Math 5331 Lecture 9

Introduction to Game Theory (I)

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What Do We Study in Game Theory?

- Players: two or more, corporative or non-corporative;
- System/environment: zero-sum or nonzero-sum;
  - Rules: description of the game (constraints);
- Objectives: each player's interest;
- Solution: how can we properly define a "solution"?
- Difference from traditional OR and stochastic OR models?

Outline:

Two-person zero-sum games: reward/payoff matrix, maximin strategy, saddle point, equilibrium point, pure strategy and randomized/mixed strategy, graphic solution, LP formulation.

Non-coorperative and coorperative two-person games:

n-person game theory:

Two-Person Zero-Sum Games:

We have two players, the row player (the 1st player) who must choose 1 of m strategies and the column player (the 2nd player) who must choose 1 of n strategies. For any combination of the row player's strategy i and column player's strategy j, the reward for the row player is a_{ij} and the column player's reward is $-a_{ij}$ (zero-sum). Matrix $A = (a_{ij})_{m \times n}$ is called the reward/payoff matrix (for the row player). Rule of the game: two players must choose their strategy independently without knowing the opponent's choice. There is no information exchange between the players in any way.

Game 1. Two players, each has two cards: 1, 2. Each select one card (face down). The two cards are compared.

1.
- If the sum is even, player 1 wins the sum from player 2.
- If the sum is odd, player 2 wins the sum from player 1.

The reward matrix (for Player 1) is:

Player 1 \ Player 2	1	2
1	2	-3
2	-3	4

Game 2. In Game 1, assume that player 1 has the third card 3. The reward matrix will have one more row.

Player 1 \ Player 2	1	2
1	2	-3
2	-3	4
3	4	-5

Game 3. In Game 1, assume that before the selected cards are compared, player 1 has to declare "even" or "odd". Then the cards are compared. Player 1 will win the sum only if the claim matches the outcome. In this case, the reward matrix is

	1	2
1; odd	-2	3
1; even	2	-3
2; odd	3	-4
2; even	-3	4

Game 4. This time player 1 must declare "even" or "odd" BEFORE player 2 putting up his card. What are player 2's strategies?

1.
- (1,1): select 1 always;
- (1,2): select 1 if hear "odd", 2 if hear "even";
- (2,1): select 2 if hear "odd", 1 if hear "even";
- (2,2): select 2 always.

The reward matrix is therefore:

	(1,1)	(1,2)	(2,1)	(2,2)
(1,odd)	-2	-2	3	3
(1,even)	2	-3	2	-3
(2,odd)	3	3	-4	-4
(2,even)	-3	4	-3	4

Game 5. Player 1 has 2 cards: 1, 2.

1. ● Player 1 selects a card.

● Player 2 guesses the card.

● Player 1 may say "stop" or "double".

- If player 1 chooses "stop", the card is revealed.
  - * Player 2 wins the face value if the guess is correct.
  - * Player 1 wins the face value if the guess is incorrect.
- If player 1 chooses "double", Player 2 says "accept" or "reject".
  - * If player 2 says "accept", the card is revealed.

    Player 2 wins twice the value if the guess is correct;

    Player 1 wins the face value if the guess is incorrect;
  - * If player 2 says "reject", the card is revealed. And the rules are as before.

Now what is the reward matrix?

Now questions. **What is a solution**? What is a solution? when two players' interest are conflict. More often we study **row player's reward/payoff matrix** in the general form

	Column 1	Column 2		Column n
Row 1	a_{11}	a_{12}	…	a_{1n}
Row 2	a_{21}	a_{22}	…	a_{2n}
	…	…	…	…
Row m	a_{m1}	a_{m2}	…	a_{mn}

Maximin strategy: If we determine the least possible payoff for each strategy, and choose the strategy for which this minimum payoff is largest, we have the maximin strategy. Equivalent definition is the saddle point. (A conservative way of acting!)

Existence: A two-person zero-sum game has a **saddle point** (**or maximin strategy**) if and only if

$$Max_{\text{all rows}}(\text{row minima}) = min_{\text{all columns}}(\text{column maxima}).$$

Or

$$\max_{1 \le i \le m} \min_{1 \le j \le n} a_{ij} = \min_{1 \le j \le n} \max_{1 \le i \le m} a_{ij}.$$

Practical Interpretation. If a two-person zero-sum or constant-sum game has a saddle point, the row player should choose any strategy (row) attaining the maximum among the row minima. The column player should choose any strategy (column) attaining the minimum on the column maxima. In this case, we say that the game reaches an **equilibrium point**. For Game 1, we find the equilibrium (or maximin) strategy is 2 and 2 for row and column players respectively. For Game 5, we will see that there is no saddle point!

Reward/Payoff matrix reduction: We say Row j is dominated by Row i, if

$$a_{jk} \le a_{ik}, \text{ for all } k = 1, 2, \ldots, n.$$

In this situation, Row player should never choose strategy j, and therefore Row j can be deleted from the payoff matrix. Similarly, Column j is dominated by Column i, if

$$a_{kj} \le a_{ki}, \text{ for all } k = 1, 2, \ldots, m.$$

In this case, Column player should never choose strategy j, and therefore Column j can be deleted from the payoff matrix. For example: the game

Column Player

Row Player		A	B	C
	1	6	5	-4
	2	9	7	-2
	3	9	8	-3

can be reduced to the equivalent one

Column Player

Row Player				C
	2			-2
	3			-3

Randomized or mixed strategies and LP formulation: In the last game, since there is no equilibrium strategy, both player will think about something else. One solution is to use the random combination of all the possible strategy. A mixed, or randomized, strategy for Player 1 (row player) is a vector

$$X = (x_1, x_2, \dots\ x_m)$$

of nonnegative real numbers satisfying the condition

$$x_1 + x_2 + \dots + x_m = 1,$$

with the interpretation that Player 1 plays strategy i with probability x_i, $1 \leq i \leq m$. Similarly, a mixed, or randomized, strategy for Player 2 is a vector

$$Y = (y_1, y_2, \dots\ y_n)$$

of nonnegative real numbers satisfying the condition

$$y_1 + y_2 + \dots + y_n = 1,$$

with the interpretation that Player 1 plays strategy j with probability $y_j, 1 \leq j \leq n$.

Pure strategies are the special cases of mixed strategies. If Player 1 chooses a mixed strategy $X = (x_1, x_2, \dots\ x_m)$ and Player 2 chooses a mixed strategy $Y = (y_1, y_2, \dots\ y_n)$, then the expected payoff (for Player 1) is

$$XAY^T = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j a_{ij}.$$

When mixed strategies are considered, Player 1's goal is to maximize the expected worst-case reward, i.e.

$$\max_{0 \leq x_i \leq 1, i=1,2,\dots,m} \sum_{i=1}^{m} x_i \left[\min_{0 \leq y_j \leq 1, j=1,2,\dots,n} \sum_{j=1}^{n} a_{ij} y_j \right] = \max_{0 \leq x_i \leq 1, i=1,2,\dots,m} \left[\min_{0 \leq y_j \leq 1, j=1,2,\dots,n} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j \right],$$

which leads to the LP formulation:

$$\begin{aligned} \max Z &= u \\ s.t.\ :\ & \sum_{i=1}^{m} a_{ij} x_i \geq u,\ j = 1,2,\dots,n, \\ :\ & \sum_{i=1}^{m} x_i = 1. \\ :\ & 0 \leq x_i \leq 1,\ i = 1,\dots,m \end{aligned}$$

we can actually simplify the LP form: note that $u > 0$,

$$
\begin{aligned}
\max Z &= u \\
s.t.\ :\ & \sum_{i=1}^{m} a_{ij}\frac{x_i}{u} \ge 1,\ j = 1,2,\ldots,n, \\
:\ & \sum_{i=1}^{m} \frac{x_i}{u} = \frac{1}{u}. \\
:\ & 0 \le \frac{x_i}{u} \le \frac{1}{u},\ i = 1,\ldots,m
\end{aligned}
$$

we introduce the new set of variables $\frac{x_i}{u} \to x_i$, then the LP becomes

$$
\begin{aligned}
\min Z &= \sum_{i=1}^{m} x_i \\
s.t.\ :\ & \sum_{i=1}^{m} a_{ij}x_i \ge 1,\ j = 1,2,\ldots,n, \\
:\ & x_i \ge 0,\ i = 1,\ldots,m
\end{aligned}
$$

Similarly, the Player 2's goal is to minimize the expected best-case reward for Player 1, i.e.

$$
\min_{0\le y_j\le 1, j=1,2,\ldots,n} \sum_{j=1}^{n} y_j \left[\max_{0\le x_i\le 1, i=1,2,\ldots,m} \sum_{i=1}^{m} x_i a_{ij} \right] = \min_{0\le y_j\le 1, j=1,2,\ldots,n} \left[\max_{0\le y_j\le 1, j=1,2,\ldots,n} \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_i y_j \right],
$$

and the LP formulation is the dual:

$$
\begin{aligned}
\max Z &= \sum_{j=1}^{n} y_j \\
s.t.\ :\ & \sum_{j=1}^{n} a_{ij}y_j \le 1,\ j = 1,2,\ldots,n, \\
:\ & y_j \ge 0,\ j = 1,\ldots,n.
\end{aligned}
$$

From the duality theorem, we know that that two values are the equal.

Two-Person Non-Constant Games:

Most game-type models in the business situations are not zero-sum or constant-sum games, because it is unusual for business competitors to be in total conflict.

Our first example is so call "The Prisoner's Dilemma". Two members of a criminal gang are arrested and imprisoned. They are placed under solitary confinement and have no chance of communicating with each other. The district attorney would like to charge them with a recent major crime but has insufficient evidence. He does have sufficient evidence to convict each of them of a lesser charge (1 year in prison.) However, if he obtains a confession from one or both of the criminals, he can convict either or both on the major charge. The district attorney offers each the chance to confess: If only one prisoner confesses and testifies against his partner he will go free while the other will receive a 3 year sentence. If both prisoners confess, they will each receive a 2 year sentence. Each prisoner knows the other has the same offer. But if both refuse, each will be imprisoned for 1 year on the lesser charge. What would you do if you were Prisoner A? or B?

Because this is not a zero-sum game, we need the separate reward matrices for two prisoners. Combine the two matrices, we have

	Prisoner B refuses	Prisoner B confesses
Prisoner A refuses	1 year, 1 year	3 years, 0 year
Prisoner A confesses	0 years, 3 years	2 years, 2 years

As we see in the above matrix, one player's gain is not equal to the other's loss. Prisoner A thinks: If the other prisoner refuses the deal then I am better off refuse too. If B confesses, I am also better off confessing. Prisoner B thinks similarly.

Because there is no communication and no mutual trust, the rational prisoners who choose the maximin strategy will obtain outcomes that are worst off than if they had cooperated.

	Prisoner B refuses	Prisoner B confesses
Prisoner A refuses	1 year, 1 year	3 years, 0 year
Prisoner A confesses	0 years, 3 years	2 years, 2 years

(2,2) is also the equilibrium point.

In the following examples, we will see how human beings play games among themselves in more practical ways.

Game 7: (Communication and collaboration). Consider the following game:

	y1	y2
x1	(0,0)	(12,-12)
x2	(-12,12)	(6,6)

Without communication, they end up with (0,0) outcome, which is the equilibrium. However, after communication, the players would coordinate their strategies to assure a (6,6) payoff to themselves.

Game 8: (Even dominance is no more helpful). Consider this game:

	y1	y2
x1	(10,1)	(2,2)
x2	(2,2)	(1,10)

For P1, R1 dominates R2, so he choose R1. For P2, C2 dominates C1, so he choose C2. They end up with only $2+2=4$ reward. A better strategy, if they both agree to collaborate, is that both players choose strategy 1 and then 2 in turn.

Game 9: (Threatening). Consider this game

	y1	y2
x1	(1,10)	(10,1)
x2	(0,-10)	(0,-9)

P1 will prefer the (10,1) outcome and P2 will prefer the (1,10) outcome. Without discussion, P1 has no reason to use R2 and he choose R1; P2 is fully aware of P1’s situation, and he will choose C1. The game may result in (1,10) outcome. If two players can communicate, then P1 would demand P2 to use C2 by threatening that he would use R2 otherwise. In this case, threatening becomes a component in the theory of cooperative games.

Game 10: (Side-payment). Consider this game

	y1	y2
x1	(50,0)	(1,5)
x2	(1,0)	(1,5)

.

Individually, P2 has to use C2, which is extremely unfavorable to P1. If side payment is possible, P1 will convince P2 to use C1 by promising to transfer some payoff to P2 after the (50,0) outcome is realized.

Modeling nonzero-sum games. Given the many variety of difficulties with nonzero-sum games, how can we develop a mathematical model that in some way reflects rational behavior of the players? What is the objective function? First of all, the conditions (rules) under which the game is played must be made precise.