Some OR Models in Graphs and Networks

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In this talk, we will present some optimization/OR models in graph or network setting through examples.

Model 1. Transportation Problem (TP)

Powerco has 3 electric power plants that supply the needs of 4 cities. Other parameters are given as below:

Shipping cost ($)	City 1	City 2	City 3	City 4	Supply from plant (kw/hr)
Plant 1	8	6	10	9	35
Plant 2	9	12	13	7	50
Plant 3	14	9	16	5	40
Demand from city (kw/hr)	45	20	30	30	

In general, a TP has the following parameters:

- m **supply points** from which a good is shipped. Each supply has **capacity** s_i, $i = 1, 2, \ldots, m$.
- n **demand points** to which the good is shipped. Each has the minimum **demand** d_j, $i = 1, 2, \ldots, n$.
- **Shipping costs** c_{ij} to ship one unit good from supply point i to demand point j.

  Objective: find the optimal transportation plan that minimizes the total transportation/shipping cost.

LP formulation:

we first have to define the decision variables, which will be used to fully describe the solution (how will the solution be described?):

x_{ij} = Amount of produced at supplier i and sent to demand j.

Objective function:

$$\text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Supply constraints:

$$\sum_{j=1}^{n} x_{ij} \leq s_i,\ i = 1, 2, \ldots, m$$

Demand constraints:

$$\sum_{i=1}^{m} x_{ij} \geq d_j,\ i = 1, 2, \ldots, n$$

Sign constraints:

$$x_{ij} \geq 0,\ i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n.$$

Due to its special structure of the parameter matrix in LP, a more efficient version of simplex method are generally used to solve a transportation problem.

An application: inventory problem.

Sailco Corporation must determine how many sailboats should be produced during each of the next four quarters. Demands that have to be met are: 40 boats for the 1st quarter, 60 boats for the 2nd quarter, 75 boats for the 3rd quarter, and 25 boats for the 4th quarter. Production capacities: 40 boats per quarter at the unit cost of $400, and over production is $450 per boat. Other conditions:

- Boats that are produced during a quarter can be used to meet the demand of the same quarter.
- Initial inventory: 10 boats;
- Carrying or holding cost: $20 per boat per quarter.

Objective: make a production plan for the next 4 quarters to minimize the total cost, including production costs and inventory costs.

Transportation formulation:

	1st qt	2nd qt	3rd qt	4th qt	Dummy	Supply
Initial inv.	0	20	40	60	0	10
1 qt reg	400	420	440	460	0	40
1 qt over	450	470	490	510	0	**150**
2 qt reg	**M**	400	420	440	0	40
2 qt over	**M**	450	470	490	0	**150**
3 qt reg	**M**	**M**	400	420	0	40
3 qt over	**M**	**M**	450	470	0	**150**
4 qt reg	**M**	**M**	**M**	400	0	40
4 qt over	**M**	**M**	**M**	450	0	**150**
Demand	40	60	75	25	**570**	**770**

Model 2. (Assignment Problem)

Machineco has four jobs to be completed. Each machine must be assigned to complete one job. The time required to setup each machine for completing each job depend on machine-job combinations. Machineco wants to minimize the total setup time needed to complete the four jobs. The cost matrix (or AP Table) is

$$\begin{bmatrix} 14 & 5 & 8 & 7 \\ 2 & 12 & 6 & 5 \\ 7 & 8 & 3 & 9 \\ 2 & 4 & 6 & 10 \end{bmatrix}$$

General problem (use the machine-job setting): There are m machines and n jobs. Each machine can process at most one job, and each job has to be processed on exactly one machine. c_{ij} is the cost to have job j processed on machine i. The objective is to find the best ”match” (assignment) so that the total cost of processing all jobs is minimized.

IP Formulation of AP:

$$\min Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$$s.t.: \sum_{i=1}^{m} x_{ij} = 1,\ j = 1,2,\ldots,n,$$

$$: \sum_{j=1}^{n} x_{ij} \le 1,\ i = 1,2,\ldots,m,$$

$$: x_{ij} \in \{0,1\}.$$

Although the problem is an integer programming problem, it can be shown that it is equivalent to its LP relaxation.

Assignment Problem in its classical version is rather simple. More general versions include Nonlinear Assignment Problem, Continuous Assignment Problem, and Multi-objective Assignment Problem.

Model 3. (Shortest Path Problem)

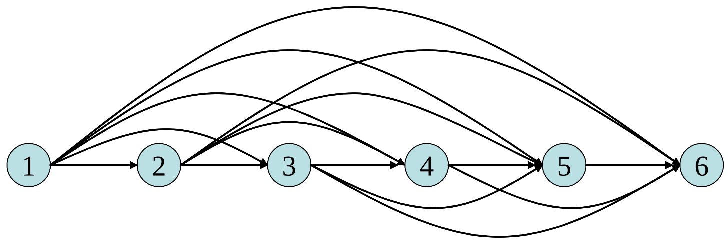
Given a graph (with undirected edges) or a network (with directed arcs). Each edge/arc is associated with an nonnegative weight (length of the road, traveling time, cost of traveling, etc.) The Shortest Path Problem (SPP) is to find a path from one node s to another node d that has the minimum total length. The problem can be easily solved by some sort of labeling method called the Dijkstra's Algorithm.

Application: Machine/Equipment Replacement Problem.

Assume that we have just purchased a new car (or machine) for $12,000 at year 0. The cost of maintaining the car during a year depends on the age of the car at the beginning of the year. In order to avoid the high maintenance cost associated with an older car, we may trade in the car and purchase a new car at beginning of any year. To simplify the computations we assume that at any time it costs $12,000 to purchase a new car. Our goal is to minimize the net cost incurred during the next five years.

Age of Car (Years)	**Annual Maintenance cost**	**Age of Car (Years)**	**Trade-in Price**
0	$2,000	1	$7,000
1	$4,000	2	$6,000
2	$5,000	3	$2,000
3	$9,000	4	$1,000
4	$12,000	5	$0

Let's formulate this problem as a shortest path problem. Our network will have six nodes: Node i is the beginning of year i, $i = 1,2,3,4,5,6$. For each pair $i < j$, an arc (i,j) corresponds to purchasing a new car at the beginning of year i and keeping it until the beginning of year j.



The length of the are (i,j), called c_{ij}, is the total net cost incurred from years i to j, i.e.

c_{ij} = purchasing cost for a new car at beginning of year i

$+$ maintenance cost to keep the car from year i to year j

$-$ trade-in revenue at the end of year j.

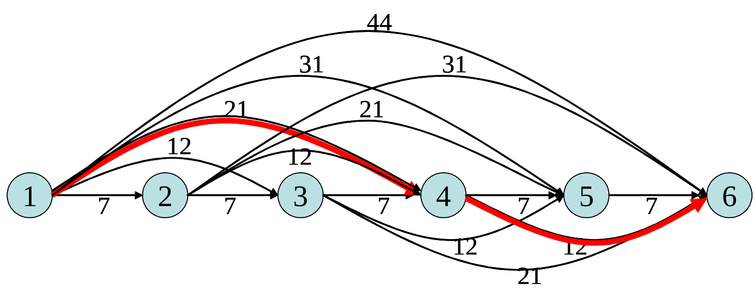
For example

$c_{25} = \$12000$ purchasing a new car at year 2

$+ (\$2000 + \$4000 + \$5000)$ three years of maintenance cost

$- \$2000$ trade-in revenue for a 3 years old car $= \$21000$

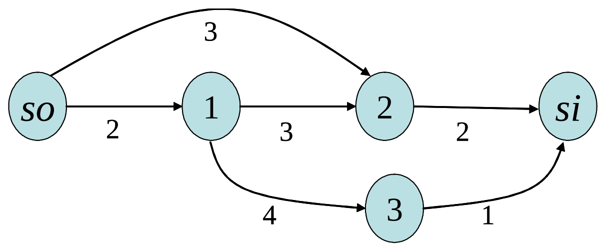
An example of solutions



Model 4. Maximum Flow Problem

In a network, we have a source node so and a sink node si. Each arc in the network has a capacity constraint. We need to send maximum amount of goods (flow) from so to si along the arcs within their capacities.

As an example, Sunco Oil wants to ship the maximum possible amount of oil (per hour) via pipeline from node so, source, to node si, sink, as shown in the figure below.



LP formulation:

Decision variables:

$$x_{ij} = \text{amount of oil passing pipeline } (i,j),\ i = 0,1,2,3,\ j = 1,2,3,4.$$

Arc capacity constraints:

$$0 \le x_{01} \le 2,\ 0 \le x_{02} \le 3,\ 0 \le x_{12} \le 3,\ 0 \le x_{13} \le 4,$$

$$0 \le x_{24} \le 2,\ 0 \le x_{34} \le 1,\ x_{ij} = 0,\ \text{for all other arcs.}$$

Actually, we just need 6 variables.

Node flow constraints:

$$\text{For Node 1 : } x_{01} = x_{12} + x_{13},$$

$$\text{For Node 2 : } x_{02} + x_{12} = x_{24},$$

$$\text{For Node 3 : } x_{13} = x_{34}.$$

Now what is the objective? We may maximize

$$z = x_{24} + x_{34}$$

or

$$z = x_{01} + x_{02}$$

We assume that there is no "leaking" in all pipes.

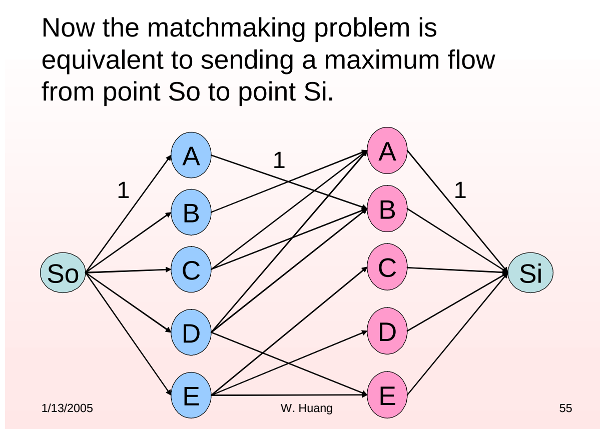
Application: Match Making

Five male and five female entertainers are at a dance. The goal of the matchmaker is to match each woman with a man in a way that the number of compatible mates is maximized. The compatibility table is as following

	Ann	Beth	Cindy	Doris	Emma
Al		**C**			
Bob	**C**				
Chris	**C**	**C**			
Dale	**C**	**C**			**C**
Ed			**C**	**C**	**C**

We can construct a network that is equivalent to the problem:

- Nodes represent the 5 male and 5 females;
- Arcs represent compatible pairs, all with capacity of 1;
- We also introduce an artificial source, an artificial sink, and artificial arcs as shown below.



Model 5. CPM and PERT

CPM stands for Critical Path Method. For a project that consists a number of activities. Each activity's processing time is known in advance. There are precedence constraints between activities (a partial order). CPM is used to determine minimum amount of time needed to finish the project by identifying the critical path in the project.

PERT stands for Project Evaluation and Review Technique. In real world, the processing time for the individual activities are to some degree random. When the minimum expected time required to finished the project is found by CPM, PERT is used to access the probability that the project can actually be finished within certain time frame. When a dead line is imposed, "crashing" some activities are usually needed. A LP can be set up to determine which activities need to be crashed and the amounts of the crashes in order to minimize the crashing cost.]]

Model 6. Minimum Cost Network Flow Problem (MCNFP)

A general MCNFP has the following parameters: Given a network (N, A), where N is the set of nodes and A is the set of arcs (directed edges),

- For each $i \in N$, b_i is the net supply (outflow - inflow) at node i (or requirement)
  Special cases:
  - The source node has only outflow, so the net supply is positive;
  - The sink node has only inflow, so the net supply is negative;
  - The transshipment point with no warehouse capacity would have the net supply of zero.
- $c_{ij} =$ cost of transporting one unit of flow from node i to node j via arc (i,j). If the arc does not exist, use a large number (in really, the arc can be created with additional cost.)
- $L_{ij} =$ Lower bound on flow through arc (i,j). In most cases, we have $L_{ij} = 0$.
- $U_{ij} =$ Upper bound on flow through arc (i,j). If there is no upper bound, choose $U_{ij} = \infty$ or a large number in practice.

The objective is the arrange the flow along all arcs to meet all the supply/demand requirement at the minimum cost.

An LP can be easily set up by letting $x_{ij} =$ number of units of flow sent from node i to note j via arc (i,j). We have

$$\min Z = \sum_{\text{All } (i,j)} c_{ij} x_{ij}$$

$$s.t. : \sum_{\text{All } j} x_{ij} - \sum_{\text{All } k} x_{ki} = b_i, \text{ for all } i,$$

$$: L_{ij} \le x_{ij} \le U_{ij}, \text{ for all } (i,j).$$

MCNFP include the following problems as special cases:

- Transportation problem;
- Assignment problems;
- Transshipment problems;
- Shortest path problems;
- Maximum flow problems;
- CPM problems, etc.