LINEAR PROGRAMMING (LP), GRAPHICAL PRESENTATION

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INTRODUCTION

Linear Programming a is combination of a linear objective function and set of linear constraints. The linear constraints express restrictions on the values that the variable can assume. The goal of a LP is to either maximize profit (for example, increase the yield of a crop or to improve productivity) or to minimize cost ,(for example, to lower the risk of a disease). The difference between maximization and minimization lies in the sign of the inequality. Consider a standard LP which seeks to minimize an objective function.

$$\text{maximize} : d_1 y_1 + d_2 y_2 + \ldots + d_n y_n$$

subject to :

$$e_{11} y_1 + e_{12} y_2 + \ldots + e_{1n} y_n \geq f_1 \qquad\qquad \#$$
$$e_{21} y_1 + e_{22} y_2 + \ldots + e_{2n} y_n \geq f_2$$

$$e_{m1} y_1 + e_{m2} y_2 + \ldots + e_{mn} y_n \geq f_n$$
$$y_1 \geq 0, y_2 \geq 0, \ldots, y_n \geq 0.$$

DEFINITION 1.

A function, $f(x_1, x_2, \ldots, x_n)$ of $x_1, x_2, \ldots, x_n$ is called a linear function if and only if for some sets of constants $c_1, c_2, \ldots, c_n$, $f(x_1, x_2, \ldots, x_n) = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$. For example, $f(x_1, x_2) = 2x_1 + x_2$ is a linear function.

DEFINITION 2.

For any linear function, $f(x_1, x_2, \ldots, x_n)$ and any number b, $f(x_1, x_2, \ldots, x_n) \leq b$ and $f(x_1, x_2, \ldots, x_n) \geq b$ are linear inequalities. For example, $4x_1 + x_6 \geq 6$. Therefore equation (1) is called an LP because all the terms are linear and programming from the fact that the system can be solved by an algorithm. From the system the first equation (the maximize equation) is called the objective function of the system and it is the function that the decisions variables (variables that completely describe the decision to be made) seek to maximize. The linear equations are called the constraints and these are restrictions that the decisions variables must obey. From the system, the variables $y_1, y_2, \ldots, y_n$ are the decision variables.

FEASIBLE SOLUTION, FEASIBLE REGION AND OPTIMAL SOLUTION OF AN LP

DEFINITIONS 3.

A feasible solution of an LP is a solution in which all the constraints are satisfied, both the sign constraints and the functional constraints of the LP. If there is any solution which violates any of the constraints, that solution is said to be an infeasible solution.

A feasible region of a LP is the set of all points satisfying all the LP's constraints and all the sign constraints. In other words, it is the collection of all feasible solutions of the LP. It is possible for a LP's feasible region to be empty, resulting in an infeasible LP. An infeasible LP has no optimal solution. See example in figure 1.

An optimal solution is a point in the feasible region with the largest objective function value. In the case of a minimization problem, it is the point in the feasible region with the smallest objective function value. Most LPs have one optimal solution, but some can have more than one optimal solution. An example of an LP with more than one optimal solution is illustrated in figure 2. The thick line comprised a number of feasible points all qualified to be optimal solutions.
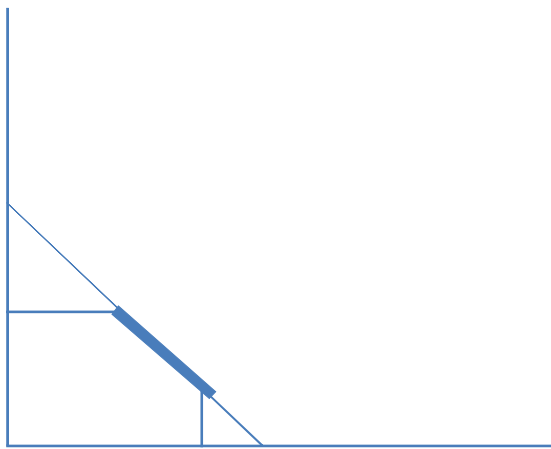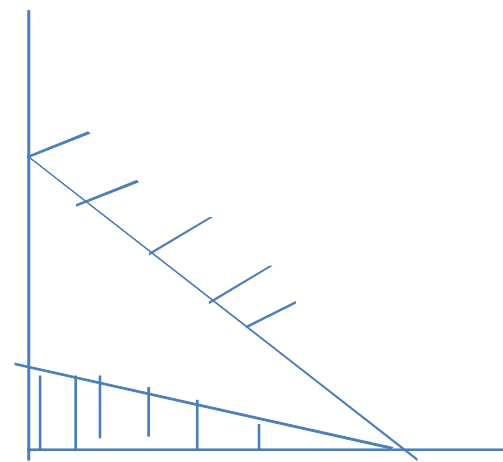

FIGURE 1

FIGURE 2

Graphical illustrations of a LP.

We shall take an example of a maximization problem. We are required to maximize,

maximize:
$$20x_1 + 30x_2$$
subject to:
$$2x_1 + 4x_2 \leq 10$$
$$-x_1 + x_2 \leq 1$$
$$x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

In this problem, x_1, x_2 are the decision variables. We shall plot each of the constraints on the same x and y plane. For example, we draw the line $2x_1 + 4x_2 = 10$ and determine the portion to be shaded by considering the inequality and taking any point on the plane which lies below the line. We substitute the point into the inequality and if the point satisfies the inequality, we shade below the line; otherwise, we shade above the line. For example, the

point $(0,0)$ the origin, lies below the line $2x_1 + 4x_2 = 10$ and when we substitute this point into the inequality, $2x_1 + 4x_2 \leq 10$ we have $0 \leq 10$. Thus we shade below the line. The same is done for all the inequalities and the graph in figure 4, is obtained.
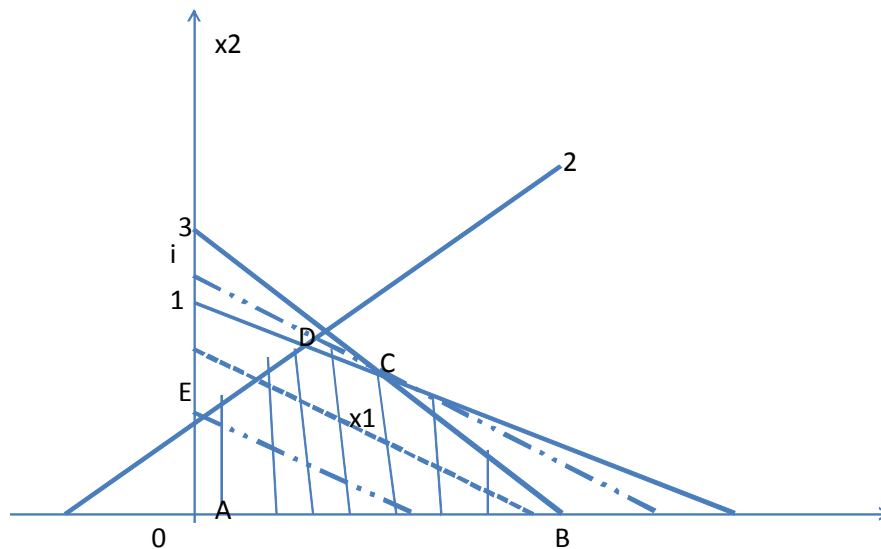
From the diagram the feasible region is bounded by the lines $1, 2, 3$ that passes through the points $A, B, C,$ and D. These points are called the **corner points** of the feasible region of the LP. The feasible region is the shaded region and the lines designated by broken lines are called **isoprofit lines or lines of equal revenue**. For a better understanding of the feasible region, there is the need to distinquish between corner points and noncorner points.

CONVEX SET, EXTREME POINTS AND LP

The feasible region in our maximization problem is an example of a convex set. We shall see why this is true at the end of the definitions and illusrations. We first take a look at some definitions.

DEFINITION 4.

A set of points is called a convex set if a line segment joining any pair of points in S is wholly contained in S. Let the shaded area to be the set S in figure 4 and figure 5. The figures are examples of a convex set and a nonconvex sets. Figure 5 is convex because any line segment joining any two points in S (the shaded portion), contains only points in S. In figure 6, S is not convex because for any two points in S, there will be points on the line segment that will not be contained in S. Relating the idea to linear programming, a certain type of point in a convex set is called an **extreme point**, usually called **corner points**, shall be of great importance.

DEFINITION 5.

For any convex set a point p in S is an extreme point if each line segment that lies completely in S and contains the point p has p as an end point of the line segment . For example, for the circle in figure 5, each point on the circumference of the circle is an

extreme point of the circle and that will never be true considering figure 6. We can therefore conclude that the feasible region in figure 4 is a convex set and the points $A, B. C, D,$ and D are the corner points.

    The feasible region of any LP has only a finite number of extreme points and the optimal solution must be an extreme point and belong to the feasible region. It is clear from the example that, the optimal solution (point c) is an extreme point and has a share in the feasible region. Any LP that has optimal solution has extreme point that is optimal. This result is very important because it reduces the set of points yielding the an optimal solution from the entire feasible region (which generally contains an infinite number of points) to a set of extreme points (a finite number).
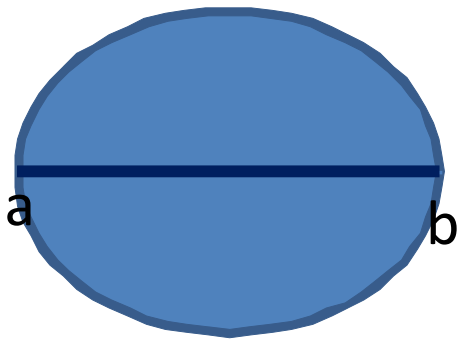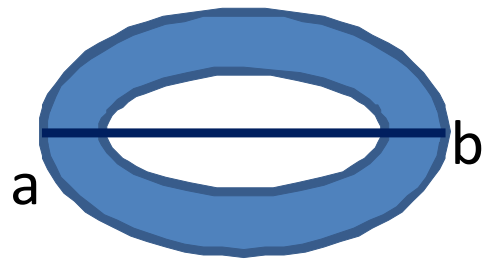

Fig.5

Fig.6

    Having identified our feasible region, we have to now locate the optimal point and from definition 3, that point should have the largest value of the objective function. A series of lines is drawn parallel to one another in the feasible region and the last line to touch the feasible region will be the best isoprofit line. This is done using the concept of parallel lines having equal gradients. Starting from the Objective function, you make x_2 the subject of the equation, for example, our objective function from the example is $w = 20x_1 + 30x_2$. Making x_2 the subject will give us; $x_2 = \frac{w}{30} - \frac{2}{3}x_1$. The gradient is $-\frac{2}{3}$. you draw the first line having a gradient of $-\frac{2}{3}$ and any other line parallel to the first line will have the same gradient. The last line to touch the feasible region will be the line with the maximum value of the objective function. You then locate and trace the point of contact. In figure 4 the point c is the last point where the last isoprofit line (line i) intersects the feasible region.

    This basically describes how to use the graphical approach to solve a LP. Though we use a maximization problem as an example, a minimization problem will take almost the same approach.The graphical approach can be used to solve any LP with two decision variables and with considerable difficulty, it is possible to extend the method to three variables. For three or more variables other methods, such as the simplex method, is more appropriate.

REFERENCE.
1.Russel C. Walker, *Introduction to Mathematical Programming. Prentice Hall Inc.,1999.*

2.Wayne L. Winston, *Operational Research and Algorithm* (Second Edition). PWS Kent publishing company, 1991.