ALGORITHM FOR TRANSPORTATION AND ASSIGNMENT PROBLEMS

GASPAR ASAMPANA

1. TRANSPORTATION PROBLEM

1.1. INTRODUCTION

A transportation problem involves the shipment of goods from various origins or sources of supply to a set of destinations (e.g. retail outlets, cities), each destination demanding a specified level of the commodity. For a typical transportation problem, there are:

1. m origins with the ith having a supply of s_i units, $i = 1\dots m$
2. n destinations with the jth demanding d_j units, $j = 1, \dots n$
3. cost c_{ij} for shipping a unit of the commodity from origin i to destination j. The cost of distributing units from any origin to any particular location s_i is proportional to the number of units distributed.

The objective function is to minimize total transportation cost or to maximize the total profit contribution from the allocation.

1.2. CLASSICAL TRANSPORTATION MODEL

The transportation problem can be formulated into a LP problem. Let x_{ij}, $i = 1\dots m$, $j = 1\dots n$ be the number of units transported from origin i to destination j. The LP problem is as follows

$$\text{Minimize} \; : \; \sum_{i=i}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$$\text{Subject to} \; : \; \sum_{j=1}^{n} x_{ij} \le s_i, (i = 1\dots,m) \tag{1}$$

$$\sum_{i=1}^{m} x_{ij} \ge d_j, (j = 1,\dots,n)$$

$$x_{ij} \ge 0, (i = 1,\dots m, \; j = 1,\dots n).$$

A transportation problem is said to be *balanced* if

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j,$$

All constraints in any balanced transportation problem must be binding. For a balanced TP, the original problem is equivalent to

$$\text{Minimize} : \sum_{i=i}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{subject to} : \sum_{j=1}^{n} x_{ij} = s, (i = 1\ldots, m)$$

$$: \sum_{i=1}^{m} x_{ij} = d_j, (j = 1, \ldots, n)$$

$$: x_{ij} \geq 0, (i = i, \ldots m; j = 1, \ldots n).$$

1.3. BALANCING AN UNBALANCED TP

In real situations, the total supply may not be equal to the total demand. It is possible to reformulate the problem to a balanced one. If total supply is more than total demand, we introduce a dummy destination with a capacity equal to excess supply and the cost involved in shipping a commodity from the origin to the dummy destination is zero. Likewise if total demand is more than total supply, a dummy origin or source is introduced with a capacity exactly equal to the excess demand.

1.4. EXAMPLE

Cape Coast has three water pumping machines that supply the water needs of four schools. Each machine can supply the following number of water gallons: machine A - 35 million gallons, machine B - 50 milion gallons, and machine C- 40 million gallons. The peak water demand in these schools which occur on weekends are as follows: School 1- 45 million, School 2- 20 million, School 3- 30 million and School 4- 30 million. The cost of pumping one million gallon of water from a machine to a school depends on the distance the water travels (see Table 1). Formulate an LP to minimize the cost of meeting each school peak water supply.

from	sch1	sch2	sch3	sch4	supply(mill. gal.)
machine A	$8	$6	$10	$9	35
machine B	$9	$12	$13	$7	50
machine C	$14	$9	$16	$5	40
demand (million gallons)	45	20	30	30	
Table 1					

We first of all, check whether the problem is balance. From Table1, we can see that $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j = 125$, and there are $m \times n = 3 \times 4$ decisions to make. From the example, there are two kind of constraints, the demand constraints and the supply constraints. Let x_{ij}

= the number of million gallons of water from machine i to school j. Then our objective function to be minimized is,

z = [(cost of pumping water from machine A) + (cost of pumping water from machine B) + (cost of pumping water from machine C)]. Thus,

2

$$z = 8x_{11} + 6x_{12} + 10x_{13} + 9x_{14} + 9x_{21} + 12x_{22} + 13x_{23} + 7x_{24} + 14x_{31} + 9x_{32} + 16x_{33} + 5x_{34}.$$

THE DEMAND CONSTRAINTS:

Each school is a demand point and expects to receive a quantity of water not less than its peak demand. From (1) above, the demand constraints can be written as shown below. Each variable with second subscript 1 represents the pumping of water from machine $i, (i = 1, 2, 3)$ to the school 1, so we have the constraint $x_{11} + x_{21} + x_{31} \geq 45$. Similarly the constraint for each of School $2, 3,$ and 4 follow the same pattern.

$$x_{11} + x_{21} + x_{31} \geq 45,$$

$$x_{12} + x_{22} + x_{32} \geq 20$$

$$x_{13} + x_{23} + x_{33} \geq 30$$

$$x_{14} + x_{24} + x_{34} \geq 30.$$

THE SUPPLY CONSTRAINTS:

Each machine can supply no more wate than its capacity. From Machine 1 to School j, $(j = 1, 2, 3, 4)$, we have the constraint $x_{11} + x_{12} + x_{13} + x_{14} \leq 35$. Similarly from machine $2, 3,$ and 4 we have the following constraints.

$$x_{11} + x_{12} + x_{13} + x_{14} \leq 35$$

$$x_{21} + x_{22} + x_{23} + x_{24} \leq 50$$

$$x_{31} + x_{32} + x_{33} + x_{34} \leq 40.$$

We can see that total demand and supply are equal. Hence this an example of a balanced transportation problem.

1.5. THE TRANSPORTATION TABLEAU

The transportation tableau consist of an $m \times n$ array of rectangles called cells, one for each variable. The rows correspond to supplies and the columns to demands. The costs are place in boxes in the upper right hand corner of the corresponding cells. The supplies are written at the right hand and the demand are written across the bottom row of the tableau. The placement of the demand and supply on the tableau varies from author to author. Values of the basic variables x_{ij} will be placed at the lower left hand corner of the cells as we will see through the algorithm.

	dest. 1	dest. 2	dest. 3	dest. 4	supply
source1	c_{11}	c_{12}	c_{13}	c_{14}	s_1
source2	c_{21}	c_{22}	c_{23}	c_{24}	s_2
source3	c_{31}	c_{32}	c_{33}	c_{34}	s_3
demand	d_1	d_2	d_3	d_4	

The TP tableau.

Where the d_1, d_2, d_3, d_4, and s_1, s_2, s_3, are the total demands and supplies, respectively

3

and dest. = destinations. We illustrate the transportation tableau using the example above with the 3 supply points and 4 demand points.

	sch1	sch2	sch3	sch4	supply
mach. 1	x_{12} 8	6	10	9	35
mach. 2	x_{21} 9	x_{22} 12	x_{23} 13	7	50
mach. 3	14	9	x_{33} 16	x_{34} 5	40
demand	45	20	30	30	

<center>Table 3.</center>

- Definition

A feasible solution is a solution containing variables called basic variables whose values satisfy the structural constraints and the nonnegativity constraints. (Basic variables are those variables selected for solution. Variables not selected for solution are called non - basic variables and are set to zero. Given a balanced transportation problem with m origins and n destinations, a basic feasible solution (bfs) has $m + n - 1$ basic variables.

(How to find feasible solution using the transportation tableau.)
- Transportation model once formulated can be solved by several methods. We shall engage only two of the methods in this presentation.

  1. the simplex algorithm

  2. heuristic algorithm

The second one can be used to determine initial basic solution. It consist of several methods namely; the north west corner method, the minimum cost method and the Vogels method. These methods are primarily used to generate basic solutions unless otherwise tested to be optimal. We take a look at the northwest corner method and introduce the minimum cost method.

DEFINITION

An order sequence of at least four different cells is called a *loop* if:

1. Any two consecutive cells lie in either the same row or column

2. No three consecutive cells lie in the same row or column.

3. The last cell in the sequence has a row or column in common with the first cell in the sequence.

Here a loop is represented as a closed path. For example, $(2,1), (2,4), (4,4), (4,1)$ is a loop. Also $(1,1), (1,2), (2,2), (2,3), (4,3), (4,5), (3,5), (3,1)$ is a loop **but** $(1,1), (1,2), (2,3), (2,1)$ and $(1,2), (1,3), (1,4), (2,4), (2,2)$ is not a loop.

1.6. GENERATING INITIAL BASIC SOLUTION.

THE NORTHWEST CORNER METHOD

Begin in the upper left corner of the transportation tableau and set the cell to its min (d_1, s_1). If $x_{11} = s_1$, cross out the first row of the tableau and change the first column to $d_1 - s_1$, otherwise, cross out the the first column and change the first row to $s_1 - d_1$. If $x_{11} = s_1 = d_1$, cross out either s_1 or d_1 but not both of the tableau and set the other to zero. Continue applying the same procedure to the

most upper left corner of the cells in the tableau that does not lie in a crossed out row or column. Eventually you will come to a point where there is only one cell that can be assigned a value. Assign this cell a value equal to its row or column demand and cross out the cell's row and column. A basic feasible solution has now been obtained.

We use our example to illustrate this method.

	sch1	sch2	sch3	sch.4	supply
mach.1	x_{11} 8	x_{12} 6	x_{13} 10	x_{14} 9	35
mach.2	x_{21} 9	x_{22} 12	x_{23} 13	x_{24} 7	50
mach.3	x_{31} 14	x_{32} 9	x_{33} 16	x_{34} 5	40
demand	45	20	30	30	

We begin by setting $x_{11} = \min(35, 45) = 35$. Then we cross out row one and change the value in the demand

row of column one to $45 - 35$.

	sch1	sch2	sch3	sch.4	supply
mach.1	$x_{11} = 35$ 8	x_{12} 6	x_{13} 10	x_{14} 9	$\times$
mach2	x_{21} 9	x_{22} 12	x_{23} 13	x_{24} 7	50
mach3	x_{31} 14	x_{32} 9	x_{33} 16	x_{34} 5	40
demand	10	20	30	30	

Next, we set the most north west cell not in the crossed out row. Thus, $x_{21} = \min(50, 10) = 10$. Then we cross out column one and change the value in the supply column of row two to $50 - 10$.

	sch1	sch2	sch3	sch.4	supply
mach.1	$x_{11} = 35$ 8	x_{12} 6	x_{13} 10	x_{14} 9	$\times$
mach2	$x_{21} = 10$ 9	x_{22} 12	x_{23} 13	x_{24} 7	40
mach3	x_{31} 14	x_{32} 9	x_{33} 16	x_{34} 5	40
demand	$\times$	20	30	30	

The next most northwest cell not in the cross - out row or column $x_{22} = \min(40, 20) = 20$. Then we cross out column two and change the value of the supply column of row two to $40 - 20$.

5

	sch1	sch2	sch3	sch.4	supply
mach.1	$x_{11}=35$ 8	x_{12} 6	x_{13} 10	x_{14} 9	×
mach2	$x_{21}=10$ 9	$x_{22}=20$ 12	x_{23} 13	x_{24} 7	20
mach3	x_{31} 14	x_{32} 9	x_{33} 16	x_{34} 5	40
demand	×	×	30	30	

The next is $x_{23}=\min(20,30)=20$. Again we cross out row two and change the value of the demand row of column three to $30-20$.

	sch1	sch2	sch3	sch.4	supply
mach.1	$x=35$ 8	x_{12} 6	x_{13} 10	x_{14} 9	×
mach2	$x_{21}=10$ 9	$x_{22}=20$ 12	$x_{23}=20$ 13	x_{24} 7	×
mach3	x_{31} 14	x_{32} 9	x_{33} 16	x_{34} 5	40
demand	×	×	10	30	,

Then $x_{33}=\min(40,10)=10$, cross out column three, change the value of the supply column of row three to $40-10$.

	sch1	sch2	sch3	sch.4	supply
mach.1	$x_{11}=35$ 8	x_{12} 6	x_{13} 10	x_{14} 9	×
mach2	$x_{21}=10$ 9	$x_{22}=20$ 12	$x_{23}=20$ 13	x_{24} 7	×
mach3	x_{31} 14	x_{32} 9	$x_{33}=10$ 16	x_{34} 5	3o
demand	×	×	×	30	

lastly you assign either the row or the column value to $x_{34}=30$ and cross out both the row and the column.

	sch1	sch2	sch3	sch.4	supply
mach.1	$x_{11}=35$ 8	x_{12} 6	x_{13} 10	x_{14} 9	×
mach2	$x_{21}=10$ 9	$x_{22}=20$ 12	$x_{23}=20$ 13	x_{24} 7	×
mach3	x_{31} 14	x_{32} 9	$x_{33}=10$ 16	$x_{34}=30$ 5	×
demand	×	×	×	×	,
			Table 6		

Usually the last row and column will be the same and $m+n$ rows and columns must be crossed out. Since the last variable assigned a value results in a row and column being crossed out, the northwest corner method will assign values to $m+n-1$ variables. Note that the variable chosen by the northwest method cannot form a loop implying that the northwest method yields a basic feasible solution. Also note that the method does not take the costs into consideration and only uses the right hand side of the demand and supply constraints. Thus it does not utilize shipping cost and can yield a basic feasible solution with high

shipping cost. This is one disadvantage of the method.

THE MINIMUM COST METHOD

The minimum cost method uses the shipping costs in an effort to produce a bfs that does not have an extremely high total cost. This method uses fewer pivot points to find the problem's optimal solution. We begin the minimum cost method by finding the variable with the smallest shipping cost, say x_{ij}. Then assign x_{ij} its largest possible value. Thus as in the northwest corner method, $\min(s_i, d_j)$. Cross out row i or column j and reduce the supply or demand of the crossed-out row or column by the value of x_{ij}. Then choose from the cells that do not lie in the crossed-out row or column the cell with the minimum shipping cost and repeat the procedure. Continue until there is only one cell that can be chosen. In this case cross - out both the cell's row and column. We assume G and V to be demand points and supplies points respectively. we let sup denote supply and deman. denote demand in the tables below.

	G_1	G_2	G_3	sup.
V_1	7	6	8	10
V_2	15	80	78	15
deman.	15	5	5	.

Table 7

.

	G_1	G_2	G_3	sup.
V_1	7	6	8	5
V_2	15	80	78	15
deman.	15	×	5	

Table 8

Here the minimum cost is coming from row two. So we set $x_{12} = \min(10, 5) = 5$. Then we cross out column two and change row one to $10 - 5$.(See Table 8) The next minimum cost not in the crossed - out row or column is x_{11}. So set $x_{11} = \min(5, 15) = 5$. Cross out row one and change column one to $15 - 5$, in Table 8. You continue and end at the last one as we did in the northwest corner method. The next point we shall look into is how to determine whether the solution generated is optimum, and if not how do we obtain the optimal solution.

1.7. OPTIMAL CRITERIA. Is the current basic solution optimal?.

A bfs is said to be optimal if all the non - basic variables in the tableau assume a nonpositive coefficient. However, if there are non -basic variables with non- negative coefficients, the most postive one is chosen to enter the basis. We first determine by algebra, the coefficient of the variable x_{ij}, say k_{ij}. The coefficient of the variable x_{ij}, in row zero is given by

$$k_{ij} = C_{BV}B^{-1}a_{ij} - c_{ij}$$

where c_{ij} is the objective function coefficient (costs) of the variable x_{ij} and a_{ij} is the column for x_{ij} in the original LP (assuming we have $m + n - 1$ constraints). For the minimization problem the basic feasible solution will be optimal if all the k_{ij} are nonpositive. Otherwise, we enter into the basis the variable with the most positive k_{ij}. After determining $C_{BV}B^{-1}$, we can calculate k_{ij}. Since the first constraints has been dropped, $C_{BV}B^{-1}$ will have $m + n - 1$

elements, where

$$C_{BV}B^{-1} = [u_2 \; u_3 \ldots u_m \; v_1 \; v_2 \ldots v_n]$$

$u_2, \; u_3, \ldots, u_m$ are the elements of $C_{BV}B^{-1}$ corresponding to the $m - 1$ supply constraints, and $v_1, v_2, \ldots, v_n$ are the elements of $C_{BV}B^{-1}$ corresponding to n demands constraints. To determine $C_{BV}B^{-1}$ we employ the principle that in any tableau, each basic variable x_{ij} must have $k_{ij} = 0$. Thus, for each $m + n - 1$ BV,

$$C_{BV}B^{-1}a_{ij} - c_{ij} \quad = 0.$$

Mathematically, the basic feasible solution is optimal, if for each nonbasic variable $k_{ij} \leq 0$

The basic variables in the transformation tableau of our example are:

$$BV = [X_{11}, X_{21}, X_{22}, X_{23}, X_{33}, X_{34}]$$

To determine our $k_{ij}s$, for the basic variables,

$$k_{11} = [u_2 \; u_3 \; v_1 \; v_2 \; v_3 \; v_4] \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - 8 = v_1 - 8 = 0$$

$$k_{21} = [u_2 \; u_3 \; v_1 \; v_2 \; v_3 \; v_4] \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - 9 = u_2 + v_1 - 9 = 0$$

$$k_{22} = [u_2 \; u_3 \; v_1 \; v_2 \; v_3 \; v_4] \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - 12 = u_2 + v_2 - 12 = 0$$

$$k_{23} = [u_2 \ u_3 \ v_1 \ v_2 \ v_3 \ v_4] \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - 13 = u_2 + v_3 - 13 = 0$$

$$k_{33} = [u_2 \ u_3 \ v_1 \ v_2 \ v_3 \ v_4] \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - 16 = u_3 + v_3 - 16 = 0$$

$$k_{34} = [u_2 \ u_3 \ v_1 \ v_2 \ v_3 \ v_4] \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - 5 = u_3 + v_4 - 5 = 0$$

Then we solve the system of $m + n - 1$ equation, for $u_1 = 0$, $u_i + v_j = c_{ij}$ for all basic variables. From the system we have, $u_1 = 0$, $u_2 = 1, u_3 = 4, v_1 = 8, v_2 = 11, v_3 = 12, v_4 = 1$. From this result, we can now compute k_{ij} for each nonbasic variable after knowing our $C_{BV}B^{-1}$. Thus, $k_{12} = 5$, $k_{13} = 2$, $k_{14} = -8$, $k_{24} = -5$, $k_{31} = -2$, $k_{32} = 6$. If all the coefficients of the nonbasic variable x_{ij}, as computed above were negative, then we could conclude that an optimal solution is obtained.

However, some of the coefficients are nonnegative so we enter into the basis with the most positive coefficient and that is $k_{32} = 6$ corresponding to the variable x_{32}.

Pivoting (moving from one bfs to another)
1. Determine the variable by criteria that should enter the basis as above.
2. Find a loop involving the entering variable and some of the basic variables. Only one loop is available.
3. Counting only cells in the loop, label the cells in the loop found in step 2, that are even numbered $(0, 2, 4, ...)$
of cells away from the entering as even cells. Also label those cells in the loop that are an odd number of cells
away from the entering variable as odd cells.
4. Find the odd cell whose variable assumes the smallest value. Call this value θ . The

9

variable corresponding to these odd cell will leave the basis. To perform the pivot, decrease the value of each odd cell by θ and increase the value of each even cell by θ. The values of variables not in the loop will remain unchanged. The pivot is now complete. If $\theta = 0$, the entering variable will equal to zero and an odd value that has a value of zero will leave the basis. In this case a degenerate basic solution existed before and will result after pivoting. If more than one odd cell in the loop equals to zero, you may arbitrarily choose one of these odd cells to leave the basis and again a degenerate basic solution will result.

Since we now know by criteria the nonbasic variable that should enter the basis, we go through the process as listed above using our example. The loop involving x_{32} and some of the basic variables is $(3,2) - (3,3) - (2,3) - (2,2)$. The odd ones in the loop are, $(3,3) - (2,2)$. since $x_{33} = 10$ and $x_{22} = 20$. Our $\theta = 10$ meaning x_{33} will leave the basis. Then we decrease x_{33} and x_{22} by 10 and increase x_{23} and x_{32} by 10. The result is shown in Table 10.

	sch1	sch2	sch3	sch.4	supply
mach.1	35 ^{8}	^{6}	^{10}	^{9}	35
mach2	10 ^{9}	20 ^{12} ·	20 · ^{13}	^{7}	50
mach3	^{14}	→· ^{9}	10 · ^{16}	30 ^{5}	40
demand	45	20	30	30	

Table 9

The four dots in the table indicates the formation of the loop involving x_{32} and some of the basic variables as described above.

Table 10.

	sch1	sch2	sch3	sch.4	supply
mach.1	35 · ^{8}	· ^{6}	^{10}	^{9}	35
mach2	10 · ^{9}	· 10 ^{12}	30 ^{13}	^{7}	50
mach3	^{14}	10 ^{9}	^{16}	30 ^{5}	40
demand	45	20	30	30	

Again when we compute k_{ij} for each nonbasic variable x_{12} will have the most positive value of $k_{ij} = 5$. Thus x_{12} will enter the basis and the loop involving x_{12} and some of the basic variables is $(1,2),(2,2),(2,1),(1,1)$. The odd cells are $(2,2),(1,1)$ and the even cells are $(1,2),(2,1)$. X_{22} is the smallest entry and we add 10 to each even cell and take away 10 from each odd cell as show on the table 11.

10

	sch1	sch2	sch3	sch.4	supply
mach.1	25 · ⟮8⟯	10 · ⟮6⟯	· ⟮10⟯	⟮9⟯	35
mach2	20 · ⟮9⟯	⟮12⟯	· 30 ⟮13⟯	⟮7⟯	50
mach3	⟮14⟯	10 ⟮9⟯	⟮16⟯	30 ⟮5⟯	40
demand	45	20	30	30	,

Table 11

We have one more table to compute but since we know how to pivot, we shall move to the last table where the optimal table is presented.

	sch1	sch2	sch3	sch.4	supply
mach.1	⟮8⟯	10 ⟮6⟯	25 ⟮10⟯	⟮9⟯	35
mach2	45 ⟮9⟯	⟮12⟯	5 ⟮13⟯	⟮7⟯	50
mach3	⟮14⟯	10 ⟮9⟯	⟮16⟯	30 ⟮5⟯	40
demand	45	20	30	30	

Here we obtain an optimal solution since each k_{ij} for the nonbasic variables is less than zero. The optimal solution is $x_{12} = 10$, $x_{13} = 25$, $x_{21} = 45$, $x_{23} = 5$, $x_{32} = 10$, $x_{34} = 30$. This will yield an objective function value of $\$1020$.

2.0 THE ASSIGNMENT PROBLEMS (AP)
An assignment problem is a special class of the transportation problem in which all demands and supplies are equal to 1. APs occur in applications in which each supply corresponds to an assignee to be assigned a task; and each demand corresponds to a task to be performed. The assignee may not necessarily be a person, it could be a machine or even time slots to be assigned tasks. To fit the definition of an assignment problem, the following assumptions are required.
1. The number of assignees and the number of tasks are the same.
2. Each assignee is to be assigned to exactly one task.
3. Each task is to be performed by exactly one assignee.
4. There is a cost c_{ij} associated with assignee i $(i = 1,2,\ldots,n)$ performing job j $(j = 1,2,\ldots,n)$
5. The objective is determine how all n assignments should be made to minimize the total cost .
Many applications in the real sense do not fit the first requirement, however, it is possible to formulate or reformulate the problem, irrespective of the calibre, to fit the requirement. As described in the transportation problem, we introduce dummy assignees or dummy tasks so as make the problem balanced ($n \times n$). In this case, each assignee can hold on to a task and vice versa.

2.1 ASSIGNMENT PROBLEM MODEL

In assignment problems, each x_{ij} is a binary variable, it is either yes or no decision. Thus , $x_{ij} = 1$, if demand point j $(j = 1, 2, \ldots, n)$ is performed by supply point i ($i = 1, 2, \ldots, n$). Otherwise it is 0. Formulation of a ILP for assignment problem may take the following form:

minimize;

$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij},$$

subject to;

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, 2, \ldots, n,$$

$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, 2, \ldots, n,$$

$$x_{ij} = 0, \text{ or } 1 \text{ for all } i \rightarrow j$$

The structure fits that of any transportation model where $s_i = 1$ and $d_j = 1$ and can be solved by the simplex algorithm. However, a special type of algorithm called the Hungarian algorithm that takes advantage of the structure and efficiently solves the system using principle of opportunity costs.

2.2 THE HUNGARIAN METHOD.

2.2.1 Determine the opportunity cost table.

a. Using the costs table provided, determine the row - reduced cost table by subtracting the least cost element in each row from all elements in the row .

b. Using the row reduced cost table, identify the least cost element in each column, and subtract it from all elements in the column

2.2.2. Determine whether or not an optimal assignment can be made. Draw the minimum number of straight lines necessary to cover all the zero elements in the opportunity cost table. If the number of straight lines is less than the number of rows or columns in the table, an optimal assignment cannot be made. Go to step 3. If the number of straight lines is equal to the number of rows or columns an optimal assignment **may** be available.

2.2.3 Revise the opportunity cost table. Identify the smallest element in the opportunity cost table that is not cover by a straight line. Subtract this element from every element not covered by the a straight line and add this element to any element(s) found at the intersection of two straight lines. Now go back to step 2.2.2 and determine the optimal solution.

Example.
Five jobs are to assigned to five people; each person will do one job only. The expected times (in hour) required for each person to complete each job have been estimated and are

shown in the following table. Use the hungarian method to determine the optimal assignments.

.

persons

job	1	2	3	4	5
1	12	15	13	14	15
2	16	18	15	14	16
3	18	16	15	18	20
4	15	20	18	17	19
5	16	15	18	14	15
Table O					

Solution

We define our decision variables as;

$x_{ij} = 1$ if job i $(i = 1,\dots,5)$ is assigned to person j and 0 if not and our objective function:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij},$$

will take only the assigned ones and the rest (of x_{ij}) will be zero.

people.

job	1	2	3	4	5
1	0	3	1	2	3
2	2	4	1	0	2
3	3	1	0	3	5
4	0	5	3	2	4
5	2	1	4	0	1

Table A

people

job	1	2	3	4	5
1	0	2	1	2	2
2	2	3	1	0	1
3	3	0	0	3	4
4	0	4	3	2	3
5	2	0	4	0	0

Table B.

13

job	1	2	3	4	5
1	0	2	1	2	2
2	2	3	1	0	1
3	3	0	0	3	4
4	0	4	3	2	3
5	2	0	4	0	0

job	1	2	3	4	5
1	0	1	0	2	1
2	2	2	0	0	0
3	1	0	0	4	4
4	0	3	2	2	2
5	2	0	4	1	0

Table C.

people

Table D

job	1	2	3	4	5
1	0	1	3	2	1
2	2	2	0	0	5
3	4	2	0	4	4
4	1	3	2	2	2
5	3	0	4	1	5

In table A   we applied the rule of 2.21a and in Table B   2.2.1b. So 2.2.1b is our opportunity cost table but you realize that only four lines were able to cover all the zeros which is not   the same as the number of rows or columns, implying an optimal solution cannot be identified. So we had to revise the values by moving to step 3 and table C was the outcome of step 2.2.3. Here five lines were able to cover all the zeros and the number of rows or columns are five. Hence an optimal solution (assignment) exist.

Procedure for identifying and making an optimal assignment.

From the optimal assignment table, as in table (D), you locate any row or column with only one zero element and make assignment to that cell. After making assignment to that cell the row and column connected to that cell should be cancelled out signifying that row and column are exhausted. Then you check other rows and columns with only one zero element and make assignment to the corresponding cells until each job is given an assignee. For our example above, row 4 has only one zero. We assign the first person to job 4 and cancel cross out column 1 and row 4. With column 1 and row 4 out, the first zero

element in row 1 is not part of row one again. Meaning row one will now have one zero element so we assign the third person to the first job and cross out column 3 and row one, no more assignment can be made in that column and row. The next assignment, will occur in cell $(2,3)$, cell $(4,2)$ and cell $(5,5)$. Hence, we have;

person	assigned	job		original table value
1	→	4		15
2	→	3		16
3	→	1		13
4	→	2		14
5	→	5		15
				7 3

The objective function value can then be calculated by picking the cell corresponding values in the original Table (O) summing to obtain the minimum time required for the five people to complete the work and those values represent the $x_{ij} = 1.$ Hence 73 hours, an equivelents of 3 days and an hour.

Observations.

1. It is not strictly a necessary that if the number of lines are equal to the number of rows or columns as stated in 2.2.2, an optimal assignment exist. There are situations in which though all the zeros are covered by the lines an optimal assignment cannot be found and you still need to revised the table. The example below illustrates this point, try it.

2. In a form of checking the accuracy of the objective function value, the total deductions or subtractions made, beginning from the rows to the table where the optimal assignment exist should be equal to the objective function value.

REFERENCE

1. Frederick S. Hillier and Gerald J. Lieberman. "Introduction to Operational Research" (Seventh Editi
 McGraw-Hill Companies, Inc., 2001.

2. Frank S. Budnick, Dennis Mcleavey, and Richard Mojena. " principles of Operational Research" (S
Edition). Ricard D. Irwin Inc., 1988.

3. Wanye L. Winston. "Operational Research, Applications and Algorithm" (Second Edition). PWS - K
Publishing Company, 1987.