $P(x,y)$ and with $d_{\exp}$, where the latter means that, for each nonopen $C \in \mathcal{C}$, either C is transverse to d (at every point of C), or C is tangent to d (at every point of C).

Lemma 10.2. *Let $C \in \mathcal{C}$ be nonopen. Then $C \cap \mathrm{gr}(\exp)$ has at most one connected component.*

Proof. If $\dim C = 0$, this is obvious, so we assume $\dim C = 1$. If C is tangent to $d_{\exp}$, then either $C \subseteq \mathrm{gr}(\exp)$ or $C \cap \mathrm{gr}(\exp) = \emptyset$; so we may assume that C is transverse to $d_{\exp}$. Note that then $C \cap \mathrm{gr}(\exp)$ is discrete; assume for a contradiction that $C \cap \mathrm{gr}(\exp)$ contains at least two points. Since C is connected and hence path connected, there is a C^1-curve $\gamma : [0,1] \longrightarrow C$ such that $\gamma(0), \gamma(1) \in \mathrm{gr}(\exp)$. It follows from Lemma 10.1 that $\gamma'(t) \in d_{\exp}(\gamma(t))$ for some $t \in [0,1]$, that is, C is tangent to $d_{\exp}$ at the point $\gamma(t)$, a contradiction. $\square$

Now, if P is not the zero polynomial, then only nonopen cells in $\mathcal{C}$ are contained in the zeroset of P. Thus, the zeroset of $P(x, \exp(x))$ has at most as many connected components as $\mathcal{C}$ has nonopen cells.

This argument generalizes to all exponential polynomials: for instance, to study the zeroset of $P(x_1, x_2, \exp x_1, \exp x_2)$, we consider the graph of $(x_1, x_2, y_2) \mapsto \exp x_1$ as tangent to the 3-plane field associated to the 1-form $d_1 := dy_1 - y_1 dx_1$, and we consider the graph of $(x_1, x_2, y_1) \mapsto \exp x_2$ as tangent to the 3-plane field associated to the 1-form $d_2 := dy_2 - y_2 dx_2$. Corresponding versions of Lemmas 10.1 and 10.2 go through; but at this point, it is worth introducing some general terminology.

Pfaffian functions

We fix an o-minimal expansion $\mathcal{R}$ of the real field. A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is **pfaffian over $\mathcal{R}$** if there are definable functions $P_i : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ such that

$$\frac{\partial f}{\partial x_i}(x) = P_i(x, f(x)) \quad \text{for } i = 1, \dots, n \text{ and } x \in \mathbb{R}^n. \tag{10.1}$$

Examples 10.3. For instance, exp is pfaffian over the real field, and hence over $\mathcal{R}$. The function log is not pfaffian over $\overline{\mathbb{R}}$ (because log is not total, i.e., defined on all of $\mathbb{R}$), but the function $x \mapsto \log\left(1 + x^2\right)$ is pfaffian over $\overline{\mathbb{R}}$. Similarly, the function arctan is pfaffian over $\overline{\mathbb{R}}$. Every antiderivative of a definable function from $\mathbb{R}$ to $\mathbb{R}$ is pfaffian over $\mathcal{R}$, but not necessarily definable: $\log\left(1 + x^2\right)$ is not definable in $\overline{\mathbb{R}}$ by quantifier elimination and analytic continuation. Finally, the functions $(x_1, x_2, y_2) \mapsto \exp x_1$ and $(x_1, x_2, y_1) \mapsto \exp x_2$ are pfaffian over the real field, and hence over $\mathcal{R}$.

The connection between pfaffian functions and Rolle leaves is a straightforward generalization of Lemma 10.1: Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be pfaffian over $\mathcal{R}$, and let $P_1, \dots, P_n : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ be definable such that $\partial f/\partial x_i(x) = P_i(x, f(x))$ for all $x \in \mathbb{R}^n$. Then

$$df(x) = P_1(x, f(x))dx_1 + \cdots P_n(x, f(x))dx_n;$$

thus, for $(x, y) \in \mathbb{R}^{n+1}$, we let d_f be the kernel of the 1-form $\omega_f := dy - P_1dx_1 - \cdots - P_ndx_n$. Note that d_f is definable and that $\mathrm{gr}(f)$ is an analytic submanifold of $\mathbb{R}^{n+1}$ that is tangent to d_f (at every point).

Lemma 10.4 (Khovanskii [3]). *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be pfaffian over $\mathcal{R}$. Then the graph $\mathrm{gr}\, f$ of f is a Rolle leaf over $\mathcal{R}$.*

Proof. The proof now goes exactly as the proof of Lemma 10.1, with $d = d_f$ and $\omega = \omega_f$. □

Thus, to prove Lemma 10.2 for arbitrary exponential polynomials, we want to repeat its proof; however, we need one other observation. To explain this, fix $n \in \mathbb{N}$ and a polynomial $P(x, y)$ with $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, and write e_i for the function $(x, y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n) \mapsto \exp(x_i)$. Then each e_i is pfaffian over $\overline{\mathbb{R}}$ with $\omega_i := \omega_{e_i} = dy_i - y_idx_i$ and $d_i := d_{e_i} = \ker \omega_i$.

We call a C^1-cell $C \subseteq \mathbb{R}^{2n}$ **compatible** with $\{d_1, \dots, d_n\}$ if, for every $I \subseteq \{1, \dots, n\}$, the definable map $d_{C,I} : C \longrightarrow G_n$ given by

$$d_{C,I}(x, y) := \dim T_{(x,y)}C \cap \bigcap_{i \in I} d_i(x, y)$$

has constant dimension, and we denote this dimension by $\dim d_{C,I}$. In this situation, it follows from the rank theorem that, for $I \subseteq \{1, \dots, n\}$, the set $C_I := C \cap \bigcap_{i \in I} \mathrm{gr}(e_i)$ is a C^1-submanifold of C of dimension $\dim d_{C,I}$. A cell decomposition $\mathcal{C}$ is **compatible** with $\{d_1, \dots, d_n\}$ if each $C \in \mathcal{C}$ is compatible with $\{d_1, \dots, d_n\}$.

Exercise 10.5. Prove that there exists a C^2-cell decomposition that is compatible with the zeroset of P and with $\{d_1, \dots, d_n\}$.

We now fix a C^2-cell decomposition $\mathcal{C}$ as obtained from the previous exercise.

Lemma 10.6. *Let $C \in \mathcal{C}$ and $I \subseteq \{1, \dots, n\}$. Then the number of components of C_I is finite.*

Proof. We proceed by induction on $\dim C$ and $|I|$; if $\dim C = 0$ or $|I| = 0$, there is nothing to do, so we assume $\dim C > 0$ and $|I| > 0$ and the claim holds for lower values of $\dim C$ or $|I|$. We are now distinguishing two cases:

Case 1: $\dim d_{C,I} = 0$. In this case, we proceed as in Lemma 10.2: fix an $i \in I$, and put $I' := I \setminus \{i\}$. Then $\dim d_{C,I'} \in \{0, 1\}$, because d_i is a codimension one subspace field. If $\dim d_{C,I'} = 0$, then $C_{I'}$ is finite by the inductive hypothesis, so C_I is finite as well. So we assume that $\dim d_{C,I'} = 1$; therefore, $C_{I'}$ is a 1-dimensional submanifold of C, and by the inductive hypothesis has finitely many connected components. Arguing as in the proof of Lemma 10.2, using Lemma 10.4 applied to $\mathrm{gr}(e_j)$, shows that C_I has finitely many connected components as well.

Case 2: $\dim d_{C,I} > 0$. In this case, we want to find a definable set $S \subseteq C$ such that $\dim S < \dim C$ and every component of C_I contains a point of S. Assuming such an S can be found, we then finish by refining $\mathcal{C}$ compatibly with S and $\{d_1, \dots, d_n\}$ and applying the inductive hypothesis.

To find such an S, we use a definable variant of Morse functions: a definable C^1-function $f : C \longrightarrow \mathbb{R}$ is a **carpeting function** if $f(x) > 0$ for all $x \in C$, $\lim_{x \to y} f(x) = 0$ for all $y \in \mathrm{fr}(C)$ and $f^{-1}(K)$ is compact for every compact $K \subseteq (0, \infty)$.

Given a carpeting function $f : C \longrightarrow \mathbb{R}$, we obtain a candidate for S by the Lagrange multiplier principle: since each component D of C_I is a closed submanifold of C, the function $\phi|_D$ attains a maximum at some point $x \in D$. This point also belongs to the definable set

$$S_f := \{x \in C : \ \nabla f(x) \text{ is orthogonal to } d_{C,I}(x)\}.$$

Thus, it suffices to show that there exists a carpeting function f for which $\dim S_f < \dim C$. By definition of C^2-cell, there exists a definable C^2-diffeomorphism $\phi : \mathbb{R}^{\dim C} \longrightarrow C$. Pulling back via ϕ (this is where we use C^2 rather than C^1), we may assume that $C = \mathbb{R}^m$ for some $m \le 2n$, and we write $d_I = d_{C,I}$. In this situation, for each $u \in (0, \infty)^m$ the map $\phi_u : \mathbb{R}^m \longrightarrow (0, \infty)$ given by

$$\phi_u(x) := 1/(u_1 x_1^2 + \cdots + u_m x_m^2)$$

defines a carpeting function on $\mathbb{R}^m$, and the family of all ϕ_u is clearly definable. Thus, we are done once we establish the following

Claim. *There exists* $u \in (0, \infty)^m$ *such that* $\dim S_{\phi_u} < m$.

To see this, assume for a contradiction that $\dim S_{\phi_u} = m$ for all $u \in (0, \infty)^m$. Then $\dim S = 2m$, where

$$S := \{(u, x) \in \mathbb{R}^m \times \mathbb{R}^m : \ u_1 > 0, \dots, u_m > 0, \ x \in S_{\phi_u}\},$$

so there are nonempty open $V \subseteq (0, \infty)^m$ and $W \subseteq \mathbb{R}^m$ such that $V \times W \subseteq S$. Fix some $x \in W$ with all $x_i \neq 0$ and let u range over V. Note that

$$\nabla \phi_u(x) = -\frac{(2u_1 x_1, \dots, 2u_m x_m)}{(1 + u_1 x_1^2 + \cdots + u_m x_m^2)^2}.$$

Therefore the vector space generated by all $\nabla \phi_u(x)$ as u ranges over V has dimension m, that is, the intersection of all their orthogonal complements, as u ranges over V, is trivial. But by assumption, $d_I(x)$ is contained in this intersection, which contradicts $\dim d_I > 0$. $\square$

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