DETERMINISTIC INVENTORY MODELS (DIM)

1. Introduction

In this presentation, we introduce another type of models known as Inventory Models. There are several versions of inventory models which space and time will never permit us to cover. We therefore focus on a small section of the inventory models: Deterministic Inventory Models (DIM). In DIM, we shall look at only three areas:

- Components of Inventory Models,
- Basic Economic Order Quantity (EOQ) Models and
- EOQ models with Back Ordering.

Inventory or inventories are stocks of goods being held for future use or sale. The demand for a product in inventory is the number of units that will need to be removed from inventory for use or sale during a specific period. If the demand for future periods can be predicted with considerable precision, it will be reasonable to use an inventory rule that assumes that all predictions will always be completely accurate. This is the case where we say that demand is deterministic.

2. Components of Inventory Models

The components of inventory models include the various types of costs. One straight forward inventory criteria is the minimization of cost.

2.1 Ordering or setup cost

An ordering cost is the cost incurred whenever an order is made. It is independent of the quantity being ordered. It is primarily clerical and administrative in nature. Typical elements of this cost include the cost associated with processing, labour, overhead (telephone, postage, etc.), and transport (delivery charges). In other environments, the setup cost could be termed clerical and administrative cost.

2.2 Carrying or Holding cost

The holding cost represents all the costs associated with the storage of the inventory until it is sold or used. Included are the cost of insurance, capital, space, protection, and taxes attributed to the storage. It is proportional to the amount of an inventory and the time over which it is held. It is unavoidable, but with good management it can be reduced.

2.3 Shortage cost

The shortage cost is the cost incurred when the amount of the inventory demanded exceeds the available stock. This cost depends on whether back ordering is allowed or not. For instance, if a customer demands a product and the demand is not met on time, a *stockout,* or *shortage*, is said to occur. If customers will accept delivery at later days, no matter how long it takes them to wait, we say that demand may be back-ordered. The case in which back-ordering is allowed is often referred to as the *backlogged* demand. If no customer will allow late delivery, we are in a lost-sales case.

2.4 Unit purchasing cost

This is the variable associated with purchasing a single unit. Typically, the unit purchasing cost includes the cost of raw materials associated with purchasing or producing a single unit. This cost may be a constant for all quantities, or it may vary with quantity purchased or produced.

3. Assumptions in Economic Order Quantity (EOQ) Models

3.1 Repetitive ordering

Repetitive ordering is the system whereby the placement of an order follows a regular fashion.

3.2 Demand is deterministic

Demand is assumed to occur at a known constant rate. For example, if demand occurs at a rate of 500 units per month, then at any particular t week period, we shall have $\frac{500t}{4}$ demand.

3.3 Constant lead time

The lead time for each order is a known constant, say L. By the lead time we mean the length of time between the instant when an order is placed and the of its arrival. If $L = 3$ month, say, then after each order will arrive exactly 3 months after the order is placed.

3.4 Continuous ordering

Here an order may be placed at any time. For example, you placed an order and wait till it gets to its reorder point (level prescibe by the ordering system that allows an order to be made when inventory falls to that level) before you place another other. In contrast continuous ordering is periodic ordering, i.e., you only review your inventory at the end of each period and decide whether to place an order or not at the time of review.

Although the constant lead time and the constant demand assumptions may seem unrealistic, there are many situations that deterministic inventory models provide a real good approximation to reality.

4. The Basic Economic Order Quantity Models

Assumptions of The Basic Economic Order Quantity Model

i. Demand is deterministic and occurs at constant rate. If D is the number of units demanded per year, then at any time interval of length t years, an amount Dt is demanded.

ii. If an order of any size, say q units is placed, an ordering or setup cost K is incurred. The setup cost is in addition to a cost pq of purchasing or producing the q units ordered. *Assuming* that the unit purchase cost p, does not depend on the size of the order. In other words the model does not permit quantity discount.

iii. The lead time for each order is zero. Each order arrives as soon as it is placed.

iv. No shortages are allowed. That is all demands must be met on time; a negative inventory is not allowed either.

v. The cost per unit year of holding inventory is h. A carrying cost of h dollars will be incurred if 1 unit is held for one year, 2 units is held for half a year, or if $\frac{1}{4}$ units is held for four years. In short, if I units are held for T years, a holding cost of ITh is incurred.

From five assumptions, the EOQ model determines an ordering policy that minimizes the yearly sum of ordering cost, purchasing cost, and holding cost.

5. Derivation of Basic EOQ Models

In addition to the assumptions, we begin the derivation by making the following observations.

Let $I(t)$ to be the inventory level at time t. From assumption iii, since the lead time is zero, the arrival of orders follows immediately when an order is placed. In that case, we will incur an unwanted holding cost if for $I > 0$, an order is placed.Therefore the policy that minimizes yearly costs must place an order when $I = 0$. If this is of constant rate, then we need the same quantity q, every time an order is placed.

So, what value of q is needed to minimize annual cost? Let q^* be that value of q that minimizes annual cost. If $C(q)$ is the cost involved when q units are ordered for $I = 0$, then the total annual cost for ordering q units is denoted $TC(q)$. $TC(q)$, comprises the annual cost of placing an order, the annual purchasing cost, and the annual holding cost, i.e.,

$$TC(q) = \text{Ordering cost/year} + \text{purchasing cost/year} + \text{Holding cost/year}. \quad (5.1)$$

To determine the number of orders, we take into consideration the quantity q units per order and the annual demand. Since each order is for q units, $\frac{D}{q}$ orders per year will have to be placed so that an annual demand of D units is met. Therefore the ordering cost per year is given by:

$$\text{Ordering cost/year} = \text{(Ordering cost/order)(number of orders/year)} = K(\frac{D}{q}). \quad (5.2)$$

Also for the purchasing cost per year, we have,

$$\text{Purchasing Cost/year} = \text{(purchasing cost/unit)(units purchase/year)} = pD.$$

For the annual holding cost, given the analysis in assumption $3.1\ v$, we have that, if I units are held for a period of one year, a holding cost of $1hI$ is incurred. Now suppose the inventory level is not constant, and varies over time. If the average inventory level during the length of time T is I^1, then the holding cost for the time period will be hTI^1.

Proof

If we define $I(t)$ to be the inventory level at time t, then during the interval $[0, T]$ the average inventory level is given by:

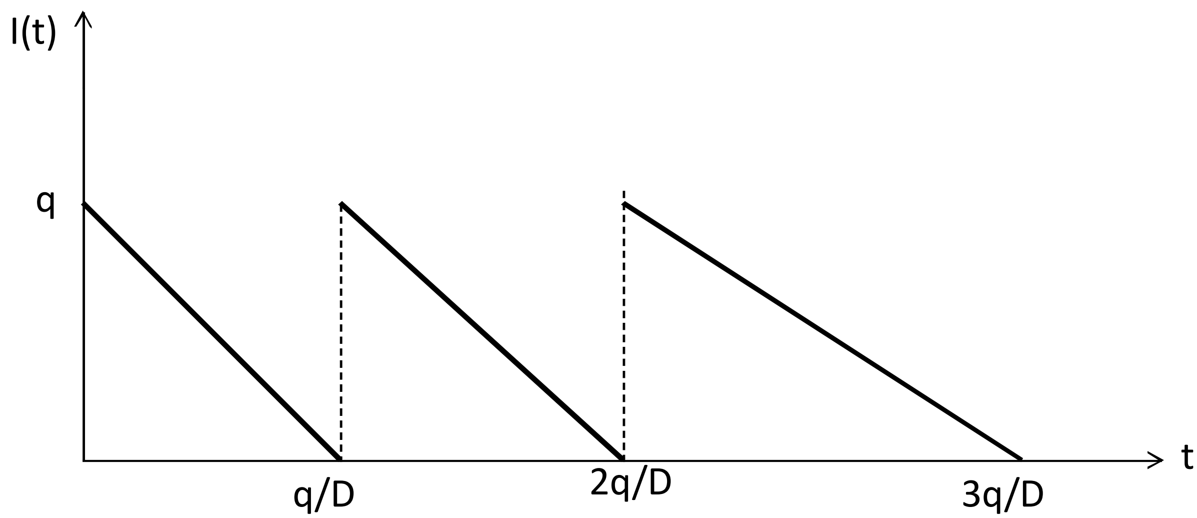
$$I^1(t) = \frac{\int_0^T I(t)dt}{T},$$

and the total holding cost incurred between time 0 and time T is

$$\int_0^T hI(t)dt = hTI^1(t). \qquad *$$

The total holding cost incurred between a particular period in time is given by *. What will be the annual holding cost? To determine this cost we need to know the behavior of I over a patternable time. Consider the graph below:

Figure 1.



Behaviour of I(t) in Basic EOQ Model

Assuming an order of size q has just arrived at time 0. Since demand occurs at a rate of D per year, it will take $\frac{q}{D}$ years for inventory to reach zero again. Since demand during any period of length t is Dt, the inventory level over any time interval will decline along a straight line of slope $-D$. When the inventory reaches zero, an order of size q is placed and arrives instantaneously, raising the inventory level back to q as depicted on the graph above. On the graph, there is a pattern of time and this pattern in EOQ models is called a *cycle*.

A cycle is an interval of time that begins with the arrival of an order and ends the instant before the next order is received. Each cycle has a length equal to the inverse of the number of orders per year that is needed to satisfy an annual demand of D units. Therefore each year will have $\frac{D}{q}$ cycles. The average inventory during any cycle will be $\frac{q}{2}$, i.e., half of the maximum inventory level attained during the cycle. For all models in which no shortage is allowed and demand occurs at a constant rate, the average inventory level during any cycle is $\frac{q}{2}$ units. Since a cycle has a length $\frac{q}{D}$, the average inventory level during each year is the average of the maximum inventory level.

To calculate our annual holding cost:

$$\text{Holding cost/year} = \text{(holding cost/cycle)(cycles/year), given that,}$$

$$\text{holding cost/cycle} = (\frac{q}{2})\frac{q}{D}h = \frac{q^2h}{2D}.$$

The holding cost per year is,

$$\frac{q^2h}{2D}(\frac{D}{q}) = \frac{qh}{2}.$$

Recall that this computation became necessary because we needed to compute the total annual cost incurred if q units are ordered each time that $I = 0.$ We can now obtain $TC(q)$:

$$TC(q) = \frac{KD}{q} + pD + \frac{hq}{2}.$$

To find the value of, q say, q^* that minimizes this function, we equate the first differential of this function to zero i.e., $TC'(q) = 0.$

$$\frac{-KD}{q^2} + \frac{h}{2} = 0,$$

$$q^* = \pm\sqrt{\frac{2kD}{h}}.$$

From assumption 4, we said no shortages are allowed, hence we do not desire a negative inventory. Hence we discard the negative value and accept the positive value as the value that minimizes the total cost function. We take our EOQ to be,

$$q^* = \sqrt{\frac{2KD}{h}}.$$

Example 1.

An airline uses 500 taillights per year. Each time an order is placed, an ordering cost of \$5 is incurred. Each light costs 40¢, the total holding cost is 8 cents /light/year. Assume that demand occurs at a constant rate and that no shortages are allowed. what is the EOQ? How many orders will be placed each year? How much time will elapse between the placement of orders.

Solution

We are given demand $D = 500$ lights/year, the setup cost $K = \$5$, h, the holding cost per each light in a year $= 8$ cents, p, the unit price to be, $40\ cents$.

Recall that, the EOQ is,

$$q^* = \sqrt{\frac{2kD}{h}}.$$

$$= \sqrt{\frac{2(5)(500)light}{(\frac{8}{100})light}} = 250$$

250 taillight orders should be placed every time the quantity reaches zero.

Orders per year $= \dfrac{D}{q^*} = \dfrac{500lights}{250lights} = 2$ orders per year.

Length of cycle $= \dfrac{q^*}{D} = \dfrac{250}{500} = .5$

Question 2.

Lakehead University book store sells 12,000 applied mathematics books each academic year. The book store orders its books from Media Sales, a national book supplier, which charges \$20 per book. Each order incurs an ordering cost of \$60. The owner of the book store believes that the demand for books can be backlogged and that the cost of being short one book for a year is \$20. The annual holding cost for inventory is 30¢ per dollar value of inventory.

i. What is the optimal order quantity of the book shop?

ii. What maximum level inventory can possibly occur?

iii. What is the maximum shortage that will occur?

One key feature we need in addition to the parameters D, K, h (with their usual meaning), is the shortage cost, say S. This value of S is not captured in the model developed in the last section, because shortages were not allowed. We need to modify the previous model for determining maximum quantity which minimizes total annual cost when q units are demanded. We suspend this question for a while.

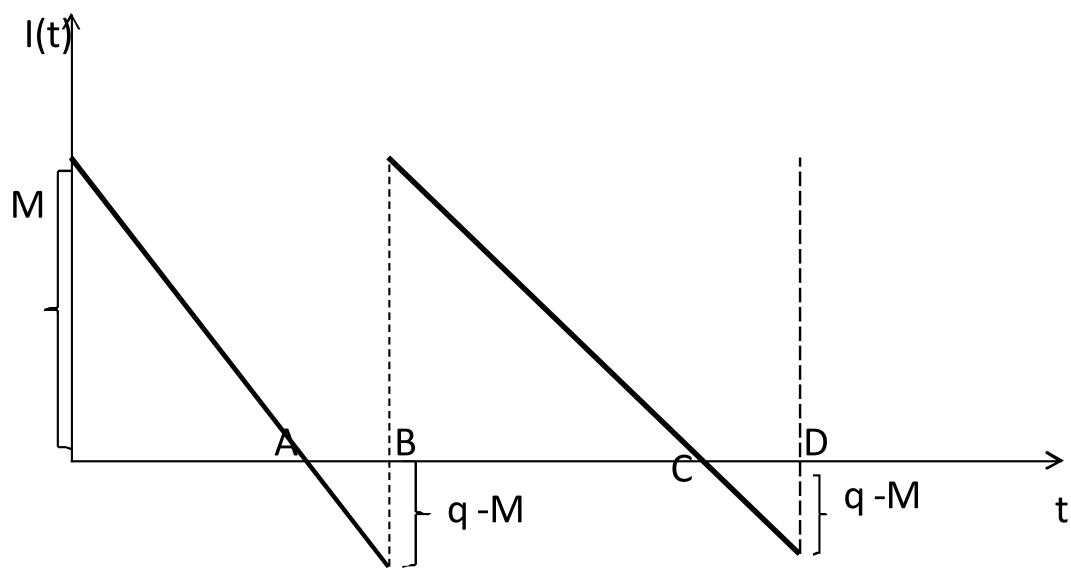
6. EOQ Model With Back Ordering Allowed

As mentioned earlier, the EOQ model to be developed here is a modification of the model in section 5. In this model, the possibility of shortages are allowed. Moreover, we assume that all demand is backlogged (defined in section 1.3) and no sales are lost.

Notice that in developing the EOQ model in the previous section, the model did not depend on the unit purchasing cost P. That is the size of each order does not change the unit purchasing cost: the total annual purchasing cost is independent of q. Its inclusion will not be considered in the model about to be developed.

Let K, D, and h be defined as in the last section and S be the cost of being short one unit for a year. To determine the value of the ordered quantity q, say q^* that minimizes total annual cost. Consider the graph below.

Figure 3.



EOQ Model With Back Order Graph.

Assume a zero lead time for each order, and let $q =$ the ordered quantity, M maximum inventory level, $q - M =$ maximum shortage that occurs under ordering rule. Since an order is placed any time the company inventory position is $q - M$, the maximum inventory level is $q - M + q = M$, i.e., the company will at all time have a backlogged demand of $q - M$ units to satisfy.

Now assuming an order is placed at time 0, and observe Figure 2. The maximum inventory level is M. Since the purchasing cost does not depend on the quantity q, it will likewise, not depend on M. We can therefore minimize the total annual cost by determining values of q and M say, q^* and M^* respectively. Let $TC(q)$ be the total annual cost.

$$TC(q,M) = \text{Holding cost/year} + \text{shortage cost/year} + \text{order cost/year}. \qquad (6.1)$$

Before we move on, take a break and compare equations 6.1 and 5.1. Notice that, the purchasing cost in 5.1 has now been replaced by the shortage cost in 6.1 as stated earlier in this development. Also compare the last two graph, the position of q in $I(t)$ axis of the graph of 5.1 is replaced by M in the graph of 6.1.

Now, to determine each of the components in 6.1, and in particular, the ordering cost and that of the shortage cost, we introduce the concept of a cycle by equivalently defining a cycle as the time interval between the placement of orders. With this definition in mind, notice in Figure 2, that what happens between time 0 and time B is the same as what happens between time B and time D. Hence you may call these time periods, $0B$ and BD cycles. That idea of a cycle sets us off to calculate the holding cost per year.

$$\text{Holding cost/year} = (\text{holding cost/cycle})(\text{cycles/year}),$$

We begin by finding holding cost per cycle. This requires that we find the lengths $0A$ and AB. From Figure 6.1, since a zero inventory level occurs after M units have been demanded, $0A = \frac{M}{D}$. Also, a cycle ends when q units have been demanded, we have $0B = \frac{q}{M}$, hence length of $AB = 0B - 0A = \frac{q-M}{D}$. Also, there are $\frac{D}{q}$ cycles for ordering q units each year, and $\frac{D}{q}$ orders are needed to satisfy the demand for the year. Here the average inventory level is $\frac{M}{2}$. Hence,

$$\text{Holding cost/cycle} = (\frac{M}{2})(\frac{M}{D})h = \frac{M^2h}{2D}$$

Also we have $\frac{D}{q}$ cycles per year, so

$$\text{Holding cost/year} = (\frac{M^2h}{2D})(\frac{D}{q}) = \frac{M^2h}{2q}. \qquad (6.2)$$

Similarly going through the same analysis,

$$\text{Shortage cost/year} = (\text{shortage cost/cycle})(\text{cycles/year}).$$

The shortage course per cycle is the shortage course incurred during time AB, since demand occurs at constant rate, the average shortage level during time AB is $\frac{q-M}{2}$. The time interval of AB, is of length $\frac{q-M}{D}$. Hence,

$$\text{Shortage cost/cycle} = (\frac{q-M}{2})(\frac{q-M}{D})S = \frac{(q-M)^2S}{2D}.$$

Moreover there are $\frac{D}{q}$ cycles, so

$$\frac{\text{Shortage cost}}{\text{Year}} = \frac{(q-M)S}{2D}(\frac{D}{q}) = \frac{(q-M)^2S}{2q}. \qquad (6.3)$$

We can obtain equation 5.1 by putting together equations $5.2, 6.2,$ and 6.3, i.e.,

$$TC(q,M) = \frac{M^2h}{2q} + \frac{(q-M)^2S}{2q} + \frac{KD}{q}.$$

We next try to find the values of q and M that will minimize $TC(q)$, we do that by equating the partial derivatives of $TC(q)$ with respect to q and M to zero.

$$\frac{\partial TC(q,M)}{\partial q} = \frac{\partial TC(q,M)}{\partial M} = 0$$

Computing this values for $q^*$and M^* gives us,

$$q^* = [\frac{2KD(h+S)}{hS}]^{\frac{1}{2}}$$

$$M^* = [\frac{2KDS}{h(h+S)}]^{\frac{1}{2}},$$

where $q^* - M^*$ is the maximum shortage.

We now answer example 2. Given $D = 12,000$ per year, $P = \$20$ dollars per book, $K = \$60$ per order $S = \$20$ per book in a year and $h = \$(\frac{30}{100})(20)$.

i. The order quantity is

$$q^* = [\frac{2KD(h+S)}{hS}]^{\frac{1}{2}} = \sqrt{\frac{2(60)(12,000)(6+20)}{6(20)}} = 558.57$$

ii. The maximum inventoy level is given by,

$$M^* = [\frac{2KDS}{h(h+s)}]^{\frac{1}{2}} = \sqrt{\frac{2(60)(12,000)(20)}{6(6+20)}} = 429.45$$

iii. The maximum shortage that will occur is the difference,

$$q^* - M^* = 128.90$$

Hence the mximum shortage of the book store is 129, and the quantity and maximum level are approximately 559 and 430 respectively.

7. References.

1. Wanye L. Winston, *Introduction to Probability Models*, Volume Two, Fourth Edition. Thomson Learning Inc., 2004.