1 Introduction

1.1 Two-Person, Non-Zero-Sum Games

In the previous lecture, we saw how given a two-player, zero-sum game, we can analyze it by looking at maximin strategies (or *security level strategies*), equilibrium points, pure strategies and mixed strategies, and other properties of such games.

We now look at the class of games called two-player non-zero-sum games where one player's loss may not correspond with the other player's gain given a certain outcome of the game (and vice versa). That is, the sum of the payoffs for the players may not be zero for every outcome of the game.

Consider the following game, with Player 1 having strategies s_1 and s_2, Player 2 having strategies t_1 and t_2, and the entries of the table giving the payoff for Player 1 and Player 2 respectively.

Game 1

	t_1	t_2
s_1	(0,0)	(5,-5)
s_2	(-5,5)	(4,4)

From Player 1's perspective, we see that no matter which strategy Player 2 chooses, Player 1 is better off choosing s_1 (since $0 > -5$, and $5 > 4$). We say that s_1 **dominates** s_2. Similarly, we see that strategy t_1 dominates t_2, so Player 2 would choose t_1 (assuming players choose to play a security level strategy). This leads to the equilibrium point (s_1, t_1), with both players making no gain. Clearly, both players would have a greater payoff by choosing the strategy pair (s_2, t_2).

Consider another game.

Game 2

	t_1	t_2
s_1	(1,1)	(0,0)
s_2	(0,0)	(1,1)

Here, both players have no reason to choose one strategy over the other, unless they can communicate ahead of time. If both players use the mixed strategy (1/2,1/2), then their expected payoffs will be (1/2,1/2). If they can agree ahead of time to use the pure strategy (s_1, t_1) or (s_2, t_2), then they can increase their expected payoffs to the constant payoff (1,1).

To reach this strategy, players need not even communicate ahead of time. If the game is played repeatedly, it is likely that after several turns, a pattern would be established, resulting in the constant payoff of (1,1).

We see here that communication is another big consideration in non-zero sum games. Consider the following game.

Game 3

	t_1	t_2
s_1	(10,1)	(2,2)
s_2	(2,2)	(1,10)

Dominance would lead to the equilibrium point (s_1, t_2), with a payoff of (2,2). If the game is repeated, the players could increase their payoff over two subsequent plays of the game to (11,11) from (4,4) if they agree to first play the strategy (s_1, t_1) followed by the strategy (s_2, t_2). However, on the second play of the game, nothing keeps player 2 from deviating from this plan to playing the strategy t_1, giving himself of herself a total payoff of 12, rather than 11.

We also consider the game given in the previous lecture:

Game 4

	t_1	t_2
s_1	(50,0)	(1,5)
s_2	(1,0)	(1,5)

Dominance leads to the equilibrium strategy (s_1, t_2). If the players are able to transfer their payoff, then they can arrange to instead play the strategy (s_1, t_1), where Player 1 reimburses Player 2 in some amount. Again, this is not possible in a zero-sum game.

2 Noncooperative Games

Here we assume that there is no communication between players, either before the game or through repeated plays. In the study of zero-sum games, we concluded that a security level strategy (that is, a maximin strategy for Player 1 and a minimax strategy for Player 2) resulted in an equilibrium point if we allow players to play mixed strategies. We will consider an example that shows that this property does not hold in general in non-zero-sum games. Consider the following game:

Game 5

	t_1	t_2
s_1	(1,10)	(10,1)
s_2	(0,-10)	(0,-9)

We see that since s_1 dominates s_2, Player 1's security level strategy is the pure strategy (1,0) (that is, Player 1 uses strategy s_1 with probability 1). Player 2's security level strategy is the pure strategy (0,1). However, the strategy pair (s_1, t_2) is not in equilibrium. If Player 2 notes that s_1 dominates s_2, he of she concludes that Player 1 will choose s_1. Thus, by using the pure strategy (1,0), Player 2 will maximize his or her payoff. We see that by using this strategy, Player 1 maintains his or her security level, while Player 2 gets 19 more units than his or her security level. This seems to be a reasonable resolution of the game with no communication or cooperation (note that if communication were allowed, Player 1 could resort to threat to try to obtain a better payoff).

Consider the following game:

Game 6

	t_1	t_2
s_1	(10,1)	(0,0)
s_2	(0,0)	(1,10)

Here we see that the strategy pairs (s_1, t_1) and (s_2, t_2) are both in equilibrium. The first strategy heavily favours Player 1, while the other favours Player 2. We notice here that equilibrium points can have distinct payoffs, a situation we would not encounter in the zero-sum case. We can calculate that the security level strategies for the players are $X_0 =$ (1/11, 10/11) and $Y_0 =$(10/11, 1/11) respectively, giving both players a security level of 10/11.

Again, the security level strategy (X_0, Y_0) is not in equilibrium. If Player 1 expects Player 2 to use t_1 with probability 10/11, then Player 1 can maximize his or her payoff by selecting s_1 with probability 10/11. Player 2 will follow similar reasoning, leading to the equilibrium strategy (X_1, Y_1) with $X_1 = (10/11, 1/11)$ and $Y_1 = (1/11, 10/11)$, with expected payoff $(10/11, 10/11)$.

If the players were able to communicate, they could agree to coordinate their strategies, say by flipping a coin and choosing (s_1, t_1) if the coin lands heads, and (s_2, t_2) otherwise. This increases the expected payoff to (11/2,11/2), a considerable improvement.

3 Cooperative Games

Now we consider the case where the players of a non-zero-sum game can cooperate in such a way that includes preplay discussion and binding agreements. We can assume that if we give the players the chance to coordinate strategies at a mutual benefit, then they will. Consider Game 6 from the previous section. As pointed out, the players can cooperate to expand the set of expected payoffs for the game. We established that by coordinating, the players were able to achieve an expected payoff of (11/2,11/2) by choosing the strategy pairs $(s_1, t_1), (s_2, t_2)$ with equal probability. We can assume that the players would prefer this situation to the situation where they independently choose their strategies with no information from the other player.

In fact, if Player 1 has m strategies, and Player 2 has n strategies, and the players agree to use the strategy pair (s_i, t_j) with probability p_{ij}, then the set of all possible payoffs, denoted by M, is

$$M = \{ \sum_{1 \leq i \leq m, 1 \leq j \leq n} p_{ij}(a_{ij}, b_{ij}) : 0 \leq p_{ij} \leq 1, \sum_{1 \leq i \leq m, 1 \leq j \leq n} p_{ij} = 1 \}$$

We call this set the **cooperative payoff set**. For game 5 above, the cooperative payoff set is given by Figure 1. If we allow side payments from one player to another, then the set M can be altered to exhibit this fact. That is, if we have a point (u', v') in M, then the extended cooperative payoff set would include all points (u, v) with $u \geq 0, v \geq 0$, and $u + v = u' + v'$. The extended set for Game 6 is given by Figure 2.

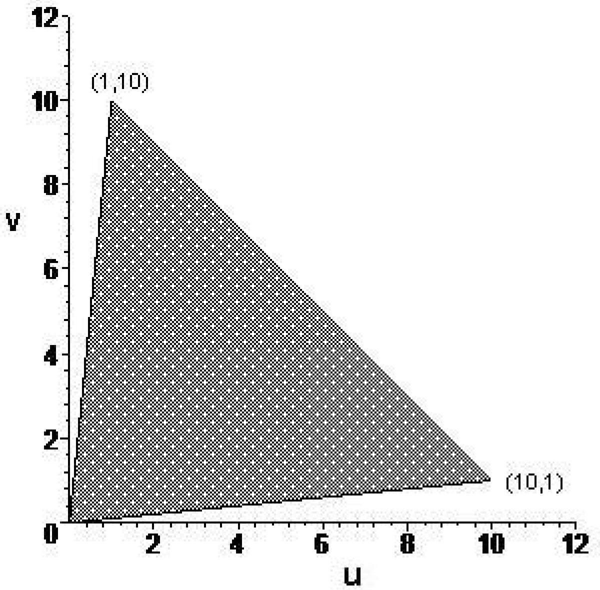

Figure 1

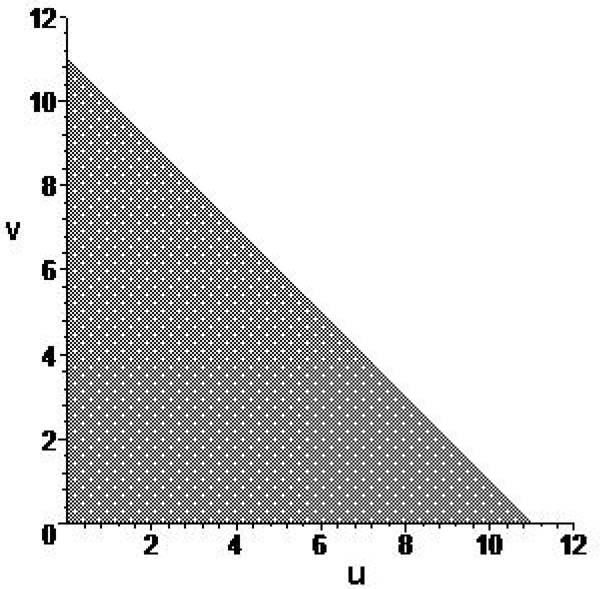

Figure 2

Now that we have this set M, how can we find a point in M that leads to a reasonable conclusion of the game? The set of points that lead to a reasonable conclusion is called the **negotiation set**, and points in this set must satisfy two properties.

Firstly, it is reasonable to assume that the players would not settle upon a point (u, v) if there exists a point (u', v') in M where $u' \geq u, v' \geq v$, and $u' + v' > u + v$. In such a case, we say that the point (u, v) is **dominated** by the point (u', v'). No dominated point may appear in the negotiation set.

Secondly, if Player 1 has a security level of value a, then Player 1 has no reason to accept a point (u, v) of the payoff set for which $u < a$. Thus, every point (u, v) in the negotiation set must have $u \geq a, v \geq b$ where a, b are the security levels of Player 1 and 2 respectively.

We now have the negotiation set that contains points corresponding to payoffs that are reasonable resolutions to a game. In the next section, we will outline procedures to find a unique point in this set that will be considered the solution to any cooperative game.

4 Axioms of Nash

The axioms of Nash provide a procedure for determining a unique solution for a given cooperative two-person game. Suppose that we have such a game with payoff set M. Suppose also that we have a point (u^*, v^*) of minimally acceptable payoffs to the players (for instance, we can take (u^*, v^*) such that u^* is the security level of Player 1 and v^* is the security level of Player 2, though in many cases this does not properly assess the threat level of the players).

Now we construct a function F, denoted $F[M, (u^*, v^*)]$ that assigns a solution to a game with payoff set M and minimally acceptable payoff point (u^*, v^*). We need such a function to satisfy some properties, given as follows.

4.1 Axioms of Nash

Let (u', v') denote the point $F[M, (u^*, v^*)]$.

1. (u', v') is an undominated point such that $u' \geq u^*, v' \geq v^*$.
2. If L is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ of the form $L(u, v) = (c_1 u + d_1, c_2 v + d_2)$, where c_1 and c_2 are positive, then
$$F[L(M), L(u^*, v^*)] = L(u', v').$$
3. If $N \subset M$, $(u^*, v^*) \in N$, and $(u', v') \in N$, then
$$F[N, (u^*, v^*)] = F[M, (u^*, v^*)] = (u', v').$$
4. If $(u, v) \in M$ implies $(v, u) \in M$, and if $u^* = v^*$, then $u' = v'$.

The first axiom simply states that the payoff of the solution to the game is at least as beneficial to each player as their respective minimally acceptable payoffs.

The second axiom deals with linear transformations of the utility functions of the players, with which we will not concern ourselves. The axiom tells us that the function is essentially invariant under utility transformations.

The third axiom is called the "independence of irrelevant alternatives". It states that if we have a solution for the payoff set M, then that point is also a solution to any subset of M that contains the point.

The fourth axiom tells us that if the two players have equivalent positions, then their payoffs should also be equal.

There exists a proof that shows that these axioms are sufficient to give a unique point (u', v') in the negotiation set. Furthermore, given a point (u^*, v^*), the solution (u', v') can be easily computed by finding the point (u', v') in M which maximizes the function $(u - u^*)(v - v^*)$ with $u \geq u^*$ and $v \geq v^*$.

4.2 Example

Consider the following game.

Game 7

	t_1	t_2
s_1	(2,7)	(-5,-1)
s_2	(0,-2)	(7,2)

Up until now, we've been able to find the security levels and strategies by inspection, but here we have that the security strategy is a mixed strategy, and we will find these strategies by treating the game as two zero-sum games.

First, we take A to be the matrix of entries of Player 1's payoffs, and B to be the matrix of entries of Player 2's payoffs. Since game 6 is not a zero-sum game, Player 2 uses a maximin strategy rather than minimax as in the zero-sum case. So, we will be treating Player 2 as the row player for the zero-sum game given by the matrix B^T.

We would like to find the strategies $X_0 = (x_1, x_2)$ and $Y_0 = (y_1, y_2)$ for Player 1 and 2 respectively that maximize each player's security level.

That is, we would like to find X_0 such that

$$\begin{aligned}
\text{min}_{Y \in T} X_0 A Y^T &= \text{max}_{X \in S} \text{min}_{Y \in T} X A Y^T \\
&= \text{max}_{(x_1, x_2) \in S} \text{min}\{2x_1, -5x_1 + 7x_2\} \\
&= \text{max}_{0 \leq x_1 \leq 1} \text{min}\{2x_1, -12x_1 + 7\} \qquad (\text{since } x_1 + x_2 = 1).
\end{aligned}$$

Employing graphical methods, we get that the maximum of the function $\min\{2x_1, -12x_1 + 7\}$ is attained at $x_1 = 1/2$. The value of the function at $x_1 = 1/2$ is 1, so Player 1 has a security level of 1 with security level strategy (1/2,1/2).

Similarly, we use this method to find the strategy Y_0 that satisfies

$$\begin{aligned}
\text{min}_{X \in S} Y B^T X^T &= \text{max}_{Y \in T} \text{min}_{X \in S} Y B^T X^T \qquad (\text{we treat P2 as a row player}) \\
&= \text{max}_{(y_1, y_2) \in T} \text{min}\{7y_1 - y_2, -2y_1 + 2y_2\} \\
&= \text{max}_{0 \leq y_1 \leq 1} \text{min}\{8y_1 - 1, -4y_1 + 2\} \qquad (\text{since } y_1 + y_2 = 1).
\end{aligned}$$

We get that the function $\min\{8y_1 - 1, -4y_1 + 2\}$ is maximized when $y_1 = 1/4$, and the value of the function at this point is 1. Thus, Player 2 has a security level of 1 with security level strategy (1/4,3/4).

Now we find the solution to this game. We see that the the payoffs (0,-2) and (-5,-1) are dominated by both (7,2) and (2,7), so we can deduce that the negotiation set is the line segment connecting the points (7,2) and (2,7) (note that if the payoff (0,9) was possible, it would not be included in the negotiation set since Player 1's payoff in this case is less than Player 1's security level, 1).

We have that $(1, 1)$ serves as a minimally acceptable payoff, so to find the solution (u', v') to the game, we find u and v that maximize $(u - 1)(v - 1)$ with $u \geq 1$ and $v \geq 1$. We find that this solution is the point (9/2, 9/2), so this point is the solution to the game given above.

5 References

[1] Paul R. Thie, *An Introduction to Linear Programming and Game Theory*, 2nd edition, 1988.