Lecture 2: Linear Programming

- Models, Theory, and Applications

Example: The Diet Problem

To determine the most economical diet that satisfies the basic minimum nutritional requirements for good health. Parameters: n different kinds of food available on the market with unit prices $c_j,\ j = 1,2,\ldots,n;$ The m basic nutritional ingredients we are interested in the study, with minimum requirements b_i, $i = 1,2,\ldots,m.$ Assume that a_{ij} is the amount of ith nutrition in jth food. We can set the mathematical model as follows

$$\begin{aligned} \min : \quad & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\ s.t. : \quad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1, \\ : \quad & a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \geq b_2, \\ & \ldots\ldots \\ : \quad & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \geq b_m, \\ : \quad & x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0. \end{aligned}$$

Or in matrix form

$$\begin{aligned} \min \ & : c^T x \\ st. : \ & Ax \geq b, x \geq 0. \end{aligned}$$

For the convenience in algorithm (mainly the simplex method), we call the following form of LP the standard form

$$\begin{aligned} \min \ & : c^T x \\ st. : \ & Ax = b, x \geq 0. \end{aligned}$$

(or in some textbook use max.) Note: all forms of LP (finite many real variables, linear objective functions, constraints consist of finitely many linear equalities and inequalities) can be made equivalent to the standard form.

Geometric interpretation: feasible set, feasible solution, optimal solution(s) (feasibility, existence, uniqueness, unboundedness) objective function contour lines (isoprofit lines).

Fundamental theorem of linear programming

- feasible set is a (convex) polygon: interior points, boundary points, corner point (basic solution);
- optimal solution, if any, is at the boundary;

- optimal solution set, if nonempty, consists at least one basic solution.

Three scenarios for any LP problem:

1. Non-feasible: the feasible set is empty, the set of constraints are self-contradictory.
2. Feasible set unbounded: the (finite) optimal solution may or may not exist. If not, there is an optimal direction...
3. Feasible set is bounded and non-empty: collect all the corner point (basic solutions), only finitely many, the one with the best objective function value is the optimal for the problem.

Finitely many, but how many?

Knowing that at least one of the basic solution is optimal, then the problem is reduced to searching over basic feasible solutions. Note that a corner point in R^n is the solution of n linear equations. Consider an LP problem with m inequality constraints (if not, reduce to subspace), then there are at most

$$\binom{m}{n} = \frac{m!}{n!(m-n)!}$$

corner points. This number is unfortunately exponential with respect to "input".

Computational issues:

According to the fundamental theory of LP, the optimal solution, if exists, can be found in finitely time. Simplex method searches for the best basic solution (optimal solution of the problem) by travelling from one basic solution to another with improved objective function value. The procedure is called pivoting. An example had be found that the simplex method has to visit every single corner point of the polygon before it lands on the last optimal one. This implies that the computational complexity of simplex method is sort of order $O(2^n)$, in other words, not the polynomial with respect to the input.

In 1979, the Russian mathematician Leonid Khatchain announced a polynomial time algorithm to solve linear programming problem. Because of the structure of the method, it is also called ellipsoid method. The new algorithm was a major theoretical development (that put LP into polynomially solvable category). However, the algorithm cannot match the simplex method in efficiency and effectively, and later continued to be the dominant practical tool to solve LP problems.

In 1984, Narendra Karmarkar in AT&T Bell Lab published another polynomial time algorithm for the LP, later many research refer it and its variations interior point method.

Worst-case analysis vs. Empirical Study:

Simplex method is very efficient **on average**.

Duality:

The primal and dual problem

$$\begin{aligned}(P)\ &:\ \min z = c^T x\\ &:\ st. Ax \geq b, x \geq 0,\\ (D)\ &:\ \max w = b^T y\\ &:\ st. A^T y \leq c, y \geq 0.\end{aligned}$$

Note:

- dimension of (P) and (D);
- min vs. max;
- variable-constraint correspondence;
- coefficients (in objective) and right-hand-side parameters (in the constraints)

Note the slightly different definitions of the primal and dual, and the equivalency between them. Dual of the dual is the primal.

Duality Theorems:

The weak version: If $\overline{x}$ is a feasible solution for (P) and $\overline{y}$ is a feasible solution for (D), then

$$c^T\overline{x} \geq b^T\overline{y}.$$

Proof. By the feasibility

$$c^T\overline{x} \geq \overline{y}^T A\overline{x} \geq \overline{y}^T b = b^T\overline{y}.$$

Corollary:

$$c^T\overline{x} \geq \min_{Ax\geq b, x\geq 0} c^T x \geq \max_{A^T y\leq c, y\geq 0} b^T y \geq b^T\overline{y}.$$

The difference between $\min_{Ax\geq b,x\geq 0} c^T x$ and $\max_{A^T y\leq c,y\geq 0} b^T y$ is called the duality gap. The concept of duality can be generalized to nonlinear programming problems and more, one of the research interest in duality theory is to study weather there is a positive duality gap between the two optimal values.

The strong version: Recall the primal and dual problems

$$\begin{aligned}(P)\ &:\ \min z = c^T x\\ &:\ st. Ax \geq b, x \geq 0,\\ (D)\ &:\ \max w = b^T y\\ &:\ st. A^T y \leq c, y \geq 0.\end{aligned}$$

Exactly one of the following situations must be true.

(**a**) Both (P) and (D) are feasible (i.e. feasible set is nonempty), and there are optimal solutions x^* and y^* for (P) and (D) and

$$c^T x^* = b^T y^*.$$

(**b**) Either of (P) or (D) has an unbounded objective function value in the feasible set, and the other problem is infeasible.

(**c**) Both (P) and (D) are infeasible.

An economic interpretation of the dual:

Consider the following diet problem

	Food 1	Food 2	Food 3	Food 4	Minimum requirement
Vit A	0	1	1	2	21
Vit B	1	2	1	1	11
Cost	20	31	11	12	

The Primal LP:

$$\begin{aligned}(P)\ &:\ \min z = 20x_1 + 31x_2 + 11x_3 + 12x_4\\ &:\ s.t. x_2 + x_3 + 2x_4 \geq 21,\\ &:\ x_1 + 2x_2 + x_3 + 2x_4 \geq 11,\\ &:\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0.\end{aligned}$$

We understand that the variables x_1, x_2, x_3, x_4 are the amount of each food we should to purchase in order to minimize the total cost.

Now we consider the dual problem

$$\begin{aligned}(D)\ &:\ \max w = 21y_1 + 11y_2\\ &:\ s.t. y_2 \leq 20,\\ &:\ y_1 + 2y_2 \leq 31,\\ &:\ y_1 + y_2 \leq 11,\\ &:\ 2y_1 + y_2 \leq 12,\\ &:\ y_1 \geq 0, y_2 \geq 0.\end{aligned}$$

Meaning: Consider the problem of setting the prices for vit A and vit B to maximize the profit. y_1 and y_2 are the unit prices of the vitamins (called shadow prices). The constraints mean that the cost to purchase the vitamins should be compatible to the cost to purchase the food.

LP Models:

- Limited resources:
- Lump sum cost: initial invest, equipment rental, building cost, license fee, etc.

- Interdependent variables:
- Other logic constraints: either-or, if-only, etc.

References:

1. David G. Luenberger, Linear and Nonlinear Programming, 2nd Edition, Addison-Wesley Publishing Company, 1984.
2. Paul R. Thie, An Introduction to Linear Programming and Game Theory, John Wiley & Sons, 1988.