Math functions in context – Differentiation

1. The following excerpt is taken from Connection between oligomeric state and gating characteristics of mechanosensitive ion channels. Christoph A. Haselwandter and Rob Phillips. PLoS Computational Biology. 9.5 (May 2013)

   The central quantity in this model is the channel opening probability

   \[ P_o = \frac{1}{1 + e^{\beta(\Delta \mathcal{G} - \tau \Delta A)}} \]  \hspace{1cm} (1)

   where \( \beta = 1/k_B T \), in which \( k_B \) is Boltzmann’s constant and \( T \) is the temperature, \( \Delta \mathcal{G} \) is the total free energy difference between the open and closed states of MscL, \( \tau \) is the membrane tension, and \( \Delta A \) is the area difference between the open and closed channel states.

   From the context we figured out that all parameters are positive numbers.

   Show that \( P_o \) is an increasing function of \( \tau \).

the average of membrane potentials of neurons in the element, that is

\[ V = \frac{N_e V_e + N_i V_i}{N_e + N_i} \]

where \( N_e, N_i \) are the numbers of excitatory and inhibitory neurons and \( V_e \) and \( V_i \) are the (average) membrane potentials of excitatory and inhibitory neuron populations respectively.

You know that the numbers \( N_e \) and \( N_i \) are positive, and the membrane potentials \( V_e \) and \( V_i \) are negative.

(a) Assume that \( V \) is a function of \( V_e \). Find its derivative and interpret your answer.
For (b) and (c) assume that $V$ is a function of $N_e$.

(b) Find the derivative of $V$ and interpret your answer.

(c) Find the relation between $V_e$ and $V_i$ which makes the graph of $V$ concave up.
3. In the paper *Evaluating the adequacy of gravity models as a description of human mobility for epidemic modelling*. James Truscott and Neil M. Ferguson. PLoS Computational Biology. 8.10 (Oct. 2012), we read the following:

Comparison of the behaviour of the epidemic model on different networks is based on the times to first infection for network nodes from a given initial infection site. Under certain simplifying assumptions, an approximation for the mean time to infection between two nodes can be calculated for the above epidemiological model.

\[
\hat{t} \approx \frac{-1}{r} \left\{ \ln \left( \frac{\beta_u \beta_h \Lambda_1 \zeta_1}{\beta_u \zeta_1 + \sigma + r} + \frac{\beta_w \beta_h \Lambda_2 \zeta_2}{\beta_w \zeta_2 + \sigma + r} \right) - \gamma \right\}
\]

(5)

The parameters \( \Lambda_{1,2} \) and \( \zeta_{1,2} \) represent network properties, such as the fraction of journeys between the two nodes. \( r = \beta_h + \beta_w + \sigma \) is the epidemic growth rate in a large node. The equation illustrates how

Assume that \( \Lambda_2 = 0 \), and that the approximately equals sign is actually an equals sign. Find the rate of change of \( \hat{t} \) with respect to \( \gamma \).
4. The resistance $R$ of the flow of blood through a blood vessel (assumed to have the shape of a cylindrical tube) is given by

$$R = \frac{Kl(\gamma + 1)^2}{d^4}$$

where $l$ is the length of the tube, $d$ is its diameter and $\gamma \geq 0$ is the curvature. The positive constant $K$ represents the viscosity of the blood (viscosity is a measure of the resistance of fluid to stress; water has low viscosity, honey has high viscosity).

(a) Find the derivative of $R$ with respect to $l$ and interpret your answer. Does it make sense?

(b) Find the derivative of $R$ with respect to $d$ and interpret your answer. Does it make sense?