DYNAMIC PROGRAMMING

1. INTRODUCTION

Dynamic programming (DP) is a recursive optimization approach to solving a sequential decision problems. Dynamic Programming has much to offer to problems requiring a sequence of related decisions. Many applications of dynamic programming reduce to finding the shortest or longest path. Moreover, unlike the transportation algorithm that presumes a standard structure, DP uses a general approach that requires few basic characteristcs in the problem to be applicable. For instance, the tool of DP can be applied to problems in which the identification of a solution, not necessary an optimal solution, can be achieved by viewing the solution as consisting of several independent decisions, called stages. The range of decisions for each stage is called the set of states for each stage. We first look at an introductory example.

A mythical fortune seeker in Missouri decided to go west to join the gold rush in California during the 19th century. The journey would require traveling by stagecoach through unsettled countries where there was serious danger of attack by marauders. Although his starting point and destination was fixed, he had considerable choice as which states to travel through en route. The possible routes are shown in Figure 1 where each state is represented by a circled letter and the direction of travel is always from left to right in the diagram, from Stage A to Stage J.

The fortune seeker was a prudent man who was quite concerned about his safety. After some thought, he came out with a rather clever way of determining the safest route. Life insurance policies were offered to stagecoach passengers. Because the cost of the policy for taking any given stagecoach was based on a careful evaluation of the safety of that run, the safest route should be the one with the cheapest total life insurance policy.

The cost for the standard policy on the stagecoach run from one state to another is given in Table 1.

	A	B	C	D	E	F	G	H	I	J
A	-	2	4	3	-	-	-	-	-	-
B	-	-	-	-	7	4	6	-	-	-
C	-	-	-	-	3	2	4	-	-	-
D	-	-	-	-	4	1	5	-	-	-
E	-	-	-	-	-	-	-	1	4	-
F	-	-	-	-	-	-	-	6	3	-
G	-	-	-	-	-	-	-	3	3	-
H	-	-	-	-	-	-	-	-	-	3
I	-	-	-	-	-	-	-	-	-	4
J	-	-	-	-	-	-	-	-	-	-

Table 1. Cost of travelling by stagecoach from *A* to *J*.

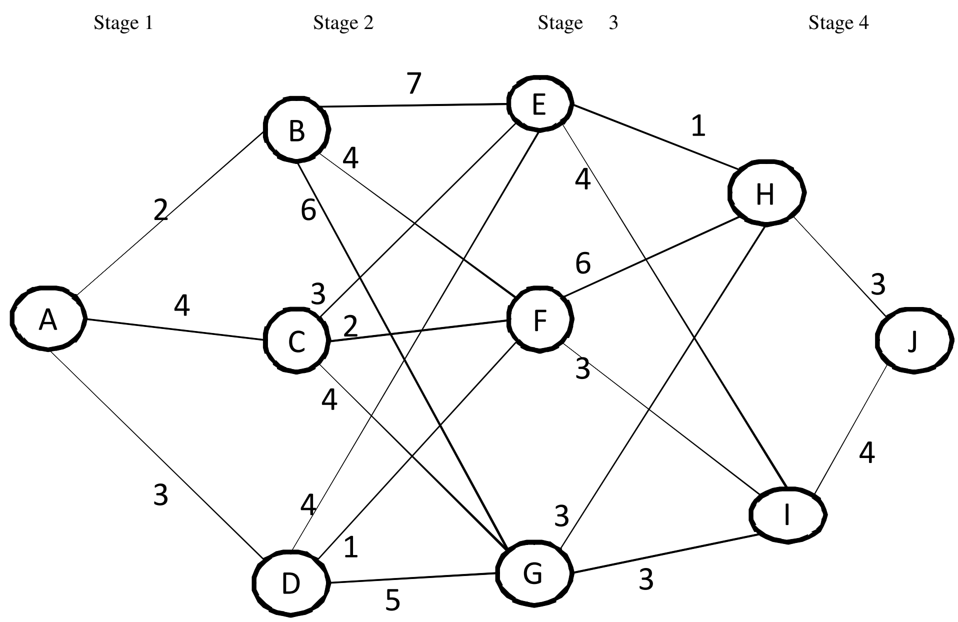


Figure1

The fortune seeker needed to find the route which minimizes the cost of the policy.

2. CHARACTERISTICS OF DYNAMIC PROGRAMMING

The problem can be divided into *stages* with a decision required at each state. The stage is the amount of time that has elapsed since the beginning of the process. In Figure 1, there are four stages connecting the first and last states, *A* and *J*. The decision to make is, which destination to select for the next stage. The fortune seeker, from state A, in the example has three possible states . That is states B, C, and D.

The effect of the decision at each stage is to transform the state in the current stage to a state in the next stage. The first decision in stage 2 the fortune seeker need to make is how to move from State A of Stage1 to a state in state 2.

Given the current state, the optimal decision for each of the remaining states must not depend on previous decision from from other states and conforms with conforms with Bellman's principle of optimality, "An optimal set of decision rules has the property that, regardless of the *nth* decision, the remaining decision must be optimal with respect to the outcome that results from the *nth* decision". Should the fortune seeker find out that from state *A* through state *C* to *J* is optimum, then from state *C* to state *J* will also be optimum. Moreover if the example above is solved and the shortest path from city *A* to city *J*, happens to take the following path. $A \to D \to E \to F \to J.$ Then $D \to E \to F \to J,$ should be the shortest path from *D* to *J*.

3. DP APPROACH TO SOLVE PROBLEMS

The DP approach to a problem is summarized as follows:

- Decomposition of the problem into stages with each stage having at least one state. In the example above, there are four stages and ten states. A is the only state of Stage 1. B, C, and D are the states of Stage 2, etc.
- Setting up a kind of recursive equation for each state of a stage to the next stage according to specific optimization objective. At each state, the decision made by solving an optimization problem defined by the recursive equation:

$$f_i(s_i,) \quad = r_i(s_i,x_i) + f_{i-1}(s_{i-1},x_{i-1}),$$

where,

$f_i(s_i,x_i)$ = value from state x_i in stage i to the last stage.

$r_i(s_i,x_i)$ = Return or result associated with a particular state.

$f_{i-1}(s_{i-1},x_{i-1})$ = The accumulative return through stage $i-1$.

x_i = Decision variable for stage i.

s_i = State of the system following stage i.

Combining the results at each stage to give the solution for the entire problem.

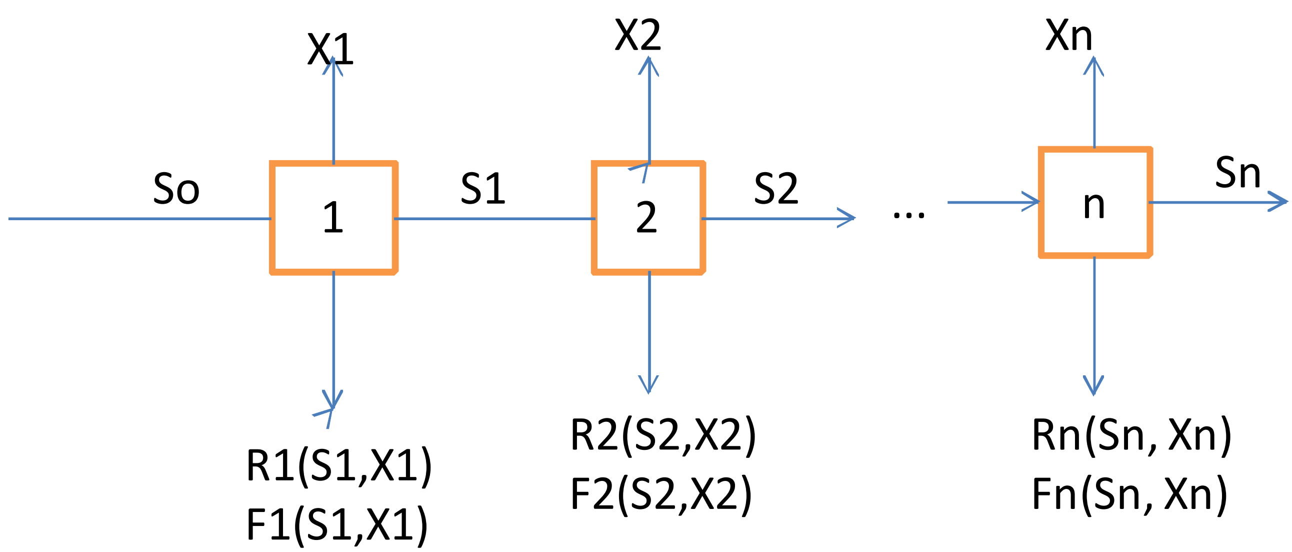
The process of combining the results is termed as composition. The act of composition, results in a set of sequentially decision rules called a policy. For example DP will optimize an n decision variable problem by decomposing it into a series of n stages with each decision variable a stage, assigning an optimal value to each variable, and combining the results from each stage to generate the solution to the entire problem.

Basically there are two ways DP can be applied to solve a problem, the forward recursion method and the backward recursion method.

4. SYMBOLIC NOTATION OF DP, FORWARD RECURSION

The forward recursion approach takes a problem and decomposes the problem into n stages (see Figure 2) and analyzes the problem starting with the first stage in the sequence, working forward to the last stage.

Stage1 Stage2 Stage n



.

Figure 2

5. SOLUTION TO INTRODUCTORY EXAMPLE

The fortune seeker needs to find the route that will minimize the total cost of the policy. This is the shortest path problem, and we have to find the shortest route connecting states A and J. From the approach of the backward recursion, we start with stage 4 states and work backwards incorporating the other states. From our recursive equation, $f_i(s_i,x_i) = r_i(s_i,x_i) + f_{i+1}(s_{i+1},x_{i+1})$, we have for stage 4, $f_4(s_4,x_4) = r_4(s_4,x_4) + f_5(s_5,x_5)$ but $f_5(s_5,x_5) = 0$ and the shortest route to J from H and I $f_4(H) = r_4(H,J) = C_{HJ} = 3$ and $f_4(I) = C_{IJ} = 4$. Here C_{SJ} is the cost connecting a particular state to state J. We next incorporate stage 3 states.

State 3 analysis.

There are several options for the fortune seeker to get to J from stage 3. First, taking state E. To move from state E to state J, he can either move from state E to H and take the shortest path from H to J or E to I and take the shortest path from I to J. The rest of that from G to J and from F to J is shown below.

$$\begin{aligned} f_3(E) &= \min\{C_{EH} + f_4(H), C_{EI} + f_4(I)\} \\ &= \min\{1 + 3, 4 + 4\} \\ &= \min\{4, 8\} = 4 \\ f_3(F) &= \min\{C_{FH} + f_4(H), C_{FI} + f_4(I)\} \\ &= \min\{6 + 3, 3 + 4\} \\ &= \min\{9, 7\} = 7 \end{aligned}$$

$$\begin{aligned} f_3(G) &= \min\{C_{GH} + f_4(H), C_{GI} + f_4(I)\} \\ &= \min\{3 + 3, 3 + 4\} \\ &= \min\{6, 7\} = 6. \end{aligned}$$

Stage 2 analysis.

We next incorporate stage two. In stage 2, there are 3 states $B, C,$ and D. What will be the shortest route from each of these states to J.

$$\begin{aligned} f_2(B) &= \min\{C_{BE} + f_3(E), C_{BF} + f_3(F), C_{BG} + f_3(G)\} \\ &= \min\{7 + 4, 4 + 7, 6 + 6\} \\ &= \min\{11, 11, 12\} = 12 \\ f_2(C) &= \min\{C_{CE} + f_3(E), C_{CF} + f_3(F), C_{CG} + f_3(G)\} \\ &= \min\{3 + 4, 2 + 7, 4 + 6\} \\ &= \min\{7, 9, 12\} = 7 \end{aligned}$$

$$\begin{aligned} f_2(D) &= \min\{C_{DE} + f_3(E), C_{DF} + f_3(F), C_{DG} + f_3(G)\} \\ &= \min\{4 + 4, 1 + 7, 5 + 6\} \\ &= \min\{8, 8, 11\} = 8. \end{aligned}$$

Stage 1: The last step is to incorporate stage 1 with only one state.

$$\begin{aligned} f_1(A) &= \min\{C_{AB} + f_3(B), C_{AC} + f_3(C), C_{AD} + f_3(D)\} \\ &= \min\{2 + 11, 4 + 7, 3 + 8\} = \min\{13, 11, 11\} = 11, \end{aligned}$$

and the minimum is attained at edge AC (by arbitrarry break the tie)

From the computations, the route(s) which minimizes the total cost of the policy can now be determined. From state A the fortune seeker can either move to stage C or D. If he moves to stage C

from A, then he has to move to state E then to state H before moving to J. Thus $A \rightarrow C \rightarrow E \rightarrow H \rightarrow J$ with minimum cost of 11. Also, if he moves from A to D, the possible routes are : $A \rightarrow D \rightarrow E \rightarrow H \rightarrow J$, $A \rightarrow D \rightarrow E \rightarrow H \rightarrow J$ with a minimum cost of 11.

REFERENCE

1. Frederick S. Hillier and Gerald J. Lieberman, Introduction To Operational Research (Seventh Edition). McGraw-Hill Companies, Inc., 2001.
2. Frank S. Budnick, Dennis Mcleavey, and Richard Mojena, Principles of Operational Research (Second Edition). Richard D. Irwin Inc., 1988.
3. Wanye L. Winston, Operational Research: Applications and Algorithm (Second Edition). PWS - Kent Publishing Company, 1987.