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Research Notes brings mathematical research ideas forth to the CMS readership in a generally accessible manner that promotes discussion of relevant topics including research (both pure and applied), activities, and noteworthy news items. Comments, suggestions, and submissions are welcome.

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Les Notes de recherche présentent des sujets mathématiques au lectorat de la SMC dans un format généralement accessible qui favorise les discussions sur divers sujets pertinents, dont la recherche (pure et appliquée), les activités et des nouvelles dignes de mention. Vos commentaires, suggestions et propositions sont les bienvenues.

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Chromatic numbers via commutative algebra

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Colouring graphs is a major subbranch of graph theory. We wish to illuminate an an overlap between this area and commutative algebra. Imagine a department where five committees must meet. Because some faculty members are on more than one committee, the meetings cannot all be scheduled at the same time (our imaginary faculty must attend their meetings!). Since faculty want to avoid meetings, we want to minimize the amount of time needed for meetings. Variations of this problem is a standard example of an application of the chromatic number of a graph. Specifically, represent each committee by a vertex, and join two vertices with an edge if there is someone who is on both committees. As an example, suppose in our fictional department, the corresponding graph is given as in Figure 1.

Figure 1. The graph representing committees and shared membership

We assign colours to each vertex so that vertices receive different colours if they are joined by an edge. We want the least number of colours needed for a valid colouring. Figure 1 can be minimally coloured with three colours (e.g., colour Committees 1 and 3 red, Committees 2 and 4 blue, and Committee 5 green). We can schedule our meetings in three hour long slots by scheduling all the committees with the same colour during the same hour slot.

More formally, a finite simple graph G is a pair $G = (V,E)$ where $V = \{x_1, \ldots, x_n\}$ are the vertices, and E consists of unordered pairs of distinct elements of V, called edges. Simple means that loops or multiple edges between vertices are not allowed. The **chromatic number** of G, denoted $\chi(G)$, is the least number of colours in a valid colouring of G. It can be quite difficult to compute $\chi(G)$ (in fact, it is an NP-complete problem).

The commutative algebra community, starting with Villarreal [7], has been interested in studying graphs algebraically. One associates with G two ideals in the polynomial ring $R = \mathbb{Q}[x_1, \ldots, x_n]$. The **edge ideal** of G is

$$I(G) = \langle x_i x_j \mid \{x_i, x_j\} \in E\rangle.$$

That is, the generators of the ideal $I(G)$ are in bijection with the edges of G. For example, the edge ideal of the graph of Figure 1 is

$$I(G) = \langle x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1\rangle.$$

The second ideal is the **cover ideal** of G:

$$J(G) = \bigcap_{\{x_i,x_j\}\in E} \langle x_i, x_j\rangle.$$

The nomenclature is due to a correspondence between the generators and the vertex covers of G (a subset $W \subseteq V$ such that $e \cap W \neq \emptyset$ for all $e \in E$). Within this framework, computing $\chi(G)$ can now be rephrased as an ideal membership problem, i.e., asking when a particular element belongs to an ideal. Below, $J(G)^d = \langle g_1 \cdots g_d \mid g_i \in J(G)\rangle$ is the d-th power of $J(G)$.

Theorem 1. ([3, Theorem 3.2]). *Let $G = (V, E)$ be a finite simple graph with cover ideal $J(G)$ and $|V| = n$. Then*

$$\chi(G) = \min\left\{d \;\middle|\; (x_1 \cdots x_n)^{d-1} \in J(G)^d\right\}.$$

The proof of Theorem 1 exploits the fact that the set of vertices that do not receive a fixed colour form a vertex cover. What is remarkable is $\chi(G)$ can be computed without finding a colouring. Moreover, programs like *Macaulay2* [4] can compute $\chi(G)$ using the ideal membership property.

We can generalize $\chi(G)$ by assigning multiple colours to each vertex. For example, the 2-fold colouring of G, denoted $\chi_2(G)$, assigns a pair of colours to each vertex so that vertices joined by an edge have the property that their corresponding pairs are disjoint. Figure 2 gives a 2-fold colouring of our running example G, and in particular, $\chi_2(G) = 5$.

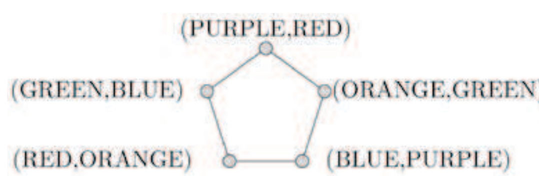

Figure 2. A minimal 2-fold colouring of our graph

For our scheduling problem, instead of one hour slots, we now use half-hour slots, and allow our meetings to take a break. For each slot, schedule all the committees where one of its two colours match the colour of the slot. We can now schedule all of our meetings in 2.5 hours!

The b-fold chromatic number $\chi_b(G)$ is defined analogously. By normalizing, the **fractional chromatic number** of G is

$$\chi_f(G) = \lim_{b\to\infty} \frac{\chi_b(G)}{b},$$

an invariant of fractional graph theory [5]. For our example, $\chi_f(G) = \frac{5}{2}$.

The fractional chromatic number also has a commutative algebra interpretation. If G is a graph, then the s-th symbolic power of $I(G)$ is

$$I(G)^{(s)} = \bigcap_{W \text{ is a minimal vertex cover of } G} \langle x_i \,|x_i \in W\rangle^s.$$

For any homogeneous ideal K, we let $\alpha(K)$ be the smallest degree of a non-zero element in K. The Waldschmidt constant of $I(G)$ is then

$$\widehat{\alpha}(I(G)) = \lim_{s\to\infty} \frac{\alpha(I(G)^{(s)})}{s}.$$

The Waldschmidt constant has origins in complex analysis [8] and is related to the "ideal containment problem" [2]. Then $\widehat{\alpha}(I(G))$ and $\chi_f(G)$ are related:

Theorem 2. ([1, Theorem 4.6]) *Let $G = (V, E)$ be a finite simple graph with edge ideal $I(G)$. Then*

$$\chi_f(G) = \frac{\widehat{\alpha}(I(G))}{\widehat{\alpha}(I(G)) - 1}.$$

To prove Theorem 2, one uses the fact that both invariants can be described in terms of linear programs, which are then related to each other. Theorems 1 and 2 hold more generally for hypergraphs. There are additional connections between colourings and commutative algebra, including the irreducible decomposition of $J(G)^s$ [3] and secant ideals of $I(G)$ [6].

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