Operations Research Seminar: Additional Examples with Gröbner Bases
Speaker: Adam Van Tuyl

Introduction

The purpose of this short note is to supplement the material of my last lecture with some examples of monomial orders and Gröbner bases. The majority of these examples come from the examples found in [1].

Monomial Orderings

The following three monomial orderings are some of the most commonly used monomial orderings in commutative algebra and algebraic geometry. A monomial ordering is a special type of total ordering on $\mathbb{N}^n$.

Definition 1 (Lexicographical Ordering)**.** For this ordering, $x^\alpha = x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n} > x^\beta = x_1^{b_1}\cdots x_n^{b_n}$ if leftmost nonzero entry of $(\alpha-\beta)\in\mathbb{Z}^n$ is positive.

Example 2. $x_1^5x_2^6x_3^2 > x_1^4x_2^8x_3^3$ since leftmost nonzero entry of $(5,6,2)-(4,8,3)=(1,-2,-1)$ is positive.

Definition 3 (Inverse Lex Ordering)**.** For this ordering, $x^\alpha = x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n} > x^\beta = x_1^{b_1}\cdots x_n^{b_n}$ if rightmost nonzero entry of $(\alpha-\beta)\in\mathbb{Z}^n$ is positive.

Example 4. $x_1^4x_2^8x_3^3 > x_1^5x_2^6x_3^2$ since leftmost nonzero entry of $(4,8,3)-(5,6,2)=(-1,2,1)$ is positive.

Definition 5 (Graded Reverse Lex Ordering)**.** For this ordering, $x^\alpha = x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n} > x^\beta = x_1^{b_1}\cdots x_n^{b_n}$ if

$$a_1+\cdots+a_n > b_1+\cdots+b_n,$$

or if $a_1+\cdots+a_n = b_1+\cdots+b_n$, then rightmost nonzero entry of $(\alpha-\beta)\in\mathbb{Z}^n$ is negative.

Example 6. $x_1^4x_2^8x_3^3 > x_1^5x_2^6x_3^2$ since $4+8+3 > 5+6+2$. Also $x_1^5x_2^6x_3^2 > x_1^4x_2^6x_3^3$ since $5+6+2 = 4+6+3$ and $(5,6,2)-(4,6,3)=(1,2,-1)$.

An Example of the Division Algorithm

In the last lecture, we presented the Division Algorithm. We now give an example. Consider the polynomial ring $R=k[x,y]$ where $k=\mathbb{R}$ or $\mathbb{C}$. Let

$$f_1 = xy+1 \text{ and } f_2 = y^2-1.$$

Fix our monomial ordering $>$ to be the lexicographical ordering. Then $LT(f_1)=xy$ and $LT(f_2)=y^2$. Consider the polynomial

$$f = xy^2 - x$$

and divide it by the set $F=\{f_1,f_2\}$. Then

$$\begin{aligned} xy^2-x &= y\cdot f_1 + 0\cdot f_2 + (-x-y)\\ &= y(xy+1)+0(y^2-1)+(-x-y). \end{aligned}$$

Note that $f^F = (-x-y)$ and no term of f^F is divisible by either $LT(f_1)$ or $LT(f_2)$.

An Example of a Gröbner Basis

Continue with the last example. The polynomials f_1 and f_2 do not form a Gröbner basis for the ideal they generate, i.e., the ideal $I = (f_1, f_2)$. To see this, we clearly have

$$(LT(f_1), LT(f_2)) = (xy, y^2) \subseteq LT(I)$$

where $LT(I) = \{LT(f) \mid f \in I\}$. On the other hand,

$$yf_1 - xf_2 = xy^2 + y - xy^2 - x = y - x$$

is an element of I with leading term $-x$. So $-x \in LT(I)$, but $-x \notin (xy, y^2)$ since everything in the ideal (xy, y^2) has degree two or higher.

We use our favorite compute algebra package to find the Gröbner basis with respect to the lex ordering:

$$\{g_1, g_2\} = \{y^2 - 1, -x - y\}$$

We then have

$$(g_1, g_2) = (y^2 - 1, -x - y) = (xy + 1, y^2 - 1) = (f_1, f_2)$$

and

$$(LT(g_1), LT(g_2)) = (y^2, -x) = LT(I).$$

References

[1] Cox, David; Little, John; O'Shea, Donal *Ideals, varieties, and algorithms. An introduction to computational algebraic geometry and commutative algebra. Second edition.* Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1997.