Optimal Investment, Consumption and Life Insurance under Mean-Reverting Returns: The Complete Market Solution

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May 14, 2012

Abstract

This paper considers the problem of optimal investment, consumption and life insurance acquisition for a wage earner who has CRRA (constant relative risk aversion) preferences. The market model is complete, continuous, the uncertainty is driven by Brownian motion and the stock price has mean reverting drift. The problem is solved by dynamic programming approach and the HJB equation is shown to have closed form solution. Numerical experiments explore the impact market price of risk has on the optimal strategies.

Keywords: Portfolio allocation; Life insurance; Mean reverting drift.

1 Introduction

The goal of this paper is to provide optimal investment, consumption and life insurance acquisition strategies for a wage earner who uses an expected utility criterion with CRRA type preferences. Existing works on optimal investment, consumption and life insurance acquisition use a financial model in which risky assets are modelled as geometric Brownian motions. In order to make the model more realistic, we allowed the risky assets to have stochastic drift parameters. Let us review the literature on optimal investment consumption and life insurance.

The investment/consumption problem in a stochastic context was considered by Merton [7] and [8]. His model consists in a risk-free asset with constant rate of return and one or more stocks, the prices of which are driven by geometric Brownian motions. The horizon $T$
is prescribed, the portfolio is self-financing, and the investor seeks to maximize the expected utility of intertemporal consumption plus the final wealth. Merton provided a closed form solution when the utilities are of constant relative risk aversion (CRRA) or constant absolute risk aversion (CARA) type. It turns out that for (CRRA) utilities the optimal fraction of wealth invested in the risky asset is constant through time. Moreover for the case of (CARA) utilities, the optimal strategy is linear in wealth.


The novelty of our work is that we allow for stochastic drift parameter. More precisely the stock price is assumed to have mean reverting drift. This stock price model was used by Kim and Omberg [5] who managed to obtain closed form solutions for the optimal investment strategies. Later, Watcher [14] added intertemporal consumption to the model considered by Kim and Omberg. Furthermore, optimal strategies are obtained in closed form and an empirical analysis has shown that under some assumptions the stock drift is mean reverting for realistic parameter values. We solve within this framework the Hamilton Jacobi Bellmann (HJB) equation associated with maximizing the expected utility of consumption, terminal wealth and legacy and this in turn provides the optimal investment consumption and life insurance acquisition. By looking at the explicit solutions we found out that: 1) when the wage earner accumulates enough wealth (to leave to his/her heirs), he/she considers buying pension annuities rather than life insurance 2) when the value of human capital is small the wage earner buys pension annuities to supplement his/her income 3) when the wage earner becomes older (so the hazard rate gets higher) the wage earner has a higher demand (in absolute terms) for life insurance/pension annuities.

In our special model, due to the same risk preference for investment, consumption and legacy, the optimal legacy (wealth plus insurance benefit) is related to the optimal consumption. More precisely the ratio between the optimal legacy and optimal consumption is a deterministic function which depends on the hazard rate, risk aversion, premium insurance ratio, the weight on the bequest, and is independent of the financial market characteristics. In a frictionless market (i.e. hazard rate equals the premium insurance ratio) it turns out that the optimal legacy equals the optimal consumption.

Our main motivation in writing this paper is to explore the impact stochastic market price of risk has on the optimal investment strategies. This could not be observed by previous works which assumed a geometric Brownian motion model for the stock prices. We found out that the optimal investment strategy (in the stock) is significantly affected by the market price of risk (MPR). Moreover, the optimal investment in the stock is increasing in the MPR. As for the optimal insurance we found two patterns depending on the risk aversion. A risk averse wage earner pays less for life insurance if MPR is increasing up to a certain threshold; when the MPR
exceeds that threshold, the amount the wage earner pays for life insurance starts to decrease. A risk seeking wage earner has a different behaviour; thus he/she pays less for life insurance when the MPR increases.

**Organization of the paper**: The remainder of this paper is organized as follows. In Section 2 we describe the model and formulate the objective. Section 3 performs the analysis. Numerical results are discussed in Section 4. Section 5 concludes. The paper ends with an appendix containing the proofs.

## 2 The Model

In this paper, we assume that a wage earner has to make decisions regarding consumption, investment and life insurance/pension annuity purchase. Let $T > 0$ be a finite benchmark time horizon, and $\{W(t)\}_{t \in [0,T]}$ a 1-dimensional Brownian motion on a probability space $(\Omega, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathcal{F}, \mathbb{P})$. The filtration $\{\mathcal{F}_t\}$ is the completed filtration generated by $\{W(t)\}_{t \in [0,T]}$. Let $\mathbb{E}$ denote the expectation with respect to $\mathbb{P}$. The continuous time economy consists of a financial market and an insurance market.

### 2.1 The financial market

The financial market contains a risk-free asset earning interest rate $r \geq 0$ and one risky asset. By some notational changes we can address the case of multiple risky assets. The asset prices evolve according to the following equations:

$$dB(t) = rB(t)dt, \quad B(0) = 1,$$

$$dS(t) = S(t) [\mu(t) dt + \sigma dW(t)].$$

Moreover, the market price of risk $\{\theta(t)\}_{t \in [0,T]} = \{\frac{\mu(t) - r}{\sigma}\}_{t \in [0,T]}$ is a mean reverting process, i.e.,

$$d\theta(t) = -k(\theta(t) - \bar{\theta})dt - \sigma \theta dW(t).$$

Here $\sigma, \sigma_\theta, k, \bar{\theta}$ are positive constants. This model for the stock price was considered by [5] and [14]. In addition, we assume that the wage earner receives income at the random rate $i$ which follows a geometric Brownian motion

$$di(t) = i(t)(\nu_i dt + \sigma_i dW(t)),$$

with positive constants $\nu_i$ and $\sigma_i$.

### 2.2 The life insurance

We assume that the wage earner is alive at $t = 0$ and his/her lifetime is a non-negative random variable $\tau$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and independent of the Brownian motion $\{W(t)\}_{t \in [0,T]}$. Let us introduce the hazard function $\lambda(t) : [0, T] \rightarrow \mathbb{R}^+$, that is, the instantaneous death rate, defined by
\( \lambda(t) = \lim_\varepsilon \to 0 \frac{\mathbb{P}(t \leq \tau < t + \varepsilon \mid \tau \geq t)}{\varepsilon}. \)

From the above definition it follows that:

\[
\mathbb{P}(\tau < s \mid \tau > t) = 1 - \exp\left\{-\int_t^s \lambda(u) \, du\right\}, \tag{2.1}
\]

and

\[
\mathbb{P}(\tau > T | \tau > t) = \exp\left\{-\int_t^T \lambda(u) \, du\right\}. \tag{2.2}
\]

Denote by \( f(s; t) \) the conditional probability density for the death at time \( s \) conditional upon the wage earner being alive at time \( t \leq s \). Thus

\[
f(s; t) = \lambda(s) \exp\left(-\int_t^s \lambda(v) \, dv\right). \tag{2.3}
\]

Here \( F(s; t) \) denotes the conditional probability for the wage earner to be alive at time \( s \) conditional upon being alive at time \( t \leq s \); consequently

\[
F(s; t) = \exp\left(-\int_t^s \lambda(v) \, dv\right). \tag{2.4}
\]

In our model the wage earner purchases term life insurance/pension annuity with the term being infinitesimally small. Premium is paid (life insurance) or received (pension annuity) continuously at rate \( p(t) \) given time \( t \). In compensation, if the wage earner dies at time \( t \) when the premium payment rate is \( p(t) \), then either the insurance company pays an insurance amount \( \frac{p(t)}{\eta(t)} \) if \( p(t) \) is positive, or the amount \( \frac{p(t)}{\eta(t)} \) should be paid by wage earner’s family if \( p(t) \) is negative. Here \( \eta: [0, T] \to \mathbb{R}^+ \) is a continuous, deterministic, prespecified function that called premium-insurance ratio, and \( \frac{1}{\eta(t)} \) is referred to as loading factor. In a frictionless market \( \eta(t) = \lambda(t) \), but due to commission fees \( \eta(t) > \lambda(t) \). In order to simplify the analysis we assume that \( \eta(t) = \lambda(t) \).

The insurance market model was pioneered by [12] who considered the problem of optimal financial planning decisions for an individual with an uncertain lifetime. Later, that model was extended by [11]. Many works followed [12] and [11] by considering instantaneous term life insurance; it means that the investor can only purchase life insurance for the next instant; if surviving the next instant the investor has to buy again the instantaneous term life insurance and so on. The purchase of an annuity could be reversed by buying a life insurance that guarantees the payment of annuity’s face amount on the death of the holder. Indeed there are situations of life annuities when premium is paid only at death (e.g. reverse mortgage). This brings up the notion of instantaneous pension annuity as the reverse of instantaneous term life insurance.

One has to admit that instantaneous term life insurance products can not be purchased in the real world and one can not find the exact same payoff structure in term life insurance products. These fictitious instantaneous term life insurance products are used in portfolio management because they offer a simplified and tractable model which can be used as a benchmark. Another drawback of this modelling approach comes from the irreversibility of pension annuities. In real world, term or lifelong survival contingent products pay a cashflow for the length of the term or as long as the buyer is alive. However, the introduction of life long payments may necessitates complicated numerical treatments and these models may not be tractable.
2.3 The Objective

The wage earner starts with wealth \( x \in \mathbb{R}^+ \) and receives income at a predictable rate \( i(t) \) during the period \([0, \min\{T, \tau\}]\); it means that the income will be terminated by the investor’s death or benchmark time \( T \), whichever happens first. At every time \( t \), the wage earner has wealth \( X(t) \) and makes the following decisions: he/she chooses \( \pi(t) \), the fractions for the wealth process invested in the risky asset, \( c(t) \) the consumption rate, and \( p(t) \), the proportion of the wealth to be paid (received) for life insurance (pension annuity). The wealth process \( \{X_{\pi,c,p}(t)\}_{t \in [0,T]} = \{X(t)\}_{t \in [0,T]} \) satisfies on \([0, \min\{T, \tau\}]\) the self-financing equation

\[
dX(t) = [X(t)(r + (\mu(t) - r)\pi(t) - c(t) - p(t)) + i(t)] \, dt + X(t)\pi(t)\sigma dW(t). \tag{2.5}
\]

The initial wealth \( X(0) = x \) is a primitive of the model. The wage earner total legacy if he/she dies at time \( t \) is the wealth plus insurance amount

\[
I(t) = X(t) \left(1 + \frac{p(t)}{\eta(t)}\right). \tag{2.6}
\]

In order to evaluate the performance of an investment-consumption-insurance strategy the wage earner uses an expected utility criterion. For a strategy \( \alpha = \{\pi, c, p\} \) and its corresponding wealth process, let us define

\[
J(t, x; \alpha) = \mathbb{E}_{t,x} \left[ \int_t^{T \wedge \tau} U_\gamma(s, c(s)X(s)) \, ds + \beta U_\gamma(\tau, I(\tau))1_{\{\tau \leq T\}} + U_\gamma(T, X(T))1_{\{\tau > T\}} \right], \tag{2.7}
\]

where \( \mathbb{E}_{t,x} \) is the conditional expectation operator, given \( X(t) = x \), and \( U_\gamma \) is the wage earner utility. It is further assumed that time preferences are exponential and risk preferences are of CRRA type, i.e., \( U_\gamma(s, x) = e^{-\rho(s-t)\frac{x^\gamma}{\gamma}} \) with \( \gamma < 1 \), and \( \rho > 0 \). Here \( 1 - \gamma \) stands for the coefficient of relative risk aversion. The positive constant \( \beta \) is the weight on the wage earner legacy’s utility. A high \( \beta \) reflects the fact that the utility of the legacy is more important. A more realistic model would perhaps allow for different consumption and legacy utilities, but in such a case one may lose tractability. The following lemma shows that the above expected utility risk criterion with random time horizon is equivalent to one with a deterministic planning horizon; for the proof see [2].

**Lemma 2.1** The functional \( J \) of (2.7) equals

\[
J(t, x; \alpha) = \mathbb{E}_{t,x} \left[ \int_t^T F(s, t)U_\gamma(s, c(s)X(s)) \, ds + \beta f(s, t)U_\gamma(s, I(s)) \, ds + F(T, t)U_\gamma(T, X(T)) \right], \tag{2.8}
\]

where \( F(s, t) \) and \( f(s, t) \) are given by (2.4) and (2.3) respectively.

The wage earner’s objective is to find the strategy \( \alpha^* = \{\pi^*, c^*, p^*\} \) which maximizes the risk criterion, i.e.,

\[
\alpha^* = \text{arg max} J(t, x; \alpha). \tag{2.9}
\]
3 The Analysis

Due to the market completeness we can compute $D(t, \theta(t), i(t))$, the present value of the income flow at rate $i(s)$ on $[t, \tau]$, as expectation under the martingale measure $\mathbb{Q}$ of the discounted future income payoff. Imagine that the wage earner buys life insurance at rate $i$ and his/her wealth is $X_t$, then at death time $\tau$ it will pay off $\frac{i(\tau)}{\bar{v}(\tau)}$. This observation leads to the following result.

**Lemma 3.1** The time $t$ value, $D(t, \theta(t), i(t))$, of the future $[t, \tau]$ income is given by

$$D(t, \theta(t), i(t)) = \mathbb{E}^Q_t \left[ \int_t^\infty e^{-r(s-t)} i(s) e^{-\int_t^s \lambda(u)du} ds \right]$$

$$= i(t) \int_t^\infty e^{(\nu_i - r)(s-t)} e^{-\int_t^s \lambda(u)du} \mathbb{E}^Q_t [e^{-\sigma_i \int_t^s i(u)du}] ds,$$  \hspace{1cm} (3.1)

and

$$\mathbb{E}^Q_t [e^{-\sigma_i \int_t^s i(u)du}] = e^{-\sigma_i ((\theta(t) - \bar{\theta}) \zeta(s-t) + \frac{\bar{k}}{\bar{\theta}} (s-t)) \varepsilon_{\tau-s} \zeta^2(s-v)dv}.$$  \hspace{1cm} Here $\delta = k - \sigma \theta$, and $\zeta(x) = \frac{1}{\delta} (1 - e^{-\delta x}).$

Appendix A gives the Proof.

Given $t$, the current time, in light of this Lemma we can assume that the wage earner does not receive income but his/her wealth is $X_t + D(t, \theta(t), i(t))$. The amount $D$ is refer to as the value of human capital. The HJB equation associated with maximizing (2.8) is the following second-order PDE:

$$\frac{\partial v}{\partial s}(s, x, \theta, i) - (\rho + \lambda(s)) v(s, x, \theta, i) + \sup_{\alpha} H(s, x, \theta, i; \alpha) = 0$$  \hspace{1cm} (3.2)

with the boundary condition

$$v(T, x, \theta, i) = \frac{x^\gamma}{\gamma}.$$  \hspace{1cm} (3.3)

Here, the Hamiltonian function $H$ is given by

$$H(s, x, \theta, i; \alpha) = \left[(r + \sigma \theta \pi - c - p)x + i\right] \frac{\partial v}{\partial x} + \frac{1}{2} \sigma^2 \theta^2 \frac{\partial^2 v}{\partial x^2} + \left(-k(\theta - \bar{\theta})\right) \frac{\partial v}{\partial \theta} + \frac{1}{2} \gamma^2 \frac{\partial^2 v}{\partial \theta^2} + \nu_i \frac{\partial v}{\partial i} + \frac{1}{2} \gamma^2 \frac{\partial^2 v}{\partial \theta^2} - \sigma_\theta \pi x \frac{\partial^2 v}{\partial x \partial \theta} + \sigma_\pi x i \frac{\partial^2 v}{\partial x \partial i} - \sigma_i \sigma_\theta \frac{\partial^2 v}{\partial i \partial \theta} + \left(\frac{cx}{\gamma}\right)^\gamma + \beta \lambda(s) \frac{(x + \frac{px}{\lambda(s)})^\gamma}{\gamma}.$$  \hspace{1cm} (3.4)

The first order conditions (FOC) are

$$\pi^* = \frac{\theta \frac{\partial v}{\partial x} - \sigma_\theta \frac{\partial^2 v}{\partial x \partial x} + \sigma_i \frac{\partial^2 v}{\partial x \partial i}}{\sigma^2 \frac{\partial^2 v}{\partial x^2}}, \quad c^* = \frac{(\frac{\partial v}{\partial x})^\gamma}{x}, \quad p^* = \lambda \left[ \frac{(\frac{1}{\beta} \frac{\partial v}{\partial x})^\gamma}{x} - 1 \right].$$  \hspace{1cm} (3.5)
We introduce the following PDE which is intimately related to (3.2):
\[
\frac{\partial h}{\partial s} + \frac{\gamma}{1-\gamma}(r + \beta \lambda(s)) - \frac{\rho + \lambda(s)}{\gamma} + \frac{\theta^2}{2(1-\gamma)}h + (\gamma - 1)\lambda(s) = 0.
\]  
(3.6)

with terminal condition
\[ h(T, \theta) = 1. \]

Here \( K(s) = 1 + \beta \lambda(s). \) We are able to solve this PDE in closed form by using Feynman-Kac Theorem.

**Lemma 3.2** The solution of (3.6) is given by
\[
h(t, \theta) = [e^{\int_t^T q(u) du} e^{\theta^2 a(t) + b(t)} + \int_t^T K(s) e^{\int_s^T q(u) du} e^{\theta^2 a(s) + b(s)} ds]
\]

Appendix B gives the Proof.

**Theorem 3.3** The function
\[
v(s, x, \theta, i) = h^{1-\gamma}(s, \theta) (x + D(s, \theta, i))(s, \theta)
\]
(3.7)

solves the HJB equation (3.2)-(3.3). The optimal admissible investment-consumption-insurance strategy \( \alpha^\star(s) = \{\pi^\star(s), c^\star(s), p^\star(s)\}_{s \in [t,T]} \) is given by
\[
\pi^\star(s) = \frac{(X^\star(s) + D(s, \theta(s), i(s))) \theta(s)}{\sigma(1-\gamma)X^\star(s)} - \frac{\sigma(1-\gamma)X^\star(s)}{\sigma(1-\gamma)X^\star(s) h(s, \theta(s))}
\]
(3.8)

\[
c^\star(s) = [h(s, \theta(s))]^{-1} \frac{X^\star(s) + D(s, \theta(s), i(s))}{X^\star(s)}
\]
(3.9)

\[
p^\star(s) = \lambda(s) \left[ \beta^{1-\gamma}[h(s, \theta(s))]^{-1} \frac{X^\star(s) + D(s, \theta(s), i(s))}{X^\star(s)} - 1 \right].
\]
(3.10)

Appendix C gives the Proof.
3.1 Optimal Consumption versus Legacy

Theorem 3.3 yields the ratio between the optimal legacy \(I^*(s)\) and optimal consumption \(C^*(s) = c^*(s)X^*(s)\).

**Corollary 3.4** For every \(\gamma < 1\), we have
\[
\frac{I^*(s)}{C^*(s)} = \beta^{1-\gamma}.
\]

This result says that the legacy/consumption ratio is independent of the financial market characteristics. It depends on the investor’s primitives: the coefficient of risk aversion \(1 - \gamma\), the bequest weight \(\beta\). If a loading factor is assumed it will also depend on it as well as on the hazard rate. The higher the the weight \(\beta\) the higher the the legacy/consumption ratio. This fact is somehow expected. Furthermore, a more risk averse wage earner consumes less in favour of the legacy. If the weight \(\beta = 1\), then the optimal legacy equals the optimal consumption. Let us now examine the relationship between wealth, consumption and life insurance/pension annuity. If the wage earner buys life insurance then he/she consumes more; this may be explained by the legacy being supplemented by the insurance payment in case of death (so he/she may consume more and save less for his/her heirs). On the other hand if a pension annuity is acquired he/she consumes less so that there will be enough money at the death time for legacy and paying off the pension annuity. In real world for the case of pension annuities, those who survive share in the mortality credit, money that are added to investment returns from the pension annuity. Let us point out that the finding of this Corollary is contingent on the special type of risk preferences considered (CRRA type with the same coefficient of risk aversion for consumption and legacy).

4 Numerical results

In this section, a 35 years old wage earner (the initial time \(t = 35\)), has an initial wealth of 50,000 dollars and a benchmark time \(T = 65\), invests (in the financial market) consumes and buys life insurance (pension annuities) as to maximize his/her expected utility. The financial market parameters are \(r = 0.01, i(t) = 4000, \nu_i = 0.02, \sigma_i = 0.04, \sigma = 0.0436, \sigma_g = 0.0189, k = 0.0226\) and \(\bar{\theta} = 0.0788\). The weight \(\beta = 1, \rho = 0.03\), and \(\lambda(t) = 0.001 + \frac{1}{10.54} \exp \frac{t - 87.24}{10.54}\), (Gompertz hazard function). We are mainly interested in the impact MPR has on the optimal insurance strategy.
A risk averse wage earner \((\gamma = -1)\) invests less in the stock than a risk seeking wage earner \((\gamma = 0.1)\) and this investment is increasing in the MPR \(\theta\). A risk averse wage earner pays less for life insurance as long as the MPR \(\theta\) is small. However when MPR \(\theta\) is large enough the wage earner pays more for life insurance. On the other hand, if the wage earner is risk seeking then he/she pays less for life insurance when MPR \(\theta\) gets large.

5 Conclusion

The purpose of life insurance is to provide financial security for the holders of such contracts and their families. Life insurance acquisition can be considered in connection with consumption and investment in the financial market. The problem of optimal investment, consumption and life insurance acquisition for a wage earner who has CRRA risk preferences is analysed in this paper. The wage earner receives income at a stochastic rate, consumes, invests in a stock and a risk free asset and buys life insurance. The MPR of the stock is assumed to follow a mean reverting process. The optimal strategies in this model are characterized by HJB equation which is solved in closed form. Numerical results explore the effect of the MPR on the optimal strategies.

6 Appendix

A Proof of Lemma 3.1 Define the martingale measure \(\mathbb{Q}\), such that

\[
\tilde{W}_t = W_t + \int_0^t \theta(s)ds
\]

is a Brownian motion under probability measure \(\mathbb{Q}\). Under \(\mathbb{Q}\), \(D(t, \theta(t), i(t))\) can be computed as follows:

\[
D(t, \theta(t), i(t)) = \mathbb{E}_t^{\mathbb{Q}}[e^{-r(\tau-t)} \frac{i(\tau)}{\lambda(\tau)}].
\]

Thus

\[
D(t, \theta(t), i(t)) = \mathbb{E}_t^{\mathbb{Q}}[\int_t^\infty e^{-r(s-t)}i(s)e^{-\int_s^\tau \lambda(u)du}ds]
\]

\[
= \int_t^\infty e^{-r(s-t)}e^{-\int_t^\tau \lambda(u)du}\mathbb{E}_t^{\mathbb{Q}}[i(s)]ds. \tag{6.1}
\]
The dynamics of stochastic income under $\mathbb{Q}$ measure is:
\[ di(t) = i(t)[(\nu_i - \sigma_i\theta(t))dt + \sigma_i d\bar{W}(t)], \]
so
\[ i(s) = i(t)e^{(\nu_i - \frac{\sigma_i^2}{2})(s-t)}e^{\sigma_i(\bar{W}(s) - \bar{W}(t))}e^{-\sigma_i \int_t^s \theta(u)du}. \]
The dynamics of MPR $\theta(t)$ is
\[ d\theta(t) = -(k - \sigma_\theta)\theta(t)dt + k\theta dt - \sigma_\theta d\bar{W}(t). \]
Recall that $\delta = k - \sigma_\theta$, so
\[ d(e^{\delta t}\theta(t)) = e^{\delta t}(k\theta dt - \sigma_\theta d\bar{W}(t)). \]
Consequently
\[ \theta_u = e^{-\delta(u-t)}\theta_t + \frac{k\theta}{\delta}(1 - e^{-\delta(u-t)}) - \sigma_\theta \int_t^u e^{-\delta(u-v)}d\bar{W}(v), \]
Integrating, we get:
\[ \int_t^s \theta_u du = (\theta(t) - \frac{k\theta}{\delta})\zeta(s-t) + \frac{k\theta}{\delta}(s-t) - \int_t^s \sigma_\theta \zeta(s-v)d\bar{W}(v), \] (6.2)
where $\zeta(x) = \frac{1}{\delta}(1 - e^{-\delta x})$. Thus
\[ D(t, \theta(t), i(t)) = i(t) \int_t^\infty e^{(\nu_i - r)(s-t)}e^{-\int_t^s \lambda(u)du}E_t\{e^{-\sigma_i \int_t^s \theta(u)du}\}ds, \]
where
\[ E_t\{e^{-\sigma_i \int_t^s \theta(u)du}\} = e^{-\sigma_i((\theta(t) - \frac{k\theta}{\delta})\zeta(s-t) + \frac{k\theta}{\delta}(s-t))}e^{-\frac{\sigma_i^2 \sigma_\theta^2}{2} \int_t^s \zeta^2(s-v)dv}. \]

**B Proof of Lemma 3.2** For every $s \in [t, T]$, we define the following process:
\[ dy(s) = (-k - \frac{\gamma}{1 - \gamma} \sigma_\theta)y(s)ds + \sigma_\theta d\bar{W}(s), \]
with the initial value $y(t) = \theta$, where $\bar{W}(s) = \frac{k\theta}{\sigma_\theta}s + W(s)$. Let
\[ G(s, y(s)) = e^{\int_t^s r(u, y(u))du}h(s, y(s)), \]
where
\[ r(s, \Theta) = q(s) + \Gamma \Theta^2, \]
with
\[ q(s) = \frac{\gamma}{1 - \gamma}(r + \beta \lambda(s) - \frac{\rho + \lambda(s)}{\gamma}), \]

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and
\[ \Gamma = \frac{\gamma}{2(1-\gamma)^2}. \]

Applying Itô formula to \( G(s, y(s)) \) leads to
\[ dG(s, y(s)) = e^\int_0^s r(u,y(u))du \left( -K(s)ds + \sigma \frac{\partial h}{\partial y}(s, y(s))d\hat{W}(s) \right). \]

Integrating from \( t \) to \( T \) and taking expectation conditioned on \( y(t) = \theta \), one gets
\[ h(t, \theta) = e^{\int_t^T \int_0^s r(u,y(u))du} E\left[ e^{\int_t^T \Gamma y^2(u)du} \right] + \int_t^T K(s)e^{\int_t^s \int_0^u r(v,y(v))du} E\left[ e^{\int_s^T \Gamma y^2(u)du} \right] ds. \quad (6.3) \]

Itô formula gives
\[ dy^2(s) = \hat{a}(\hat{b} - y^2(s))ds + \hat{c}y(s)d\hat{W}(s), \]
where \( \hat{a} = 2(k + \frac{\gamma}{1-\gamma} \sigma \theta) \), \( \hat{b} = \frac{\sigma^2}{2(1+\gamma^2 \sigma^2)} \), \( \hat{c} = 2 \sigma \theta \). Let \( Y(s) = y^2(s) \), so that
\[ dY(s) = \hat{a}(\hat{b} - Y(s))ds + \hat{c}\sqrt{Y(s)}d\hat{W}(s). \]

Let
\[ P(t, y) = E[e^{\int_t^T \Gamma y(u)du}], \]
where \( Y(t) = y = \theta^2 \). The function \( P \) satisfies the following PDE
\[ \frac{\partial P}{\partial t} + \Gamma yP + \hat{a}(\hat{b} - y) \frac{\partial P}{\partial y} + \frac{\hat{c}^2 y}{2} \frac{\partial^2 P}{\partial y^2} = 0, \quad (6.4) \]
with the terminal condition
\[ P(T, Y(T)) = 1. \quad (6.5) \]

Let us guess that
\[ P(t, y) = \exp\left( a(t)y + b(t) \right). \]
Plugging this guess into \( (6.4) \) we get the following Riccati system for the functions \( a(t), b(t) \):
\[ a'(t) = -\frac{\hat{c}^2}{2} a^2(t) + \hat{a} a(t) - \Gamma, \]
\[ b'(t) = -\hat{a} \hat{b} a(t), \]
with the boundary conditions
\[ a(T) = b(T) = 0. \]

Define \( \delta = \hat{a}^2 - 2\Gamma \hat{c}^2 \).

**Case 1:** If \( \delta > 0 \), let \( \Delta = \sqrt{\delta} \), and
\[ a(t) = \frac{2\Gamma (e^{\Delta(T-t)} - 1)}{2\Delta + (\hat{a} + \Delta)(e^{\Delta(T-t)} - 1)}, \]
Case 2: If \( \delta = 0 \), let
\[
a(t) = \frac{2}{c^2(T - t - \frac{2}{a})} + \frac{\hat{a}}{c^2},
\]
\[
b(t) = -\int_t^T \hat{a} b(u) du
\]
Case 3: If \( \delta < 0 \), let \( \Delta = \sqrt{-\delta} \), and
\[
a(t) = -\frac{\Delta}{c^2} \tan \left( \frac{(T - t)\Delta}{2} + \tan^{-1}\left( \frac{\hat{a}}{\Delta} \right) \right) + \frac{\hat{a}}{c^2},
\]
\[
b(t) = -\int_t^T \hat{a} b(u) du
\]
Finally,
\[
\hat{E}[e^{\int_t^T \gamma^2(u) du}] = e^{\theta^2 a(t) + b(t)}.
\]
Therefore
\[
h(t, \theta) = \left[ e^{\int_t^T q(u) du} e^{\theta^2 a(t) + b(t)} + \int_t^T K(s) e^{\int_t^s q(u) du} e^{\theta^2 a(s) + b(s)} ds \right].
\]

\[\Box\]

**C Proof of Theorem 3.3** By Lemma 3.1 one can think of the wage earner as having an initial wealth \( X(t) + D(t, \theta(t), i(t)) \) and no income instead of having an initial wealth \( X(t) \) and income stream at rate \( i \). Following Proposition 2 in [9], the solution of the HJB (3.2) is
\[
v(s, x, \theta, i) = h^{1-\gamma}(s, \theta) \left( x + D(s, \theta, i) \right)^{\gamma}, \quad s \in [t, T].
\]
The function \( h \) is chosen such that \( V(s, x, \theta) = h^{1-\gamma}(s, \theta) \frac{x^\gamma}{\gamma} \), solves the HJB
\[
\frac{\partial V}{\partial s}(s, x, \theta) - (\rho + \lambda(s))V(s, x, \theta) + \sup_{\alpha} H(s, x, \theta; \alpha) = 0,
\]
where
\[
H(s, x, \theta; \alpha) = (r + \sigma \theta \pi - c - p)x \frac{\partial V}{\partial x} + \frac{\sigma^2 \pi^2 x^2}{2} \frac{\partial^2 V}{\partial x^2} + (-k(\theta - \bar{\theta})) \frac{\partial V}{\partial \theta} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial \theta^2} - \sigma \theta \pi x \frac{\partial^2 V}{\partial x \partial \theta} + (\frac{cx}{\gamma})^\gamma + \beta \lambda(s) \frac{(x + \frac{px}{\lambda(s)})^\gamma}{\gamma}.
\]
Based on the FOC, we obtain

\[ c^* (s) = \frac{\left( \frac{\partial V}{\partial s} \right)^{\gamma-1}}{x}, \]

\[ \pi^* (s) = - \frac{\partial V}{\partial x} \theta - \frac{\partial^2 V}{\partial s \partial \theta} \sigma_{\theta}, \]

\[ p^* (s) = \lambda (s) \left[ \frac{\left( \frac{1}{\beta} \frac{\partial V}{\partial x} \right)^{\gamma-1}}{x} - 1 \right]. \]

Substituting \((c^*, \pi^*, p^*)\) into the HJB equation (6.6) leads to the following PDE:

\[
\frac{\partial V}{\partial s} - (\rho + \lambda (s))V + (rx + \beta \lambda (s)x) \frac{\partial V}{\partial x} + (-k(\theta - \bar{\theta})) \frac{\partial V}{\partial \theta} + \frac{\sigma_{\theta}^2 \partial^2 V}{2 \partial \theta^2} \\
- \frac{(\theta \frac{\partial V}{\partial x} - \sigma_{\theta} \frac{\partial^2 V}{\partial s \partial \theta})^2}{2 \frac{\partial^2 V}{\partial s^2}} + \frac{1 - \gamma}{\gamma} (1 + \beta \lambda (s)) \left( \frac{\partial V}{\partial x} \right)^{\gamma-1} = 0, \quad (6.8)
\]

with the terminal condition

\[ V(T, x, \theta) = \frac{x^\gamma}{\gamma}. \]

Next, guess the solution of this equation to be of the form

\[ V(s, x, \theta) = h^{1-\gamma}(s, \theta) \frac{x^\gamma}{\gamma}. \]

Let \(K(s) = 1 + \beta \lambda (s)\). Then \(h\) solves the following PDE

\[
\frac{\partial h}{\partial s} + \frac{\gamma}{1 - \gamma} (r + \beta \lambda (s)) - \frac{\rho + \lambda (s)}{\gamma} + \frac{\theta^2}{2(1 - \gamma)h} + (-k(\theta - \bar{\theta}) - \frac{\gamma}{1 - \gamma} \theta \sigma_{\theta} \frac{\partial h}{\partial \theta}) \\
+ \frac{\sigma_{\theta}^2 \partial^2 h}{2 \partial \theta^2} + K(s) = 0. \quad (6.9)
\]

with terminal condition

\[ h(T, \theta) = 1. \]

Function \(h\) was found in Lemma 3.2.

References


