Math 5331 Lecture 9 N-Person Game

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Characteristic function:

Let $N = \{1, 2, \ldots, n\}$ be the set of players.

For each subset S of N, the characteristic function v for the game gives the amount $v(S)$ that the members of S can be **sure of receiving** if they act together and form a coalition.

Example 1 (The garbage game): Each of four property owners has one bag of garbage and must dump it on somebody's property. If b bags of garbage are dumped on the coalition of property owners, then the coalition receives a reward of $-b$. Find the characteristic function for this game. It is easy to see that

$$\begin{aligned} v(\{1\}) &= v(\{2\}) = v(\{3\}) = v(\{4\}) = -3, \\ v(\{1,2\}) &= \ldots = v(\{3,4\}) = -2, \\ v(\{1,2,3\}) &= \ldots = v(\{2,3,4\}) = -1, \\ v(\{1,2,3,4\}) &= -4. \end{aligned}$$

Example 2 (Land develop game): Player 1 owns a piece of land and valued at $\$10,000$. Player 2 is a subdivider who can develop the land and increase its value to $\$20,000$. Player 3 is also a subdivider who can develop the land and increase its value to $\$30,000$. Then the characteristic function of the game is

$$\begin{aligned} v(\{1\}) &= \$10,000 \text{ (sell the land as is)}, \\ v(\{2\}) &= v(\{3\}) = 0, \\ v(\{1,2\}) &= \$20,000, \\ v(\{1,3\}) &= \$30,000, \\ v(\{2,3\}) &= 0, \\ v(\{1,2,3\}) &= \$30,000. \end{aligned}$$

Superadditivity of a characteristic function: Consider any two sets A and B of players, if A and B have no players in common, then

$$v(A \cup B) \geq v(A) + v(B)$$

Reward vector:

What are we interested in an n-person game? A peaceful solution which should make everybody happy is a good way to distribute the common wealth or total reward. We use the reward vector

$$x = (x_1, x_2, \ldots, x_n)$$

to express this distribution plan, where x_i is the reward received by player i.

Obviously, a reward vector can not be a reasonable solution unless

1. $\sum_{i=1}^{n} x_i = v(N)$, (total wealth/reward restriction);
2. $x_i \geq v(\{i\})$, for all $i \in N$, (individual satisfaction);

A reward vector x that satisfies the above two conditions is called an **imputation**.

In the land develop game, we can see that

- $(5000, 2000, 5000)$ is not an imputation, because $x_1 < v(\{1\})$ and Player 1 will leave the union and act on his own.
- $(12000, 19000, -1000)$ is not an imputation, because $x_3 < v(\{3\})$ and Player 3 is not fairly treated.
- $(11000, 11000, 11000)$ is not an imputation, where can we get that additional \$3000?
- $(10000, 10000, 10000)$ is an imputation, so are $(10000, 20000, 0)$.

Concept of domination:

Given an imputation x, we say that the imputation y dominates x through a coalition S if

$$\sum_{i \in S} y_i \leq v(S), \text{ (y is achievable)}$$

$$y_i \geq x_i, \text{ for all } i \in S \text{ (everybody in S will prefer y to x).}$$

and we write

$$y >^S x.$$

We can see that in this case, x should not be considered as a possible solution for the game because the coalition S will go for strike.

In the last example, although $(10000, 20000, 0)$ is an imputation, we see that it is dominated by $(15000, 0, 15000)$ with respect to the coalition $\{1, 3\}$.

Core of the game:

The **core** of an n-person game is the set of all undominated imputations.

In the land developed game $(15000, 0, 15000)$ is not in the core, because

$$(16000, 4000, 10000) >^{\{1,2\}} (15000, 0, 15000).$$

Theorem 1: An imputation $x = \{x_1, x_2, \ldots, x_n\}$ is in the core of an n-person game if and only if for each subset $S \subset N$,

$$\sum_{i \in S} x_i \geq v(S).$$

How can we find the core? In the land develop game, we have

$$\begin{aligned} v(\{1\}) &= \$10,000, \\ v(\{2\}) &= v(\{3\}) = v(\{2,3\}) = 0, \\ v(\{1,2\}) &= \$20,000, \\ v(\{1,3\}) &= v(\{1,2,3\}) = \$30,000. \end{aligned}$$

A reward vector in the core should satisfy the following conditions

$$\begin{aligned} x_1 &\geq 10000, \\ x_2 &\geq 0, x_3 \geq 0, x_2 + x_3 \geq 0, \\ x_1 + x_2 &\geq 20000, \\ x_1 + x_3 &\geq 30000, x_1 + x_2 + x_3 = 30000. \end{aligned}$$

Solutions for this set of inequalities are

$$x_1 \geq 20000, x_3 = 30000 - x_1, x_2 = 0.$$

Example 3 (Drug Game): Player 1 invented a new drug but cannot manufacture it. Player 2 can manufacture the drug and split $1 million profit with Player 1. Player 3 can also manufacture the drug and split $1 million profit with Player 1. What is the core of the game?

Solution: consider the characteristic function

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0,$$
$$v(\{1,2\}) = v(\{1,3\}) = v(\{1,2,4\}) = 1.$$

Solve the set of inequalities

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_2 + x_3 \geq 0,$$
$$x_1 + x_2 \geq 1, x_1 + x_3 \geq 1, x_1 + x_2 + x_3 = 1.$$

The only imputation in the core is (1, 0, 0)!! Is it fair?

The Shapley Value – an alternative solution concept:

For any characteristic function, Lloyd Shapley proved that there is a unique reward vector $x,$ called the Shapley value, satisfying the following 4 axioms:

Axiom 1: Relabeling of players interchanges the player's rewards. Suppose the Shapley value is $x = (10, 15, 20)$. If we interchange the roles of Player 1 and Player 3, then the Shapley value for the new game should be $x = (20, 15, 10)$.

Axiom 2: All wealth are distributed. i.e.

$$\sum_{i=1}^{n} x_i = v(N).$$

Axiom 3: Lazy guy should not get paid!

$$v(S - \{i\}) = v(S)$$

for all coalitions S, then the Shapley value has $x_i = 0$. Player i is useless for any coalition!

Before we introduce the fourth axiom, we define the sum of two games: Let u and v be two characteristic functions for two games with identical players. Define a new game $(u+v)$ to be the game with the characteristic function $(u+v)$, such that

$$(u+v)(S) = u(S) + v(S), \forall S \subset N.$$

Axiom 4: Let x be the Shapley value for game u, y be the Shapley value for game v, then the Shapley value for game $(u+v)$ is $x+y$. (Additivity).

OK, sounds reasonable! but how can we find, or compute, the Shapley value?

Theorem 2. Given any n-person game with the characteristic function v, there is a unique reward vector x satisfying Axioms 1 – 4. The reward of ith player x_i is given by

$$x_i = \sum_{S \subset N \setminus \{i\}} p_n(S)[v(S \cup \{i\}) - v(S)],$$

where

$$p_n(S) = \frac{|S|!(n - |S| - 1)!}{n!}.$$

Interpretation: Imagine that n players arrive in a random order. If when Player i arrives, there is already a coalition S. When Player i join the coalition, the reward for the coalition is improved (hopefully). Then Player i should receive this improvement, which is

$$v(S \cup \{i\}) - v(S).$$

Since the order is random, Player i should receive the expectation. $p_n(S)$ is exactly the probability that when Player i arrive, the existing coalition is S. (Why?)

We now calculate the Shapley values for the drug game. We use following the tables.

For player 1 (the inventor): Recall

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0,$$
$$v(\{1,2\}) = v(\{1,3\}) = v(\{1,2,4\}) = 1.$$

S	$p_n(S)$	$v(S \cup \{i\}) - v(S)$
$\{\ \ \}$	$\frac{0!(3-0-1)!}{3!} = \frac{1}{3}$	$v(\{1\}) - v(\emptyset) = 0 - 0 = 0$
$\{2\}$	$\frac{1!(3-1-1)!}{3!} = \frac{1}{6}$	$v(\{1,2\}) - v(2) = 1 - 0 = 1$
$\{3\}$	$\frac{1!(3-1-1)!}{3!} = \frac{1}{6}$	$v(\{1,3\}) - v(3) = 1 - 0 = 1$
$\{2,3\}$	$\frac{2!(3-2-1)!}{3!} = \frac{1}{3}$	$v(\{1,2,3\}) - v(2,3) = 1 - 0 = 1$

Therefore the Shapley value for Player 1 is

$$x_1 = \frac{1}{3} \times 0 + \frac{1}{6} \times 1 + \frac{1}{6} \times 1 + \frac{1}{3} \times 1 = \frac{2}{3}.$$

Similarly, we can calculate the Shapley value for Players 2:

S	$p_n(S)$	$v(S \cup \{i\}) - v(S)$
$\{\ \ \}$	$\frac{0!(3-0-1)!}{3!} = \frac{1}{3}$	$v(\{2\}) - v(\emptyset) = 0 - 0 = 0$
$\{1\}$	$\frac{1!(3-1-1)!}{3!} = \frac{1}{6}$	$v(\{1,2\}) - v(1) = 1 - 0 = 1$
$\{3\}$	$\frac{1!(3-1-1)!}{3!} = \frac{1}{6}$	$v(\{2,3\}) - v(3) = 0 - 0 = 0$
$\{1,3\}$	$\frac{2!(3-2-1)!}{3!} = \frac{1}{3}$	$v(\{1,2,3\}) - v(1,3) = 1 - 1 = 0$

$$x_2 = \frac{1}{3} \times 0 + \frac{1}{6} \times 1 + \frac{1}{6} \times 0 + \frac{1}{3} \times 0 = \frac{1}{6}.$$

and the same for Player 3: $x_3 = \frac{1}{6}$.

Alternative method for determining Shapley values:

Arrival Order	Value added by P1	Value added by P2	Value added by P3
$\{1,2,3\}$	0	1	0
$\{1,3,2\}$	0	0	1
$\{2,1,3\}$	1	0	0
$\{2,3,1\}$	1	0	0
$\{3,1,2\}$	1	0	0
$\{3,2,1\}$	1	0	0

Each divided by 6 and we will get the same Shapley values.

Example 5 (Airport Pricing): Three types of planes (Piper Cubs, DC-10s, and 707s) use an airport. A Piper Cub requires a 100-yd runway, a DC-10 requires a 150-yd runway and a 707 requires a 400-yd runway. The cost of maintaining a runway is $1 per yard per year. Because of 707, the airport has a 400 yd runway. For simplicity, assume that only one of each type plane use the airport in the year. How should the airport charge the $400 maintenance fee to each of the plane?

Solution: We define a three-person game. Let P1 be the Piper Cub, P2 be the DC-10, and P3 be the 707. The value of the coalition is the cost associated with the runway length needed to serve the planes in the coalition. The characteristic function is

Coalition	Payoff
{ }	0
1	-100
2	-150
3	-400
1, 2	-150
1, 3	-400
2, 3	-400
1, 2, 3	-400

Find the Shapley value:

Arrival	**Value added by P1**	**Value added by P2**	**Value added by P3**
1,2,3	**-100**	**-50**	**-250**
1,3,2	**-100**	**0**	**-300**
2,1,3	**0**	**-150**	**-250**
2,3,1	**0**	**-150**	**-250**
3,1,2	**0**	**0**	**-400**
3,2,1	**0**	**0**	**-400**

$$\text{Shapley value for P1} = \frac{1}{6}(-100-100) = -\frac{200}{6} = -\$33.33$$

$$\text{Shapley value for P2} = \frac{1}{6}(-50-150-150) = -\frac{350}{6} = -\$58.33$$

$$\text{Shapley value for P3} = \frac{1}{6}(-250-300-250-250-400-400) = -\$308.33$$

and we see that they are not proportional to the lengths.