An appendix to a paper of Catalisano, Geramita, Gimigliano: The Hilbert function of generic sets of 2-fat points in $\mathbb{P}^1 \times \mathbb{P}^1$

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Abstract. Implicit in the paper of Catalisano, Geramita, and Gimigliano [1] is a formula for the Hilbert function of generic sets of 2-fat points in $\mathbb{P}^1 \times \mathbb{P}^1$. We make this result explicit in this note.

We shall use the definitions and notation as found in [1].

Let d_1, d_2 be two positive integers and consider the fat point scheme $W = d_1 Q_1 + d_2 Q_2 + 2R_1 + \cdots + 2R_s \subseteq \mathbb{P}^2$ where $Q_1, Q_2, R_1, \ldots, R_s$ are $s + 2$ points in generic position. Proposition 2.1 of [1] describes some properties of the Hilbert function of W. Furthermore, this proposition is used (see [1, Corollary 2.2]) to compute the dimensions of the secant varieties of $\mathbb{P}^1 \times \mathbb{P}^1$ embedded into $\mathbb{P}^{(d_1+1)(d_2+1)-1}$ by the morphism given by $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(d_1, d_2)$.

By Terracini's Lemma (see §1 of [1]) there is a relationship between the dimensions of these embedded varieties and the Hilbert functions of generic sets of 2-fat points in $\mathbb{P}^1 \times \mathbb{P}^1$. Thus, implicit in [1, Corollary 2.2] is a formula for $H(Z, (d_1, d_2))$ when $Z = 2P_1 + \cdots + 2P_s \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is a generic set of 2-fat points and $(d_1, d_2) \geq (1, 1)$.

Because of the recent work on the Hilbert functions of points (for the reduced case, see [2, 6] and for the nonreduced case, see [3, 4]) in multiprojective spaces, it is of interest to have an explicit description of this formula. In fact, by coupling [1, Proposition 2.1] with a result from [5], we can describe $H(Z, (i, j))$ for all $(i, j) \in \mathbb{N}^2$.

Theorem 1. *Suppose $Z = 2P_1 + \cdots + 2P_s \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is a generic set of 2-fat points. Then*

$$
H(Z,(i,j)) = \begin{cases}
\min\{(i+1), 2s\} & \text{if } j = 0, \\
\min\{(j+1), 2s\} & \text{if } i = 0, \\
\min\{(i+1)(j+1), 3s\} & \text{if } (i,j) \geq (1,1) \text{ and} \\
& \quad (i,j) \notin \{(2, s-1), (s-1, 2)\}, \\
3s & \text{if } s \equiv 0 \pmod 2 \text{ and} \\
& \quad (i,j) \in \{(2, s-1), (s-1, 2)\}, \\
3s-1 & \text{if } s \equiv 1 \pmod 2 \text{ and} \\
& \quad (i,j) \in \{(2, s-1), (s-1, 2)\}.
\end{cases}
$$

Proof. Suppose that $(i,j) \in \mathbb{N}^2$ with $j = 0$. Let $\pi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection morphism given by $\pi_1(P \times Q) = P$. If we set $Z_1 := \pi_1(Z) \subseteq \mathbb{P}^1$, then by [5, Lemma 4.1] we have

$$H(Z,(i,0)) = H(Z_1, i) \quad \text{for all } i \in \mathbb{N}.$$

Because the support of Z is in generic position, the first coordinates of the points $P_1, \ldots, P_s$ must all be distinct. Thus Z_1 is a set of s double points in $\mathbb{P}^1$, and hence

$$H(Z,(i,0)) = H(Z_1, i) = \min\{(i+1), 2s\} \quad \text{for all } i \in \mathbb{N}.$$

Notice that if $(i,j) \in \mathbb{N}^2$ with $i = 0$, then the proof is the same except that the projection morphism $\pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ defined by $\pi_2(P \times Q) = Q$ is used.

So, we need to compute $H(Z,(i,j))$ when $(i,j) \geq (1,1)$. By [1, Theorem 1.1] (and the discussion at the start of §2 in [1]) we have

$$
\begin{aligned}
H(Z,(i,j)) &= \dim_k R_{(i,j)} - \dim_k (I_Z)_{(i,j)} \\
&= (i+1)(j+1) - \dim_k (I_W)_{i+j}
\end{aligned}
\tag{1}
$$

where I_W is the defining ideal in $S = k[x,y,z]$ of the fat point scheme

$$W = iQ_1 + jQ_2 + 2R_1 + \cdots + 2R_s \subseteq \mathbb{P}^2$$

and $Q_1, Q_2, R_1, \ldots, R_s$ are $s+2$ points in generic position.

The value of $H(W, i+j)$ can be computed using [1, Proposition 2.1]. In particular,

$$H(W, i+j) = \min\{\dim_k S_{i+j}, \deg W\} \tag{2}$$

except when

$$s = 2a+1 \quad \text{and} \quad (i,j) = (2, 2a) \quad \text{or} \quad (2a, 2).$$

(Note that although [1, Proposition 2.1] assumes that $i \geq j$, if $(i,j) \in \mathbb{N}^2$ with $i < j$, then we can swap the roles of i and j to compute $H(W, i+j)$.) In the two exceptional cases, [1, Proposition 2.1] demonstrated that

$$h^0(\mathcal{I}_W(i+j)) = \dim_k (I_W)_{i+j} = 1. \tag{3}$$

Using the fact that $\deg W = 3s + \binom{i+1}{2} + \binom{j+1}{2}$, we can use (2) and (3) to derive the following formula for $\dim_k(I_W)_{i+j}$:

$$\dim_k(I_W)_{i+j} = \begin{cases} \max\{0, (i+1)(j+1) - 3s\} & \text{if } (i,j) \geq (1,1) \text{ and} \\ & \quad (i,j) \notin \{(2, s-1), (s-1, 2)\}, \\ 0 & \text{if } s \equiv 0 \pmod 2 \text{ and} \\ & \quad (i,j) \in \{(2, s-1), (s-1, 2)\}, \\ 1 & \text{if } s \equiv 1 \pmod 2 \text{ and} \\ & \quad (i,j) \in \{(2, s-1), (s-1, 2)\}. \end{cases}$$

We now substitute the above values into (1). If $(i,j) \geq (1,1)$ and $(i,j) \notin \{(2, s-1), (s-1, 2)\}$, then

$$\begin{aligned} H(Z, (i,j)) &= (i+1)(j+1) - \max\{0, (i+1)(j+1) - 3s\} \\ &= \min\{(i+1)(j+1), 3s\}. \end{aligned}$$

When $(i,j) \in \{(2, s-1), (s-1, 2)\}$, then $(i+1)(j+1) = 3s$. So, if $s \equiv 0 \pmod 2$, then

$$H(Z, (i,j)) = 3s - \dim_k(I_W)_{i+j} = 3s.$$

However, if $s \equiv 1 \pmod 2$, then

$$H(Z, (i,j)) = 3s - \dim_k(I_W)_{i+j} = 3s - 1.$$

This completes the proof. $\square$

References

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