Operations Research Seminar: The Simplex Method
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## Introduction

The simplex method is an algorithm for solving linear programs. It was invented by G.B. Dantzig in the 1940's. It has many features similar to Gaussian Elimination used in Linear Algebra.

The goal of this lecture is to introduce the basics of the simplex method. If time permits, we will also discuss some of the finer details of the algorithm. I have based my explanation of the simplex method on a number of different texts. In particular, this presentation was heavily influenced by the textbooks [1, 2, 3].

## Basic Terminology

We begin by introducing the needed terminology. Linear programs (LP) of the following form are said to be in *standard form*:

$$\begin{array}{rrcl} \text{Maximize:} & c_1x_1 + \cdots + c_nx_n & = & z \\ \text{Subject to:} & a_{11}x_1 + \cdots + a_{1n}x_n & \leq & b_1 \\ & a_{21}x_1 + \cdots + a_{2n}x_n & \leq & b_2 \\ & & \vdots & \\ & a_{m1}x_1 + \cdots + a_{mn}x_n & \leq & b_m \\ & x_1, \ldots, x_n & \geq & 0. \end{array}$$

*Remark.*
1. Each textbook has their own definition of standard from.
2. $z = c_1x_1 + \cdots + c_nx_n$ is the *objective function.*
3. The inequalities are called *constraints*; the first $m$ inequalities are *functional constraints*; the last $n$ inequalities, i.e., $x_1, \ldots, x_n \geq 0$, are *non-negative constraints.*

**Definition 1.** A tuple $(r_1, \ldots, r_n) \in \mathbb{R}^n$ is a *feasible solution* if all the constraints are satisfied by this tuple. The *feasible region* is the set of all feasible solutions.

*Remark.* Since $x_1, \ldots, x_n \geq 0$, all feasible solutions live in

$$\mathbb{R}^n_{\geq 0} = \{(r_1, \ldots, r_n) \mid r_i \geq 0\}.$$

**Definition 2.** An *optimal solution* is a feasible solution that maximizes the value of the objective function.

*Example* 3. Throughout this lecture, we will illustrate the main points with the following linear program:

$$\begin{array}{rrcl} \text{Maximize:} & 20x_1 + 30x_2 & = & z \\ \text{Subject to:} & x_1 + x_2 & \leq & 4 \\ & -x_1 + x_2 & \leq & 1 \\ & 2x_1 + 4x_2 & \leq & 10 \\ & x_1, x_2 & \geq & 0. \end{array}$$

## Slack Variables

The first step in the simplex method is to introduce new variables, called *slack variables*. The original variables are sometimes called *decision variables*. To each functional constraint, we add a new variable, the slack variable, which turns the inequality into an equality. For example:

$$x_1 + 2x_2 + 4x_5 \leq 5 \;\leftrightarrow\; x_1 + 2x_2 + 4x_3 + s_1 = 5.$$

We do this because there is a bijection (a one-to-one and onto) correspondence between the feasible solutions of the two equations. More precisely, let

$$F_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + 2x_2 + 4x_3 \leq 5 \text{ and} x_1, x_2, x_3 \geq 0\}$$

and

$$F_2 = \{(x_1, x_2, x_3, s_1) \in \mathbb{R}^4 \mid x_1 + 2x_2 + 4x_3 + s_1 = 5 \text{ and} x_1, x_2, x_3, s_1 \geq 0\}.$$

We then define a bijective map $\varphi : F_1 \to F_2$ by

$$\varphi(x_1, x_2, x_3) = (x_1, x_2, x_3, 5 - x_1 - 2x_2 - 4x_3) \in F_2.$$

The inverse map $\varphi^{-1} : F_2 \to F_1$ is given by

$$\varphi^{-1}(x_1, x_2, x_3, s_1) = (x_1, x_2, x_3).$$

We introduce $m$ slack variables, one for each functional constraint. As a consequence, to study the feasible region of the original linear program, one can study the feasible region of a new linear program involving slack variables.

*Example* 4. Return to the linear program of Example 3. Our equations, with slack variables, become

$$\begin{array}{rrcrcrrrcl}
\text{Maximize} & 20x_1 & + & 30x_2 & & & & & = & z \\
\text{Subject to} & x_1 & + & x_2 & + & s_1 & & & = & 4 \\
 & -x_1 & + & x_2 & + & & s_2 & & = & 1 \\
 & 2x_1 & + & 4x_2 & + & & & s_3 & = & 10 \\
 & & & x_1, & x_2, & s_1, & s_2, & s_3 & \geq & 0.
\end{array}$$

## Tableau Format

Just as a matrix encodes the required information to solve a system of linear equations, a tableau is commonly used to solve a linear program. We begin by rewriting our objective function:

$$z - c_1x_1 - \cdots - c_nx_n = 0.$$

Our linear program, with slack variables, can be expressed as

$$\begin{array}{rcccrcccccl}
a_{11}x_1 & + & \cdots & + & a_{1n}x_n & + & s_1 & & & = & b_1 \\
 & & & & & & & & & \vdots & \\
a_{m1}x_1 & + & \cdots & + & a_{mn}x_n & + & & s_m & & = & b_m \\
-c_1x_1 & - & \cdots & - & c_nx_n & + & & & z & = & 0
\end{array}$$

Omitting the coefficient of $z$, we get a tableau

$$\begin{array}{ccccccc|c} a_{11} & \cdots & a_{1n} & s_1 & 0 & \cdots & 0 & b_1 \\ a_{21} & \cdots & a_{2n} & 0 & s_2 & \cdots & 0 & b_2 \\ \vdots & & & & & & & \\ a_{m1} & \cdots & a_{mn} & 0 & 0 & \cdots & s_m & b_m \\ \hline -c_1 & \cdots & -c_n & 0 & & \cdots & 0 & 0 \end{array}$$

The last row is called the *objective row.*

*Example* 5. The tableau associated to Example 3 is

$$\begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 & 4 \\ -1 & 1 & 0 & 1 & 0 & 1 \\ 2 & 4 & 0 & 0 & 1 & 10 \\ \hline -20 & -30 & 0 & 0 & 0 & 0 \end{array}$$

## Basic Solutions and Initial Basic Solutions

A *basic solution* has the following four properties:

1. Each variable is either a basic variable or a non-basic variable.
2. The number of basic variables equals the number of functional constraints. The number of non-basic variables is the total number of variables minus the number of basic variables.
3. Non-basic variables are set to zero.
4. Values of basic variables are obtained by solving remaining system of linear equations.

The *initial basic solution* is the case that we take $s_1, \ldots, s_m$ to be basic variables, $x_1, \ldots, x_n$ are the non-basic variables, and $s_i = b_i$ for each $i$.

*Example* 6. Consider Example 3. We let $s_1, s_2, s_3$ be our basic variables, and set $x_1 = x_2 = 0$. We then get the system

$$\begin{array}{ccccc} s_1 & 0 & 0 & = & 4 \\ 0 & s_2 & 0 & = & 1 \\ 0 & 0 & s_3 & = & 10 \end{array}$$

So, our initial basic solution is $(0, 0, 4, 1, 10)$. This is a feasible solution. This is the case that $x_1 = x_2 = 0$ and when our objective function equals 0.

*Key Observation* 7. I've been a bit vague about picking the basic variables. However, note that the columns corresponding to $s_1, s_2, s_3$ are all columns of an identity matrix, i.e., they only contain one nonzero entry, and this entry is a one. As we will see, the basic variables will correspond to the columns of the tableau that resemble the columns of an identity matrix.

*Warning* 1. Suppose that $b_i < 0$ for some $i$. In this case, if we try to construct the initial basic solution described above, we get $0 \leq s_i = b_i < 0$, a contradiction. In this case, we need to resort to some "mathematical trickery" to get an initial basic solution. For the purpose of this talk (and many elementary books) this problem is avoided by assuming that $b_i \geq 0$ for all $i$ at the onset.

## Pivot Rows and Columns

We now describe the main iterative step in the simplex method.

**Definition 8.**

1. Pick a variable with largest negative entry in the objective row (the last row). If there is a tie, pick either variable. The corresponding column is the *pivot column* and the corresponding variable is the *entering variable*
2. For each **positive** entry $r$ in the pivot column, compute the ratio $s/r$ where $s$ is the rightmost entry in the row containing $r$. The row that gives the smallest ratio is the *pivot row*. The current basic variable that has a 1 in the pivot row is the *leaving variable*.

*Example* 9. We continue with our example. Since $-30$ is the smallest value in the objective row, $x_2$ is our entering variable, and the second column (which is boxed below) is the pivot column:

$$\begin{array}{c|c|cccc}
1 & 1 & 1 & 0 & 0 & 4 \\
-1 & 1 & 0 & 1 & 0 & 1 \\
2 & 4 & 0 & 0 & 1 & 10 \\
\hline
-20 & -30 & 0 & 0 & 0 & 0
\end{array}$$

For each positive entry $r$ in this column, we form the ratio $s/r$ where $s$ is the entry at the end of the row containing $r$. So, our ratios are $4/1$, $1/1$, and $10/4$. The smallest of these ratios is $1/1$. So, the second row is our pivot row, which we box below:

$$\begin{array}{c|c|cccc}
1 & 1 & 1 & 0 & 0 & 4 \\
\hline
-1 & 1 & 0 & 1 & 0 & 1 \\
\hline
2 & 4 & 0 & 0 & 1 & 10 \\
\hline
-20 & -30 & 0 & 0 & 0 & 0
\end{array}$$

Note that $s_2$ is a basic variable with a 1 in our pivot row. So $s_2$ will be our leaving variable.

To find the next basic solution, we treat entire array as a matrix. Use Gaussian Elimination to make every entry in the pivot column 0, except the entry in the pivot row, which becomes a 1.

*Example* 10. Viewing the array as a matrix, and applying row operations, we get the new array

$$\begin{array}{cccccc}
2 & 0 & 1 & -1 & 0 & 3 \\
-1 & 1 & 0 & 1 & 0 & 1 \\
6 & 0 & 0 & -4 & 1 & 6 \\
\hline
-50 & 0 & 0 & 30 & 0 & 30
\end{array}$$

Our new basic variables are $x_2, s_1$, and $s_3$ (the variable $s_2$ has left). The basic variables correspond to columns of the identity matrix. The next basic solution is $(0, 1, 3, 0, 6)$. If we return to our original linear program, this corresponds to setting $x_2 = 1$ and $x_1 = 0$. In this case, our objective function equals 30, which corresponds to the value in the bottom right-hand corner of the array.

## Stopping Criterion

The main idea of the simplex method is to repeat the steps given in the previous section until we reach a stopping criterion:

**Stopping Criterion**:

1. Each entry in the pivot column is negative. In this case, the feasible region is unbounded (the objective function can be made as large as you want).
2. All entries of the objective row are positive.

To see why the second condition is a stopping criterion, suppose that after doing a number of iterations of the above procedure, the last row has the form

$$d_1 \quad d_2 \quad \cdots \quad d_n \quad e_1 \quad \cdots \quad e_m \quad f.$$

This corresponds to the function

$$d_1x_1 + d_2x_2 + \cdots + d_nx_n + e_1s_1 + \cdots e_ms_m + z = f$$

with $d_i, e_i \geq 0$. Since each $d_i, e_i \geq 0$, after rearranging we get

$$z = f - d_1x_1 - \cdots - d_nx_n - e_1s_1 - \cdots - e_ms_m.$$

So, if try to make any of the $x_i$'s or $s_i$'s bigger, we will only make $z$ smaller. So, the optimal value is $f$, and the basic solution for this tableau gives the feasible solution.

*Example* 11. We finish working out the remaining details of our example. At our last step, our tableau looked like:

$$\begin{array}{cccccc} 2 & 0 & 1 & -1 & 0 & 3 \\ -1 & 1 & 0 & 1 & 0 & 1 \\ 6 & 0 & 0 & -4 & 1 & 6 \\ \hline -50 & 0 & 0 & 30 & 0 & 30 \end{array}$$

Since neither Stopping Criterion has been reached, we continue. The first column must be our next pivot column since this is the only column where the objective row has a negative value (the $-50$). For each positive entry $r$ in this column, we form the ratio $s/r$ where $s$ is the corresponding last entry in the row. We only have two ratios to consider: $3/2$ and $6/6$ (because we have a -1 in the second row, this row is ignored). Since $6/6$ is minimal, our pivot row is the third row.

Viewing the above array as a matrix, we use row operations to turn all entries in the pivot column (except the one in the pivot row) to zero, and the remaining entry into a 1. To do this, we first re-scale the third row by $1/6$:

$$\begin{array}{cccccc} 2 & 0 & 1 & -1 & 0 & 3 \\ -1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & -\frac{4}{6} & \frac{1}{6} & 1 \\ \hline -50 & 0 & 0 & 30 & 0 & 30 \end{array}$$

Row operations then give us:

$$\begin{array}{cccccc}
0 & 0 & 1 & \frac{2}{6} & -\frac{2}{6} & 1 \\
0 & 1 & 0 & \frac{2}{6} & \frac{1}{6} & 2 \\
1 & 0 & 0 & -\frac{4}{6} & \frac{1}{6} & 1 \\
\hline
0 & 0 & 0 & -\frac{20}{6} & \frac{50}{6} & 80
\end{array}$$

Our new basic variables are $x_1, x_2$ and $s_1$. The corresponding basic solution is $(1, 2, 1, 0, 0)$.

We have yet to reach a Stopping Criterion. So, we continue with the simplex method. The next pivot column is the column corresponding to $s_2$ since we have a negative entry $-\frac{20}{6}$ in the objective row. The new pivot row is the first row. Applying row operations to the array, viewed as a matrix, gives:

$$\begin{array}{cccccc}
0 & 0 & 3 & 1 & -1 & 3 \\
0 & 1 & -1 & 0 & \frac{1}{2} & 1 \\
1 & 0 & 2 & 0 & -\frac{3}{6} & 3 \\
\hline
0 & 0 & 0 & 10 & \frac{30}{6} & 90
\end{array}$$

We have now reached the second Stopping Criterion. The basic solution in this case is $(3, 1, 0, 3, 0)$. The optimal solution is therefore $x_1 = 3$ and $x_2 = 1$, and the maximal value of our linear program is 90.

## Final Remarks

This is simply a cursory introduction to the simplex method. Notice that we really haven't discussed the mathematics (the proofs!) for why this works. In fact, there is a glaring gap in this presentation – I have not explained why the optimal solution, if it exists, must have the form of a basic solution. Also, examples can be found where the simplex method gets stuck in a cycle (although they are quite rare). The book of Chvátal [2] gets into the "nitty-gritty" details. For example, it discusses how to get around the problems like those in the warning. As well, it discuss the complexity of the simplex method, that is, the speed of this algorithm.

## Problems from Lecture 3

1. (Exercise 2.1.c of [2]) Use the simplex method to solve

$$
\begin{array}{rrcl}
\text{Maximize:} & 2x_1 + x_2 & & \\
\text{Subject to:} & 2x_1 + 3x_2 & \leq & 3 \\
 & x_1 + 5x_2 & \leq & 1 \\
 & 2x_1 + x_2 & \leq & 4 \\
 & 4x_1 + x_2 & \leq & 5 \\
 & x_1, x_2 & \geq & 0.
\end{array}
$$

## References


[1] E. Chong, S. Żak, *An Introduction to Optimization.* John Wiley & Sons, Inc., New York, 1996.

[2] V. Chvátal, *Linear programming.* W. H. Freeman and Company, New York, 1983.

[3] R. Walker, *Introduction to Mathematical Programming.* Prentice Hall, New Jersey, 1999.