Shellability of the van der Waerden complex

by

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Abstract

The van der Waerden complex $vdW(n, k)$, defined in 2016 by Ehrenborg, Govindaiah, Park, and Readdy [**2**], has been studied in the context of topology, and in particular, homotopy equivalence. Here we look at $vdW(n, k)$ from the context of combinatorial commutative algebra. We consider the shellability of the van der Waerden complexes, and give a necessary and sufficient condition on n and k for shellability.

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CHAPTER 1

Introduction

The van der Waerden complex $vdW(n, k)$ is a simplicial complex defined in 2016 by Ehrenborg, Govindaiah, Park, and Readdy [**2**]. These simplicial complexes are defined by arithmetic progressions and have been studied in the context of topology. This project aims to look at the van der Waerden complex in a different context; that is, a combinatorial commutative algebraic perspective. In particular, we will be considering the shellability of these complexes. This chapter will introduce some key ideas which will be used throughout. More detailed explanations of the ideas discussed here can be found in Chapter 2.

We first introduce simplicial complexes. Given a set of vertices X, a *simplicial complex* on X is defined by a set Δ of subsets of X such that both of the following hold:

- for every vertex $x_i \in X$, $\{x_i\} \in \Delta$, and
- if $F \in \Delta$, then every subset of F is also in Δ.

Each set $F \in \Delta$ is called a *face* of the simplicial complex. In addition to this set notation, we can also represent simplicial complexes pictorially. We now give an example of a simplicial complex.

Example 1.1. Consider the vertex set $X = \{x_1, x_2, x_3, x_4, x_5\}$. Then consider the set of vertex subsets

$$\Delta = \left\{ \begin{array}{l} \{x_1, x_2, x_3\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \\ \{x_1, x_4, x_5\}, \{x_1, x_4\}, \{x_1, x_5\}, \{x_4, x_5\}, \\ \{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}, \varnothing \end{array} \right\}.$$

Notice that each vertex appears as the sole member of a set in Δ, and every set in Δ has each of its subsets also appearing as a set. From this observation, we can see that Δ is a simplicial complex. We often express a simplicial complex in terms of its facets; that is, the faces which are not strictly contained in any other face. In this way, we can express this simplicial complex as

$$\Delta = \langle \{x_1, x_2, x_3\}, \{x_1, x_4, x_5\} \rangle.$$

As mentioned, we can also draw this simplicial complex as shown below:

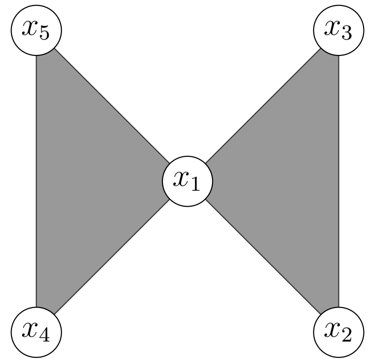

FIGURE 1.1. Simplicial complex $\Delta = \langle \{x_1, x_2, x_3\}, \{x_1, x_4, x_5\} \rangle$

Each face of a simplicial complex has a *dimension* equal to its cardinality minus one. The dimension of the whole complex is defined to be the maximum of the dimensions of all its facets. A simplicial complex is called *pure* if each of its facets have the same dimension. The simplicial complex from Example 1.1 is an example of a pure 2-dimensional simplicial complex.

The complexes of interest to us, the van der Waerden complexes, are another example of pure complexes. The *van der Waerden complex* of dimension $k > 0$ on $n > k$ vertices, denoted $vdW(n, k)$, is generated by facets given by arithmetic progressions $\{x_i, x_{i+d}, \ldots, x_{i+kd}\}$, where $0 < i < i + kd \leq n$. We now give an example to clarify.

EXAMPLE 1.2. Consider the van der Waerden complex of dimension 2 on 7 vertices. We then generate the complex

$$vdW(7,2) = \left\langle \begin{matrix} \{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_3, x_4, x_5\}, \{x_4, x_5, x_6\}, \{x_5, x_6, x_7\}, \\ \{x_1, x_3, x_5\}, \{x_2, x_4, x_6\}, \{x_3, x_5, x_7\}, \{x_1, x_4, x_7\} \end{matrix} \right\rangle.$$

Pictorially, this simplicial complex looks like the following figure.

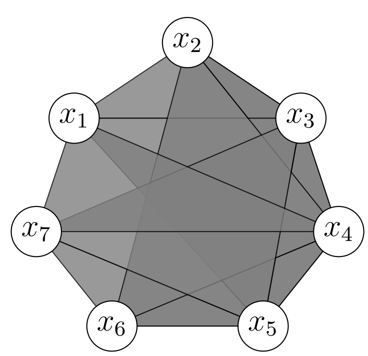

FIGURE 1.2. Simplicial complex $vdW(7, 2)$

A pure simplicial complex is called *shellable* if its facets can be put together in a particularly nice way. If the complex can be assembled one facet at a time such that each new facet joins the previous facets along a face with dimension one less than that of the complex, this condition is satisfied. The simplicial complex in Example 1.1 is then not shellable, since its only two facets join along a face with dimension zero, but the complex has dimension two. The van der Waerden complex $vdW(7,2)$ from the Example 1.2 is another example of a non-shellable complex. On the other hand, the simplicial complex on four vertices given by $\Gamma = \langle\{x_1, x_2, x_3\}, \{x_1, x_3, x_4\}\rangle$ is shellable, since it has dimension two and its two facets join along the face $\{x_1, x_3\}$ which has dimension one.

Given a simplicial complex Δ, we can find a related simplicial complex called the *Alexander dual*, which we denote by $\Delta^\vee$. This complex can be found by taking the complements of the minimal nonfaces of Δ as facets. We illustrate with an example.

EXAMPLE 1.3. Again, we consider $\Delta = \langle\{x_1, x_2, x_3\}, \{x_1, x_4, x_5\}\rangle$. The minimal nonfaces are $\{x_2, x_4\}$, $\{x_2, x_5\}$, $\{x_3, x_4\}$, and $\{x_3, x_5\}$. Taking the complements of these sets, we have $\{x_1, x_3, x_5\}$, $\{x_1, x_3, x_4\}$, $\{x_1, x_2, x_5\}$ and $\{x_1, x_2, x_4\}$ as facets of the Alexander dual. That it,

$$\Delta^\vee = \langle\{x_1, x_2, x_4\}, \{x_1, x_2, x_5\}, \{x_1, x_3, x_4\}, \{x_1, x_3, x_5\}\rangle.$$

Pictorially, this is represented by

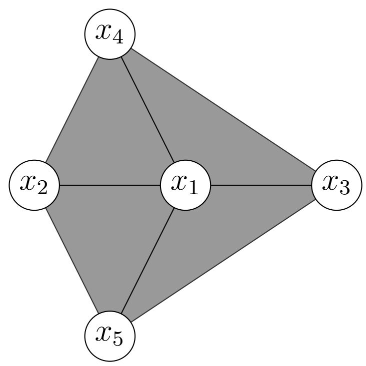

FIGURE 1.3. Simplicial complex $\Delta^\vee = \langle\{x_1, x_2, x_4\}, \{x_1, x_2, x_5\}, \{x_1, x_3, x_4\}, \{x_1, x_3, x_5\}\rangle$

The Alexander dual of the Alexander dual of a simplicial complex will always give the original simplicial complex as a result, i.e., $(\Delta^\vee)^\vee = \Delta$.

We can also take a simplicial complex Δ and create an associated monomial ideal; this is called the *Stanley-Reisner ideal*, denoted by I_Δ, and was defined by Stanley and Reisner in the 1970's. We create this ideal by assigning each vertex to a variable, and then creating products of these variables for every nonface of the simplicial complex. This will be especially useful in relation to the Alexander dual, and so we demonstrate this using $\Delta^\vee$ from the Example 1.3.

EXAMPLE 1.4. Given $\Delta^\vee = \langle\{x_1, x_2, x_4\}, \{x_1, x_2, x_5\}, \{x_1, x_3, x_4\}, \{x_1, x_3, x_5\}\rangle$, we have $\{x_2, x_3\}$ and $\{x_4, x_5\}$ as minimal nonfaces. Our Stanley-Reisner ideal will then be the ideal generated by the monomials created through assigning each vertex a variable and taking the products of each of these sets. That is, we have the set of generators $\{x_2x_3, x_4x_5\}$. Hence we have the Stanley-Reisner ideal of the Alexander dual, $I_{\Delta^\vee} = \langle x_2x_3, x_4x_5\rangle$.

We can use the Stanley-Reisner ideal of the Alexander dual of a simplicial complex to determine if the simplicial complex is shellable or not. A more precise statement and proof can be found in Theorem 3.4 of this paper. Shellability is a useful property to consider, as it carries implications for other important properties; in particular, it can determine if a simplicial complex is Cohen Macaulay. That is, if a simplicial complex is shellable, then it is also Cohen-Macaulay. In this project, we will address the following problem:

Question 1.5. *For what values of n and k is the van der Warden complex $vdW(n, k)$ a shellable complex?*

To answer this question, we begin by giving the required background information from combinatorics and algebra. This includes a general discussion of simplicial complexes and the language used to talk about them, as well as introducing the Stanley-Reisner ideal and Alexander dual, and how to find these. We then introduce the concept of shellability. We give three different equivalent definitions of shellability, as well as examples, and we show how the Stanley-Reisner ideal $I_{\Delta^\vee}$ of the Alexander dual of a simplicial complex can be used to determine shellability. Next, we look at an important property of the van der Waerden complex, and use this property to show which of these complexes are and are not shellable. In particular, we explore the way in which different increment sizes of facets of the van der Waerden complex result in important differences between the faces given by them. We use this property to make some conclusions about $I_{vdW(n,k)^\vee}$, which in turn gives us our final result. Finally, we restate the main theorem and make note of additional areas of interest related to van der Waerden complexes and shellability.

The main theorem is as follows:

Theorem 4.12. *Let $0 < k < n$ be integers and consider the van der Waerden complex $vdW(n, k)$ of dimension k on n vertices. Then $vdW(n, k)$ is shellable if and only if*

- *$n \leq 6$, or*
- *$n > 6$ and $k = 1$, or*
- *$n > 6$ and $\frac{n}{2} \leq k < n$.*

As noted above, this is proved using the the Stanley-Reisner ideal $I_{vdW(n,k)^\vee}$ of the Alexander dual of $vdW(n, k)$. We show that given this condition $vdW(n, k)$ is shellable, and that otherwise $I_{vdW(n,k)^\vee}$ does not have linear quotients.

CHAPTER 2

Background

This chapter will introduce the relevant information from combinatorics and algebra. We begin with a general discussion of simplicial complexes and some of their properties. We will also relate simplicial complexes to monomial ideals via the Stanley Reisner ideal and discuss some results which can be found using this relation. Much of the introductory material here can be found in the survey done by Francisco, Mermin, and Schweig [**4**].

1. Simplicial Complexes

In this section, we define simplicial complexes along with some of their properties. We demonstrate these properties with some examples.

DEFINITION 2.1. Fix $n > 0$ and let $\{x_1, x_2, \ldots, x_n\}$ be a set of vertices. A *simplicial complex* on $\{x_1, \ldots, x_n\}$ is a set Δ of subsets of $\{x_1, \ldots, x_n\}$ where $A \in \Delta$ if $A \subseteq F \in \Delta$, and for all $0 < i \leq n$, $\{x_i\} \in \Delta$.

Elements of Δ are called *simplices*, or *faces*. A face $F \in \Delta$ for which there is no $A \in \Delta$ such that $F \subsetneq A$ is called a *facet*. A simplicial complex can be described by its facets, i.e., $\Delta = \langle F_1, F_2, \ldots, F_s \rangle$. A set $B \notin \Delta$ is called a *minimal non-face* of Δ if for any $A \subsetneq B$, the set $A \in \Delta$.

DEFINITION 2.2. Let Δ be a simplicial complex. The *dimension* of a face of Δ is one less than its cardinality, i.e., $\dim(A) = |A| - 1$. The *dimension* of Δ is the dimension of its largest facet.

We now look at some examples of simplicial complexes.

EXAMPLE 2.3. As mentioned above, we can denote a simplicial complex Δ by the set of its facets. The following set Δ is a simplicial complex on the set $\{x_1, x_2, x_3, x_4, x_5\}$:

$$\Delta = \langle \{x_1, x_2, x_3\}, \{x_1, x_4\}, \{x_1, x_5\}, \{x_2, x_4\} \rangle.$$

Notice that the largest face of Δ has dimension 2, and hence, Δ has dimension 2.

A simplicial complex can also be represented pictorially, where each x_i is a vertex and an edge joins x_i to x_j if and only if $\{x_i, x_j\}$ is a face of Δ. Further, the area bounded by edges $x_1x_2, x_2x_3, \ldots, x_{p-1}, x_p$ is filled if and only if $\{x_1, x_2, \ldots, x_p\}$ is a face of Δ. Using this, we have the following representation of Δ:

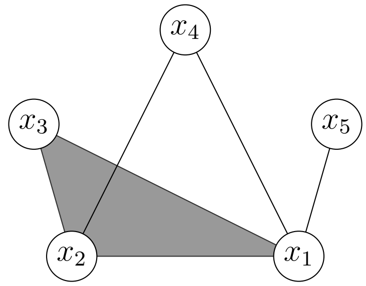

FIGURE 2.1. Simplicial complex $\Delta = \langle \{x_1, x_2, x_3\}, \{x_1, x_4\}, \{x_1, x_5\}, \{x_2, x_4\} \rangle$

EXAMPLE 2.4. The void complex, $\Delta = \{\varnothing\}$, is the simplicial complex with no faces.

EXAMPLE 2.5. Consider the picture Γ:

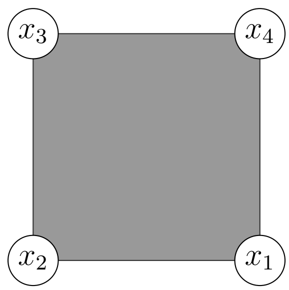

FIGURE 2.2. Picture Γ

This does *not* represent a simplicial complex, since $\{x_1, x_2, x_3, x_4\}$ would be a facet, but none of $\{x_1, x_2, x_3\}$, $\{x_1, x_2, x_4\}$, $\{x_1, x_3, x_4\}$, $\{x_2, x_3, x_4\}$, $\{x_1, x_3\}$, or $\{x_2, x_4\}$ are faces. If these faces were to be included, we would then have the simplicial complex $\Gamma' = \langle \{x_1, x_2, x_3, x_4\} \rangle$, and would have the representation:

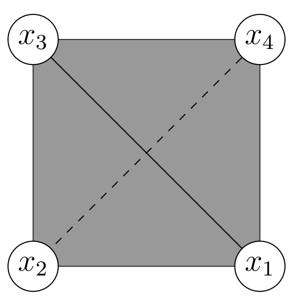

FIGURE 2.3. Simplicial complex $\Gamma' = \langle \{x_1, x_2, x_3, x_4\} \rangle$

That is, Γ' is a solid tetrahedron. In this case, Γ' is a simplicial complex with dimension 3.

An interesting property which some simplicial complexes have is purity. A simplicial complex Δ is called *pure* if all of its facets have the same dimension. We now give additonal examples of simplicial complexes and note if they possess this property or not.

EXAMPLE 2.6. Consider the simplicial complex

$$\Delta = \langle \{x_1, x_2, x_3, x_6\}, \{x_2, x_4, x_5, x_6\}, \{x_1, x_2, x_3, x_5\} \rangle.$$

This simplicial complex is pure since each facet has dimension 3.

EXAMPLE 2.7. The simplicial complex

$$\Delta = \langle \{x_1, x_2, x_3, x_4\}, \{x_1, x_2, x_5\}, \{x_1, x_4, x_5\} \rangle$$

is not pure, since there are facets with both dimension 3 and 2.

EXAMPLE 2.8. The simplicial complex given by

$$\Delta = \langle \{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_3, x_4, x_5\}, \{x_1, x_3, x_5\} \rangle$$

is a pure simplicial complex of dimension 2. In fact, this is an example of a special type of simplicial complex, called the van der Waerden complex, which will be discussed in depth later.

2. Alexander Duality and the Stanley Reisner Ideal

Here we introduce concepts which relate simplicial complexes to other simplicial complexes and to monomial ideals. We also consider some properties of these ideals.

DEFINITION 2.9. For a simplicial complex Δ on the set of vertices $X = \{x_1, x_2, \ldots, x_n\}$, the *Alexander dual* of Δ, denoted $\Delta^\vee$, is the simplicial complex whose facets are complements of the minimal non-faces of Δ. That is,

$$\Delta^\vee = \{X \backslash m \mid m \notin \Delta\}.$$

Notice that the dual of the dual is the original complex, i.e., $(\Delta^\vee)^\vee = \Delta$.

EXAMPLE 2.10. Consider the simplicial complex given by

$$\Delta = \langle \{x_1, x_2, x_4\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\} \rangle.$$

From this, we can see that the minimal non-faces are

$$\{x_1, x_3\}, \{x_1, x_5\}, \{x_2, x_5\}, \{x_3, x_5\}, \{x_2, x_3, x_4\}.$$

Finding the complements of these non-faces, we have

$$\Delta^\vee = \langle \{x_2, x_4, x_5\}, \{x_2, x_3, x_4\}, \{x_1, x_3, x_4\}, \{x_1, x_2, x_4\}, \{x_1, x_5\} \rangle.$$

We can then illustrate these complexes as done below.

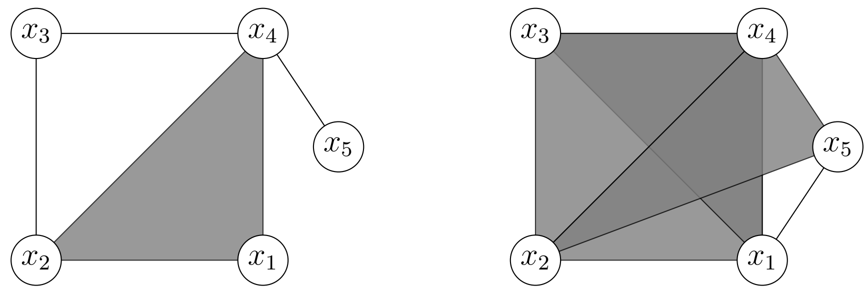

FIGURE 2.4. A simplicial complex Δ (left) and its Alexander dual $\Delta^\vee$

DEFINITION 2.11. A *monomial* in a ring $k[x_1, \dots, x_n]$ is an element which can be written uniquely (up to order) as a product of variables; a monomial m is *squarefree* if no variable appears more than once in this factorization. A *squarefree monomial ideal* is an ideal $I \subset k[x_1, \dots, x_n]$ generated by squarefree monomials of $k[x_1, \dots, x_n]$.

Given a simplicial complex, we can define a related squarefree monomial ideal.

DEFINITION 2.12. Let Δ be a simplicial complex on the set $\{x_1, \dots, x_n\}$. Then the *Stanley-Reisner ideal* of Δ is the squarefree monomial ideal in $k[x_1, \dots, x_n]$ defined by

$$I_\Delta = \langle x_{i_1} \cdots x_{i_t} : \{x_{i_1}, \dots, x_{i_t}\} \notin \Delta \rangle.$$

EXAMPLE 2.13. Again we consider the simplicial complex defined by

$$\Delta = \langle \{x_1, x_2, x_4\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\} \rangle.$$

To find its Stanley-Reisner ideal, we must find the minimal non-faces of Δ. These are $\{x_1, x_3\}$, $\{x_1, x_5\}$, $\{x_2, x_5\}$, $\{x_3, x_5\}$, and $\{x_2, x_3, x_4\}$. We then assign each vertex a variable and create monomials by taking the product of variables corresponding to each minimal non-face. This gives

$$I_\Delta = \langle x_1x_3, x_1x_5, x_2x_5, x_3x_5, x_2x_3x_4 \rangle.$$

Sometimes we want to consider the Stanley-Reisner ideal of the Alexander dual of a simplicial complex. This is particularly interesting since it has consequences related to the shellability of the complex which will be discussed in Chapter 3. Rather than taking the Alexander dual of a simplicial complex and then taking its Stanley-Reisner ideal, we can go straight from Δ to $I_{\Delta^\vee}$.

THEOREM 2.14. *Let $\Delta = \langle F_1, F_2 \dots, F_s \rangle$ be a simplicial complex. Then*

$$I_\Delta = Q_1 \cap \cdots \cap Q_s,$$

where $Q_i = \langle x \mid x \notin F_i \rangle$. See [**3**] *for a proof.*

COROLLARY 2.15. *Let $\Delta = \langle F_1, F_2 \dots, F_s \rangle$ be a simplicial complex and set $Q_i = \langle x \mid x \notin F_i \rangle$. Then*

$$I_{\Delta^\vee} = \langle m_1, \dots, m_s \rangle$$

where $m_i = \prod_{x \in Q_i} x = \prod_{x \notin F_i} x$. *See* [**6**] *for this result.*

Example 2.16. We again recall the complex defined by

$$\Delta = \langle \{x_1, x_2, x_4\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\} \rangle$$

and its Alexander dual

$$\Delta^\vee = \langle \{x_2, x_4, x_5\}, \{x_2, x_3, x_4\}, \{x_1, x_3, x_4\}, \{x_1, x_2, x_4\}, \{x_1, x_5\} \rangle.$$

Notice that the minimal non-faces of $\Delta^\vee$ are given by

$$\{\{x_3, x_5\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_5\}, \{x_1, x_4, x_5\}\},$$

and hence we have

$$I_{\Delta^\vee} = \langle x_3x_5, x_1x_2x_3, x_1x_2x_5, x_1x_4x_5 \rangle.$$

We can also compute this using Corollary 2.15. Taking Q_i for each facet of Δ and taking products, we find

$$I_{\Delta^\vee} = \langle x_3x_5, x_1x_4x_5, x_1x_2x_5, x_1x_2x_3 \rangle,$$

which is the same ideal we found using the first method.

Definition 2.17. Let I, J be ideals in the ring R. The *colon ideal* of I with J is given by

$$\langle I \rangle : \langle J \rangle = \{r \in R \mid rJ \subset I\}.$$

Proposition 2.18. *Let* I, J *be monomial ideals generated by* $\{m_1, m_2, \ldots, m_s\}$ *and* $\{n_1, n_2, \ldots, n_t\}$ *respectively. Then*

$$I : J = \bigcap_{j=1}^{t} (I : \langle n_j \rangle) = \bigcap_{j=1}^{t} \left\langle \frac{m_i}{gcd(m_i, n_j)} \;\middle|\; i = 1, \ldots, s \right\rangle.$$

See [**3**] *for a proof.*

Definition 2.19. Let I be a squarefree monomial ideal in $k[x_1, \ldots, x_n]$ with $I = \langle m_1, \ldots, m_s \rangle$. The ideal I has *linear quotients* if there is an ordering $m_{i_1}, \ldots, m_{i_s}$ such that for $1 \leq r_t \leq n$,

$$\langle m_{i_1}, \ldots, m_{i_{t-1}} \rangle : \langle m_{i_t} \rangle = \langle x_{j_1}, \ldots, x_{j_{r_t}} \rangle$$

for all $1 < t \leq s$, i.e., it is generated by a subset of the variables.

To illustrate this method of finding colon ideals and linear quotients, we give an example.

Example 2.20. Let I, J be ideals in $k[x_1, x_2, x_3, x_4, x_5]$ generated in the following way:

$$\begin{aligned} I &= \langle x_1x_2x_3, x_1x_4x_5, x_2x_5 \rangle \\ J &= \langle x_1x_2x_3, x_1x_3x_4, x_2x_3x_4 \rangle. \end{aligned}$$

We take the colon ideal below:

$$\begin{aligned}
I : J &= \langle x_1x_2x_3, x_1x_4x_5, x_2x_5\rangle : \langle x_1x_2x_3, x_1x_3x_4, x_2x_3x_4\rangle \\
&= \bigcap_{j=1}^{3}(I : \langle n_j\rangle) \\
&= \bigcap_{j=1}^{3}\left\langle \frac{m_i}{gcd(m_i, n_j)} \,\middle|\, i = 1, 2, 3\right\rangle \\
&= \left\langle \frac{x_1x_2x_3}{x_1x_2x_3}, \frac{x_1x_4x_5}{x_1}, \frac{x_2x_5}{x_2}\right\rangle \cap \left\langle \frac{x_1x_2x_3}{x_1x_3}, \frac{x_1x_4x_5}{x_1x_4}, \frac{x_2x_5}{1}\right\rangle \cap \left\langle \frac{x_1x_2x_3}{x_2x_3}, \frac{x_1x_4x_5}{x_4}, \frac{x_2x_5}{x_2}\right\rangle \\
&= \langle 1, x_4x_5, x_5\rangle \cap \langle x_2, x_5, x_2x_5\rangle \cap \langle x_1, x_1x_5, x_5\rangle \\
&= R \cap \langle x_2, x_5\rangle \cap \langle x_1, x_5\rangle \\
&= \langle x_2, x_5\rangle \cap \langle x_1, x_5\rangle \\
&= \langle x_5, x_1x_2\rangle.
\end{aligned}$$

Hence, we have the colon ideal of I with J. Now we want to check if either of our original ideals have linear quotients. We begin with I, and suggest the ordering of monomials $m_1 = x_2x_5$, $m_2 = x_1x_2x_3$, $m_3 = x_1x_4x_5$. Then consider the following colon ideals:

$$\begin{aligned}
\langle x_2x_5\rangle : \langle x_1x_2x_3\rangle &= \left\langle \frac{x_2x_5}{x_2}\right\rangle \\
&= \langle x_5\rangle \\
\langle x_2x_5, x_1x_2x_3\rangle : \langle x_1x_4x_5\rangle &= \left\langle \frac{x_2x_5}{x_5}, \frac{x_1x_2x_3}{x_1}\right\rangle \\
&= \langle x_2, x_2x_3\rangle = \langle x_2\rangle.
\end{aligned}$$

Since each of these colon ideals are generated by a single variable, the ideal I has linear quotients. Similarly, the ideal J also has linear quotients with the ordering $n_1 = x_1x_2x_3$, $n_2 = x_1x_3x_4$, and $n_3 = x_2x_3x_4$.

While each of the monomial ideals in the previous example have linear quotients, this is not always the case. We now look at an example where this does not occur.

Example 2.21. Next we consider the ideal in $k[x_1, x_2, x_3, x_4, x_5, x_6]$ given by

$$L = \langle x_1x_2x_3x_4, x_1x_3x_5x_6, x_2x_4x_5x_6\rangle.$$

We will show this ideal does not have linear quotients by showing that none of the generators may be the final monomial in an ordering. To do this, consider the following colon ideals:

$$\begin{aligned}
\langle x_1x_2x_3x_4, x_1x_3x_5x_6\rangle : \langle x_2x_4x_5x_6\rangle &= \left\langle \frac{x_1x_2x_3x_4}{x_2x_4}, \frac{x_1x_3x_5x_6}{x_5x_6} \right\rangle \\
&= \langle x_1x_3\rangle \\
\langle x_1x_2x_3x_4, x_2x_4x_5x_6\rangle : \langle x_1x_3x_5x_6\rangle &= \left\langle \frac{x_1x_2x_3x_4}{x_1x_3}, \frac{x_2x_4x_5x_6}{x_5x_6} \right\rangle \\
&= \langle x_2x_4\rangle \\
\langle x_1x_3x_5x_6, x_2x_4x_5x_6\rangle : \langle x_1x_2x_3x_4\rangle &= \left\langle \frac{x_1x_3x_5x_6}{x_2x_4}, \frac{x_2x_4x_5x_6}{x_1x_3} \right\rangle \\
&= \langle x_5x_6\rangle.
\end{aligned}$$

Notice that none of these colon ideals are generated by a set of variables; each of their generators is a product of two variables. Hence, we conclude that L cannot have linear quotients.

CHAPTER 3

Shellable Simplicial Complexes

We now focus on a specific family of simplicial complexes related to the property of shellability. We discuss shellable simplicial complexes, giving some examples and identifying strategies for determining the shellability of a given simplicial complex.

Definition 3.1. Let Δ be a d-dimensional pure simplicial complex on $\{x_1, \ldots, x_n\}$. An ordering of the facets $F_1, F_2, \ldots, F_s$ of Δ is called a *shelling* if one of the following equivalent statements are satisfied:

(1) $\langle F_i \rangle \cap \langle F_1, \ldots, F_{i-1} \rangle$ is generated by a non-empty set of maximal proper subfaces of F_i, i.e., faces of dimension $\dim(F_i) - 1$, for all i such that $1 < i \leq s$.
(2) The set $\{A \mid A \in \langle F_1, \ldots, F_i \rangle, A \notin \langle F_1, \ldots, F_{i-1} \rangle\}$ has a unique minimal element for all i such that $1 < i \leq s$.
(3) For all i, j with $1 \leq j < i \leq s$, there exists some $x \in F_i \backslash F_j$ and some $1 \leq k < i$ with $F_i \backslash F_k = \{x\}$.

We say that a simplicial complex is *shellable* if it admits a shelling.

Given this definition, we first want to show that each of these conditions are equivalent. We will then demonstrate how to use this definition to determine whether a simplicial complex is shellable, and look at other ways to determine shellability.

Theorem 3.2. *The statements (1), (2), and (3) from Definition 3.1 are equivalent.*

Proof. To show this equivalence, we follow the proof given by Bruns and Herzog [**1**].

(1) $\Rightarrow$ (2)

Under relabeling, suppose that $F_i = \{x_1, x_2, \ldots, x_p\}$ is a facet of dimension $p-1$. Then let $A_1, \ldots, A_p$ be the maximal proper subfaces of F_i defined by $A_l = \{x_1, x_2, \ldots, x_{l-1}, x_{l+1}, \ldots, x_p\}$ for $1 \leq l \leq p$, i.e., $A_l = F_i \backslash \{x_l\}$ and $\dim(A_l) = p - 2$. Now, by assumption, we have

$$\langle F_i \rangle \cap \langle F_1, F_2, \ldots, F_{i-1} \rangle = \langle A_{j_1}, A_{j_2}, \ldots, A_{j_q} \rangle$$

for $1 \leq q \leq p$, where $j_t \in \{1, 2, \ldots, p\}$ for all $1 \leq t \leq q$. Now define the face $G = \{x_{j_1}, x_{j_2}, \ldots, x_{j_q}\}$.

It is clear that $G \in \langle F_i \rangle$, and so we have $G \in \langle F_1, F_2, \ldots, F_i \rangle$. However, G was constructed in such a way that $G \notin \langle A_{j_1}, A_{j_2}, \ldots, A_{j_q} \rangle$, i.e., $G \notin \langle F_i \rangle \cap \langle F_1, F_2, \ldots, F_{i-1} \rangle$. But since $G \in \langle F_i \rangle$, it must be that $G \notin \langle F_1, F_2, \ldots, F_{i-1} \rangle$. Hence, we have that

$$G \in \mathcal{A} = \{A \mid A \in \langle F_1, \ldots, F_i \rangle, A \notin \langle F_1, \ldots, F_{i-1} \rangle\}.$$

Notice that for $1 \leq r \leq q$, removing the r-th vertex from G will give

$$G_r = G \backslash \{x_{j_r}\} = \{x_{j_1}, x_{j_2}, \ldots, x_{j_{r-1}}, x_{j_{r+1}}, \ldots, x_{j_q}\} \subset A_{j_r},$$

and so $G_r \notin \mathcal{A}$. As a result, we see that G is a minimal element.

Now suppose that there is another minimal element $H = \{x_{v_1}, x_{v_2}, \ldots, x_{v_q}\} \in \mathcal{A}$. Since $H \in \langle F_i \rangle$, it must be that $v_t \in \{1, 2, \ldots, p\}$. Further, if there is some x_{v_t} such that $x_{v_t} \notin H$, then $H \subset A_{v_t}$, i.e., $H \notin \mathcal{A}$. But this means that $G \subset H$, and since H is minimal, $G = H$. Hence, it must be that G is the unique minimal element of $\mathcal{A}$.

(2) $\Rightarrow$ (3)

Suppose that for all i such that $1 < i \leq s$, G is the unique minimal element of $\mathcal{A}$. After relabeling, suppose that $G = \{x_1, x_2, \ldots, x_p\}$. By assumption, we have $G \not\subset F_j$ for all $1 \leq j < i$, and so $G \backslash F_j \neq \varnothing$. Then for some $1 \leq q \leq p$, we have $x_q \in G \backslash F_j$. Further, since $G \subset F_i$, $x_q \in F_i \backslash F_j$. If $F_i \backslash F_j = \{x_q\}$, we are done.

Now suppose $F_i \backslash F_j \neq \{x_q\}$. Since G is minimal, we know that $G \backslash \{x_q\} \notin \mathcal{A}$, i.e., $G \backslash \{x_q\} \in \langle F_1, \ldots, F_{i-1} \rangle$. Then there is some k with $1 \leq k \leq i-1$ such that $G \backslash \{x_q\} \subset F_k$. Then, since $x_q \notin F_k$, we have that $F_i \backslash F_k = \{x_q\}$.

(3) $\Rightarrow$ (1)

Consider $\langle F_i \rangle \cap \langle F_1, \ldots, F_{i-1} \rangle$ and suppose that there is a face $A \in \langle F_i \rangle \cap \langle F_1, \ldots, F_{i-1} \rangle$. Then we have that $A \subset F_i$ and $A \subset F_j$ for some $1 \leq j < i \leq s$.

By assumption, there is some $x \in F_i \backslash F_j$, i.e., $x \in F_i$ and $x \notin F_j$, and some $1 \leq k < i$ such that $F_i \backslash F_k = \{x\}$. Then consider $F_i \backslash \{x\}$. Since $A \subset F_j$, we have that $x \notin A$, and so $A \subset F_i \backslash \{x\}$. Then $F_i \backslash \{x\}$ is a maximal proper subface of F_i containing A. Also, since $F_i \backslash F_k = \{x\}$, we have $F_i \backslash \{x\} \subset F_k$, and so $F_i \backslash \{x\} \in \langle F_i \rangle \cap \langle F_1, \ldots, F_{i-1} \rangle$. Hence, we conclude that $\langle F_i \rangle \cap \langle F_1, \ldots, F_{i-1} \rangle$ is generated by a set of maximal proper subfaces of F_i. $\square$

Example 3.3. Consider the simplicial complex defined by the following figure.

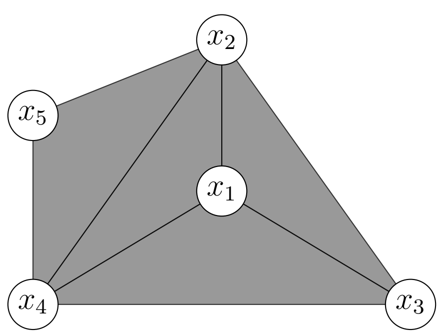

Figure 3.1. $\Delta = \langle \{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_3, x_4\}, \{x_2, x_4, x_5\} \rangle$

Consider the ordering of the facets $F_1 = \{x_1, x_2, x_3\}$, $F_2 = \{x_1, x_3, x_4\}$, $F_3 = \{x_1, x_2, x_4\}$, $F_4 = \{x_2, x_4, x_5\}$. We want to show that this ordering is a shelling. We will do this in three different ways to show how each different definition can be applied.

Using Definition 3.1 (1), we look at the following sets.

$$\begin{aligned}\langle F_2\rangle \cap \langle F_1\rangle &= \langle\{x_1,x_3\}\rangle\\ \langle F_3\rangle \cap \langle F_1,F_2\rangle &= \langle\{x_1,x_2\},\{x_1,x_4\}\rangle\\ \langle F_4\rangle \cap \langle F_1,F_2,F_3\rangle &= \langle\{x_2,x_4\}\rangle.\end{aligned}$$

Notice that we have subsets of maximal proper faces $\{x_1,x_3\} \subset F_2$, $\{x_1,x_2\},\{x_1x_4\} \subset F_3$, and $\{x_2,x_4\} \subset F_4$ which generate these sets respectively. Hence, we have that this simplicial complex is shellable.

Using Definition 3.1 (2), we look at the following sets.

$$\begin{aligned}\{A \mid A \in \langle F_1,F_2\rangle, A \notin \langle F_1\rangle\} &= \{\{x_1,x_3,x_4\},\{x_1,x_4\},\{x_3,x_4\},\{x_4\}\}\\ \{A \mid A \in \langle F_1,F_2,F_3\rangle, A \notin \langle F_1,F_2\rangle\} &= \{\{x_1,x_2,x_4\},\{x_2,x_4\}\}\\ \{A \mid A \in \langle F_1,F_2,F_3,F_4\rangle, A \notin \langle F_1,F_2,F_3\rangle\} &= \{\{x_2,x_4,x_5\},\{x_2,x_5\},\{x_4,x_5\},\{x_5\}\}.\end{aligned}$$

Notice that in each of these sets, we have the unique minimal elements $\{x_4\}, \{x_2,x_4\}$, and $\{x_5\}$ respectively, and so our simplicial complex is shellable.

Using Definition 3.1 (3), we need to consider each i,j such that $1 \leq j < i \leq 5$, and then look at the sets $F_i \backslash F_j$:

$$\begin{aligned}&j=1,i=2 \quad F_2,\backslash F_1 &&= \{\{x_4\}\}\\ &j=1,i=3 \quad F_3,\backslash F_1 &&= \{\{x_4\}\}\\ &j=1,i=4 \quad F_4,\backslash F_1 &&= \{\{x_4,x_5\},\{x_4\},\{x_5\}\}\\ &j=2,i=3 \quad F_3,\backslash F_2 &&= \{\{x_2\}\}\\ &j=2,i=4 \quad F_4,\backslash F_2 &&= \{\{x_2,x_5\},\{x_2\},\{x_5\}\}\\ &j=3,i=4 \quad F_4,\backslash F_3 &&= \{\{x_5\}\}.\end{aligned}$$

For each case where $F_i \backslash F_j$ has a single element, we choose x to be this element and $k=j$. For $F_4 \backslash F_1$, we can choose $x = x_4$ and $k = 2$. For $F_4 \backslash F_2$, we can choose $x = x_2$ and $k = 3$. We conclude that this is a shellable simplicial complex.

While each of the methods demonstrated in the previous example can be used to show shellability from the definition, it is sometimes easier to use a different method to show shellability. We now introduce an alternative method using commutative algebra for showing the shellability of a simplicial complex.

Theorem 3.4. *A simplicial complex Δ is shellable if and only if the Stanley-Reisner ideal $I_{\Delta^\vee}$ of the Alexander dual of Δ has linear quotients.*

Proof. We use the proof given by Herzog, Hibi, and Zheng [**7**] as an outline for the proof provided here.

First, suppose we have a simplicial complex Δ which is shellable, and let $F_1, F_2, \ldots, F_s$ be a shelling order of its facets. By Definition 3.1 (3), we have that for all i, j with $1 \leq j < i \leq s$, there exists some $x \in F_i \backslash F_j$ and some $1 \leq k < i$ with $F_i \backslash F_k = \{x\}$. By Corollary 2.15, we know there is a one-to-one correspondence between these facets and the monomial generators of $I_{\Delta^\vee}$. In particular, we have

$$I_{\Delta^\vee} = \langle m_1, m_2, \ldots, m_s \rangle,$$

where m_t is the monomial created by taking the product of the vertices in the complement of F_t.

Since for every $1 \leq j < i \leq s$ there is an $x \in F_i \backslash F_j$, we know that $x \nmid m_i$ and $x \mid m_j$. That is, we know that $\frac{m_j}{gcd(m_i, m_j)}$ will have x as a divisor. Further, since there is some $k < i$ with $F_i \backslash F_k = \{x\}$, we know that $x \nmid m_i$ and $x \mid m_k$, and in particular that this x is the only such element for which this holds. That is, we have that $\frac{m_k}{gcd(m_i, m_k)} = x$. Combining these two facts, we know that for each $\frac{m_j}{gcd(m_i, m_j)}$ with a factor of x, there is a $k < i$ such that $\frac{m_k}{gcd(m_i, m_k)} = x$

From Proposition 2.18, we can write

$$\langle m_1, m_2, \ldots, m_{i-1} \rangle : \langle m_i \rangle = \left\langle \frac{m_1}{gcd(m_i, m_1)}, \frac{m_2}{gcd(m_i, m_2)}, \ldots, \frac{m_{i-1}}{gcd(m_i, m_{i-1})} \right\rangle.$$

From above, we know that for each $1 \leq j < i$, $\frac{m_j}{gcd(m_i, m_j)}$ is either a variable or is divisible by some other $\frac{m_k}{gcd(m_i, m_k)}$ which is a variable, and so $\langle m_1, m_2, \ldots, m_{i-1} \rangle : \langle m_i \rangle$ is generated by a subset of variables. Hence, we have that given a shellable simplicial complex Δ, $I_{\Delta^\vee}$ has linear quotients.

Now suppose that given a simplicial complex Δ, the Stanley-Reisner ideal of its Alexander dual has linear quotients. Let $m_1, m_2, \ldots, m_s$ be an ordering of the generators of $I_{\Delta^\vee}$ which gives linear quotients. Then, for every $1 < i \leq s$, we have that $\langle m_1, m_2, \ldots, m_{i-1} \rangle : \langle m_i \rangle$ is generated by a subset of variables.

Again from Proposition 2.18, we have

$$\langle m_1, m_2, \ldots, m_{i-1} \rangle : \langle m_i \rangle = \left\langle \frac{m_1}{gcd(m_i, m_1)}, \frac{m_2}{gcd(m_i, m_2)}, \ldots, \frac{m_{i-1}}{gcd(m_i, m_{i-1})} \right\rangle.$$

Since this must be generated by a subset of variables, we know that for every $1 \leq j < i$, it must be that either $\frac{m_j}{gcd(m_i, m_j)}$ is a variable or has a variable factor x such that for some $1 \leq k < i$, $\frac{m_k}{gcd(m_i, m_k)} = x$. In particular, we have that there is some x such that $x \mid m_j$ and $x \nmid m_i$, and there is a k such that x is the only variable for which $x \mid m_k$ and $x \nmid m_i$ is true.

In terms of the facets of Δ, recall the one-to-one correspondence of monomial generators of $I_{\Delta^\vee}$ to facets of Δ given by Corollary 2.15. In particular, we can denote each of the s facets of Δ by $F_i = \{x_{j_1}, x_{j_2}, \ldots, x_{j_r}\}$, where $m_t = \frac{x_1 x_2 \cdots x_n}{x_{j_1} x_{j_2} \cdots x_{j_r}}$. Then for every $1 \leq i < j$, we have that there exists some vertex x such that $x \in F_i$ and $x \notin F_j$, i.e., $x \in F_i \backslash F_j$. Further, we know that for some such x, there is a k with $1 \leq k < i$ such that x is the only such vertex with $x \in F_i$ and $x \notin F_k$, i.e., $F_i \backslash F_k = \{x\}$. Now, for any $1 \leq j < i \leq s$, there is an $x \in F_i \backslash F_j$ and some $1 \leq k < i$ such that $F_i \backslash F_k = \{x\}$. But this then implies that $F_1, \ldots, F_s$ is a shelling order. That is, if $I_{\Delta^\vee}$ has linear quotients, then Δ is shellable.

From the above, we conclude that Δ is shellable if and only if $I_{\Delta^\vee}$ has linear quotients. □

EXAMPLE 3.5. Recall the simplicial complex from Example 3.3,

$$\Delta = \langle \{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_3, x_4\}, \{x_2, x_4, x_5\} \rangle.$$

We first find the Stanley-Reisner ideal of the Alexander dual of Δ. To do this, we take the complements of the facets and then create monomials by taking products of these complements. Doing this, we get

$$\begin{aligned}
\{x_1, x_2, x_3\} &\to \{x_4, x_5\} \to & x_4x_5 \\
\{x_1, x_2, x_4\} &\to \{x_3, x_5\} \to & x_3x_5 \\
\{x_1, x_3, x_4\} &\to \{x_2, x_5\} \to & x_2x_5 \\
\{x_2, x_4, x_5\} &\to \{x_1, x_3\} \to & x_1x_3,
\end{aligned}$$

and hence our ideal is given by $I_{\Delta^\vee} = \langle x_4x_5, x_3x_5, x_2x_5, x_1x_3 \rangle$. We will use the shelling order to get the linear quotient ordering. We propose the following ordering: $m_1 = x_1x_3$, $m_2 = x_3x_5$, $m_3 = x_2x_5$, $m_4 = x_4x_5$. To determine if $I_{\Delta^\vee}$ has linear quotients, we now check the following colon ideals:

$$\begin{aligned}
\langle x_1x_3 \rangle : \langle x_3x_5 \rangle &= \left\langle \frac{x_1x_3}{x_3} \right\rangle = \langle x_1 \rangle \\
\langle x_1x_3, x_3x_5 \rangle : \langle x_2x_5 \rangle &= \left\langle \frac{x_1x_3}{1}, \frac{x_3x_5}{x_5} \right\rangle = \langle x_3 \rangle \\
\langle x_1x_3, x_3x_5, x_2x_5 \rangle : \langle x_4x_5 \rangle &= \left\langle \frac{x_1x_3}{1}, \frac{x_3x_5}{x_5}, \frac{x_2x_5}{x_5} \right\rangle = \langle x_3, x_2 \rangle.
\end{aligned}$$

We can see that each colon ideal is generated by a subset of variables, and so $I_{\Delta^\vee}$ has linear quotients. We can use this to conclude that Δ is shellable.

CHAPTER 4

Shellability and the van der Waerden Complex

In this chapter, we define the van der Waerden complex, originally defined by Ehrenborg, Govindaiah, Park, and Readdy [**2**]. We also prove our main result in this chapter; in particular, we classify all the van der Waerden complexes that are shellable.

1. van der Waerden Complexes

Here we introduce the van der Waerden complex and give some examples.

DEFINITION 4.1. Let $0 < k < n$ and define the vertex set $\{x_1, x_2, \ldots, x_n\}$. The *van der Waerden complex* of dimension k on n vertices, denoted $vdW(n, k)$, is the pure simplicial complex on $\{x_1, \ldots, x_n\}$ whose facets are given by arithmetic progressions of the form $\{x_i, x_{i+d}, x_{i+2d}, \ldots, x_{i+kd}\}$ for $d \in \mathbb{Z}$ with $1 \le i < i + kd \le n$.

Given a facet $F = \{x_i, x_{i+d}, x_{i+2d}, \ldots, x_{i+kd}\}$, we call d the *increment* of F.

EXAMPLE 4.2. Let $n = 5$ and $k = 2$. The facets of $vdW(5, 2)$ are $\{x_1, x_2, x_3\}$, $\{x_2, x_3, x_4\}$, $\{x_3, x_4, x_5\}$, and $\{x_1, x_3, x_5\}$, and we have

$$vdW(5, 2) = \langle \{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_3, x_4, x_5\}, \{x_1, x_3, x_5\} \rangle.$$

This gives us the van der Warden complex on 5 vertices of dimension 2. In this example, we have three facets with an increment of 1, and one facet with an increment of 2. We can also represent this complex pictorially:

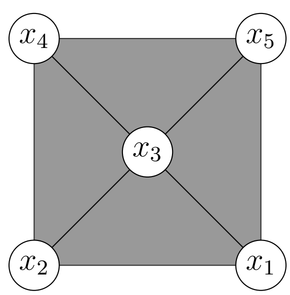

FIGURE 4.1. $vdW(5, 2)$

EXAMPLE 4.3. For $n = 8$, $k = 3$, we have

$$vdW(8, 3) = \left\langle \begin{array}{l} \{x_1, x_2, x_3, x_4\}, \{x_2, x_3, x_4, x_5\}, \{x_3, x_4, x_5, x_6\}, \{x_4, x_5, x_6, x_7\}, \\ \{x_5, x_6, x_7, x_8\}, \{x_1, x_3, x_5, x_7\}, \{x_2, x_4, x_6, x_8\} \end{array} \right\rangle.$$

This complex has five facets with an increment of 1, and two facets with an increment of 2. We can represent this complex pictorially, as shown below.

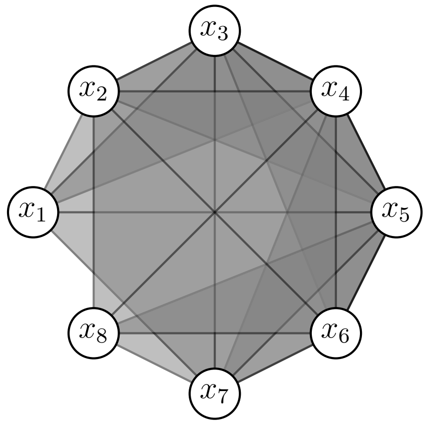

FIGURE 4.2. $vdW(8,3)$

We now describe a specific property about the facets of the van der Waerden complex. This property will be used in our main result.

LEMMA 4.4. *Let* $2 < k < \frac{n}{2}$ *and* $n > 6$*, and suppose* $F_1 = \{x_i, x_{i+d_1}, \ldots, x_{i+kd_1}\}$ *and* $F_2 = \{x_j, x_{j+d_2}, \ldots, x_{j+kd_2}\}$ *are facets of* $vdW(n,k)$ *such that* $d_1 > d_2$*. If there is an* a *with* $0 \le a \le k$ *such that* $x_{i+ad_1} \notin F_2$*, then* F_1 *and* F_2 *differ by at least two vertices.*

PROOF. Suppose by contradiction that all other vertices of F_1 are contained in F_2. We now consider the cases where $a = 0$ or $a = k$, $a = 1$, or $1 < a < k$.

CASE 1. First, suppose $a = 0$ or $a = k$. Notice that if $a = k$, we can relabel the vertices by $x_p = y_{n+1-p}$ and we will then be in the case of $a = 0$, so it is enough to consider the case $a = 0$. We have that $x_{i+d_1}, x_{i+2d_1}, x_{i+3d_1} \in F_2$. In particular, there exist integers $1 \le \alpha < \beta < \gamma \le k$ such that $x_{i+d_1} = x_{j+\alpha d_2}$, $x_{i+2d_1} = x_{j+\beta d_2}$, and $x_{i+3d_1} = x_{j+\gamma d_2}$. Then, we can calculate the following:

$$
\begin{array}{rrcl}
 & i + d_1 & = & j + \alpha d_2, \\
\text{so} & i + d_1 + d_1 & = & j + \alpha d_2 + d_1. \\
\text{But then} & i + 2d_1 & = & j + \alpha d_2 + d_1 = j + \beta d_2. \\
\text{This means} & \beta d_2 & = & \alpha d_2 + d_1. \text{ Rearranging gives} \\
 & \beta d_2 - \alpha d_2 & = & d_1 \\
 & d_2(\beta - \alpha) & = & d_1 \\
 & \beta - \alpha & = & \dfrac{d_1}{d_2}.
\end{array}
$$

Then, since $d_1 > d_2$, we have $\beta - \alpha = \frac{d_1}{d_2} > 1$, giving $\beta > \alpha + 1 > \alpha$.

Next, notice

$$\begin{aligned} i+d_1 &= j+\alpha d_2 \\ &< j+(\alpha+1)d_2 \\ &= j+\alpha d_2+d_2 \\ &= i+d_1+d_2 \\ &< i+2d_1, \end{aligned}$$

and so we have $j+(\alpha+1)d_2 \neq i+\epsilon d_1$ for any $\epsilon \in \mathbb{Z}$. Hence, $x_{j+(\alpha+1)d_2} \in F_1$ but $x_{j+(\alpha+1)d_2} \notin F_2$.

Similarly, we compute $\gamma - \beta = \frac{d_1}{d_2} > 1$, giving $\gamma > \beta + 1 > \beta$. Using this, we have

$$\begin{aligned} i+2d_1 &= j+\beta d_2 \\ &< j+(\beta+1)d_2 \\ &= j+\beta d_2+d_2 \\ &= i+2d_1+d_2 \\ &< i+3d_1, \end{aligned}$$

and so $x_{j+(\beta+1)d_2} \in F_1$ but $x_{j+(\beta+1)d_2} \notin F_2$. We now have two vertices of F_2 which do not appear in F_1.

Case 2. Suppose next that $a=1$. In this case, we have $x_i, x_{i+2d_1}, x_{i+3d_1} \in F_2$. Then there exist integers $1 \leq \alpha < \beta < \gamma \leq k$ such that $x_i = x_{j+\alpha d_2}$, $x_{i+2d_1} = x_{j+\beta d_2}$, and $x_{i+3d_1} = x_{j+\gamma d_2}$. Similarly to the previous case, we can compute the following:

$$\begin{aligned} i &= j+\alpha d_2, \\ \text{so} \qquad i+2d_1 &= j+\alpha d_2+2d_1 = j+\beta d_2. \\ \text{Then we have} \qquad \beta d_2 &= \alpha d_2+2d_1. \text{ Rearranging gives} \\ \beta d_2-\alpha d_2 &= 2d_1 \\ d_2(\beta-\alpha) &= 2d_1 \\ \beta-\alpha &= 2\frac{d_1}{d_2}. \end{aligned}$$

Since $d_1 > d_2$, we have $\beta - \alpha = 2\frac{d_1}{d_2} > 2$, and so $\beta > \alpha+2 > \alpha+1 > \alpha$.

We now consider the vertex $x_{j+(\alpha+1)d_2}$ of F_2. Notice

$$\begin{aligned} i &= j+\alpha d_2 \\ &< j+(\alpha+1)d_2 \\ &= j+\alpha d_2+d_2 \\ &= i+d_2 \\ &< i+2d_1. \end{aligned}$$

Since $i < i + d_2 < i + 2d_1$, it must be that $x_{j+(\alpha+1)d_2} \notin F_1$. We now have $x_{i+d_1} \in F_1 \backslash F_2$ and $x_{j+(\alpha+1)d_2} \in F_2 \backslash F_1$.

Next, similarly to before, we calculate the following:

$$\begin{array}{rrcl} & i + 2d_1 & = & j + \beta d_2, \\ \text{so} & i + 2d_1 + d_1 & = & j + \beta d_2 + d_1. \\ \text{But then} & i + 3d_1 & = & j + \beta d_2 + d_1 = j + \gamma d_2. \\ \text{Then we have} & \gamma d_2 & = & \beta d_2 + d_1. \text{ Rearranging gives} \\ & \gamma d_2 - \beta d_2 & = & d_1 \\ & d_2(\gamma - \beta) & = & d_1 \\ & \gamma - \beta & = & \dfrac{d_1}{d_2}. \end{array}$$

This gives that $\gamma - \beta = \frac{d_1}{d_2} > 1$, and hence $\gamma > \beta + 1 > \beta$. Then, we can easily see that the vertex $x_{j+(\beta+1)d_2} \in F_2$ is not in F_1 by checking the calculations below:

$$\begin{array}{rcl} i + 2d_1 & = & j + \beta d_2 \\ & < & j + (\beta + 1)d_2 \\ & = & j + \beta d_2 + d_2 \\ & = & i + 2d_1 + d_2 \\ & < & i + 3d_1. \end{array}$$

This implies that $x_{j+(\beta+1)d_2} \in F_2 \backslash F_1$. Since $|F_1| = |F_2|$, this means that F_1 and F_2 differ by at least two vertices.

Case 3. Finally, suppose that $1 < a < k$. Here, $x_{i+(a-2)d_1}, x_{i+(a-1)d_1}, x_{i+(a+1)d_1} \in F_2$. Then there exist integers $1 \leq \alpha < \beta < \gamma \leq k$ such that $x_{i+(a-2)d_1} = x_{j+\alpha d_2}$, $x_{i+(a-1)d_1} = x_{j+\beta d_2}$, and $x_{i+(a+1)d_1} = x_{j+\gamma d_2}$. With this, notice the following:

$$\begin{array}{rrcl} & i + (a-2)d_1 & = & j + \alpha d_2, \\ \text{so} & i + (a-2)d_1 + d_1 & = & j + \alpha d_2 + d_1. \\ \text{But then} & i + (a-1)d_1 & = & j + \alpha d_2 + d_1 = j + \beta d_2. \\ \text{This means} & \beta d_2 & = & \alpha d_2 + d_1. \text{ Rearranging gives} \\ & \beta d_2 - \alpha d_2 & = & d_1 \\ & d_2(\beta - \alpha) & = & d_1 \\ & \beta - \alpha & = & \dfrac{d_1}{d_2}. \end{array}$$

Recall that $d_1 > d_2$, and so $\beta - \alpha = \frac{d_1}{d_2} > 1$, hence $\beta > \alpha + 1 > \alpha$.

Now we have the vertex $x_{j+(\alpha+1)d_2} \in F_2$. On the other hand, we have

$$\begin{aligned} i+(a-2)d_1 &= j+\alpha d_2 \\ &< j+(\alpha+1)d_2 \\ &= j+\alpha d_2+d_2 \\ &= i+(a-2)d_1+d_2 \\ &< i+(a-1)d_1. \end{aligned}$$

Since $i+(a-2)d_1 < j+(\alpha+1)d_2 < i+(a-1)d_1$, it must be that $x_{j+(\alpha+1)d_2} \notin F_1$.

We now calculate the following:

$$\begin{aligned} && i+(a-1)d_1 &= j+\beta d_2, \\ \text{so} && i+(a-1)d_1+2d_1 &= j+\beta d_2+2d_1. \\ \text{But then} && i+(a+1)d_1 &= j+\beta d_2+2d_1 = j+\gamma d_2. \\ \text{Then we have} && \gamma d_2 &= \beta d_2+2d_1. \text{ Rearranging gives} \\ && \gamma d_2-\beta d_2 &= 2d_1 \\ && d_2(\gamma-\beta) &= 2d_1 \\ && \gamma-\beta &= 2\frac{d_1}{d_2}. \end{aligned}$$

This gives that $\gamma-\beta = 2\frac{d_1}{d_2} > 2$, and hence $\gamma > \beta+2 > \beta+1 > \beta$. Now we consider the vertex $x_{j+(\beta+1)d_2} \in F_2$. Since

$$\begin{aligned} i+(a-1)d_1 &= j+\beta d_2 \\ &< j+(\beta+1)d_2 \\ &= j+\beta d_2+d_2 \\ &= i+(a-1)d_1+d_2 \\ &< i+ad_1 \notin F_1, \end{aligned}$$

we have $x_{j+(\beta+1)d_2} \notin F_1$.

Combining the three cases above, we see that for $vdW(n,k)$ with $2 < k < \frac{n}{2}$, any two facets with different increments will differ by at least two vertices. □

The previous lemma covers many cases, but excludes any case with $k = 2$. Lemma 4.5 and Lemma 4.6 cover the remaining case.

Lemma 4.5. *Let $1 < k < \frac{n}{2}$ and suppose $F_1 = \{x_i, x_{i+d_1}, \ldots, x_{i+kd_1}\}$ and $F_2 = \{x_j, x_{j+d_2}, \ldots, x_{j+kd_2}\}$ are facets of $vdW(n,k)$ such that $d_1 > d_2$. If there is an a with $0 < a < k$ such that $x_{i+ad_1} \notin F_2$, then F_1 and F_2 differ by at least two vertices.*

Proof. Suppose for a contradiction that all other vertices of F_1 are contained in F_2. In particular, we have that $x_i, x_{i+kd_1} \in F_2$. Then for some integers $0 \le \alpha < \beta \le k$,

$x_i = x_{j+\alpha d_2}$ and $x_{i+kd_1} = x_{j+\beta d_2}$. This gives the following equalities: $i = j + \alpha d_2$ and $i + kd_1 = j + \beta d_2$. Using these, notice the following:

$$\begin{array}{rrcl} & i &=& j+\alpha d_2, \\ \text{so} & i+kd_1 &=& j+\alpha d_2+kd_1 = j+\beta d_2. \\ \text{Then we have} & \beta d_2 &=& \alpha d_2 + kd_1.\text{ Rearranging gives} \\ & \beta d_2 - \alpha d_2 &=& kd_1 \\ & d_2(\beta-\alpha) &=& kd_1 \\ & \beta-\alpha &=& k\dfrac{d_1}{d_2}. \end{array}$$

Since $d_1 > d_2$, we have $\frac{d_1}{d_2} > 1$, and hence $k\frac{d_1}{d_2} > k$. Also notice that since $0 \le \alpha < \beta \le k$, we have $\beta - \alpha \le k$. But since $\beta - \alpha = k\frac{d_1}{d_2}$, this implies $k < k\frac{d_1}{d_2} \le k$, a contradiction. We conclude that either x_i or x_{i+kd_1} does not appear in F_2, and so F_1 and F_2 differ by at least two vertices. $\square$

LEMMA 4.6. *Let $1 < k < \frac{n}{2}$ and suppose $F_1 = \{x_i, x_{i+d_1}, \ldots, x_{i+kd_1}\}$ and $F_2 = \{x_j, x_{j+d_2}, \ldots, x_{j+kd_2}\}$ are facets of $vdW(n,k)$ such that d_1 is the largest odd increment and $d_2 \ne d_1$. If $x_{i+ad_1} \notin F_2$ for either $a = 0$ or $a = k$, then F_1 and F_2 differ by at least two vertices.*

PROOF. First notice that if $a = k$, we can relabel the vertices by $x_p = y_{n+1-p}$ and will then be in the case of $a = 0$. Hence, we consider only the case $a = 0$.

Suppose $a = 0$. Then we have that $x_{i+d_1}, x_{i+2d_1} \in F_2$. That is, $i + d_1 = j + \alpha d_2$ and $i + 2d_1 = j + \beta d_2$ for some integers $0 \le \alpha < \beta \le k$. Using this, we can see the following:

$$\begin{array}{rrcl} & i+d_1 &=& j+\alpha d_2, \\ \text{so} & i+d_1+d_1 &=& j+\alpha d_2+d_1. \\ \text{But then} & i+2d_1 &=& j+\alpha d_2+d_1 = j+\beta d_2. \\ \text{This means} & \beta d_2 &=& \alpha d_2 + d_1.\text{ Rearranging gives} \\ & \beta d_2 - \alpha d_2 &=& d_1 \\ & d_2(\beta-\alpha) &=& d_1 \\ & \beta-\alpha &=& \dfrac{d_1}{d_2}. \end{array}$$

Notice that since d_1 is the largest odd increment, we have that either $d_1 > d_2$ or $d_2 = d_1+1$. But since $0 \le \alpha < \beta \le k$, we have $\beta - \alpha \ge 1$, and so $\frac{d_1}{d_2} \ge 1$, i.e., $d_2 \ne d_1 + 1$. Then $d_1 > d_2$, and so $\beta - \alpha = \frac{d_1}{d_2} \ge 2$. Further, since d_1 is odd, $\frac{d_1}{d_2} \ne 2c$ for any $c \in \mathbb{Z}$, and so $\beta - \alpha = \frac{d_1}{d_2} \ge 3$.

We now have that $\beta - \alpha \geq 3$, and hence $\alpha < \alpha + 1 < \alpha + 2 < \beta$. We now consider the vertices $x_{j+(\alpha+1)d_2}$ and $x_{j+(\alpha+2)d_2}$ of F_2. Notice that

$$\begin{aligned} j + (\alpha+1)d_2 &= j + \alpha d_2 + d_2 \\ &= i + d_1 + d_2 \\ &= i + d_1 + \frac{d_1}{\beta - \alpha} \\ &= i + (1 + \frac{1}{\beta - \alpha})d_1, \end{aligned}$$

and

$$\begin{aligned} j + (\alpha+2)d_2 &= j + \alpha d_2 + 2d_2 \\ &= i + d_1 + 2d_2 \\ &= i + d_1 + 2\frac{d_1}{\beta - \alpha} \\ &= i + (1 + \frac{2}{\beta - \alpha})d_1. \end{aligned}$$

Since $\beta - \alpha \geq 3$, we have that $\frac{1}{\beta-\alpha}, \frac{2}{\beta-\alpha} \notin \mathbb{Z}$, and so $x_{j+(\alpha+1)d_2}$ and $x_{j+(\alpha+2)d_2}$ are vertices of F_2 which do not appear in F_1. Hence, if the first vertex of F_1 is omitted from F_2, then F_1 and F_2 differ by at least two vertices. $\square$

Corollary 4.7. *Let $1 < k < \frac{n}{2}$ and $n > 6$, and suppose F_1 and F_2 are facets of $vdW(n, k)$ such that F_1 has the largest odd increment and F_2 has any other increment. Then F_1 and F_2 differ by at least two vertices.*

Proof. Notice that if the largest odd increment is 3 or more, the result follows easily from Lemmas 4.5 and 4.6. Further, notice that for $n > 6$, $vdW(n, 2)$ will always have a facet with an increment of 3, since $1 + 3 \cdot 2 = 7$. Hence, if there is no facet with increment 3, we have $k > 2$. Then the largest odd increment will be 1. In this case, Lemma 4.4 gives that any two facets with different increments will differ by at least two vertices; and in particular, that a facet with the largest odd increment will differ by at least two vertices from a facet with any other increment. $\square$

2. Shellability

We now classify the van der Waerden complexes that are shellable. We break our proof into a number of steps.

Theorem 4.8. *For all integers $n \geq 1$, the van der Waerden complex $vdW(n, k)$ is shellable if $k = 1$ or $\frac{n}{2} \leq k < n$.*

Proof. First we consider the case $k = 1$. By the definition of the van der Waerden complex, we have that

$$vdW(n, 1) = \langle \{x_i, x_j\} | 1 \leq i < j \leq n \rangle.$$

We now want to find a shelling of these facets. Consider the ordering of the facets defined by

$$\begin{array}{rclrclcl} F_1 &=& \{x_1,x_2\} & F_n &=& \{x_2,x_3\} & \dots & F_{\frac{n(n+1)}{2}} = \{x_{n-1},x_n\}. \\ F_2 &=& \{x_1,x_3\} & F_{n+1} &=& \{x_2,x_4\} & & \\ F_3 &=& \{x_1,x_4\} & F_{n+2} &=& \{x_2,x_5\} & & \\ & \vdots & & & \vdots & & & \\ F_{n-3} &=& \{x_1,x_{n-2}\} & F_{2n-4} &=& \{x_2,x_{n-1}\} & & \\ F_{n-2} &=& \{x_1,x_{n-1}\} & F_{2n-3} &=& \{x_2,x_n\} & & \\ F_{n-1} &=& \{x_1,x_n\} & & & & & \end{array}$$

For each $1 < j < n$, we have $\langle F_j\rangle \cap \langle F_1, F_2, \dots, F_{j-1}\rangle = \langle\{x_1\}\rangle$. For each $n \le j \le \frac{n(n+1)}{2}$ with $F_j = \{x_p, x_q\}$, we have $\langle F_j\rangle \cap \langle F_1, F_2, \dots, F_{j-1}\rangle = \langle\{x_p\},\{x_q\}\rangle$. Further, notice $\langle\{x_1\}\rangle = \{\{x_1\},\varnothing\}$ and $\langle\{x_p\},\{x_q\}\rangle = \{\{x_p\},\{x_q\},\varnothing\}$. Since $\dim(vdW(n,1)) = 1$, any $\{x_i\}$ is a maximal proper subface. Hence, we have that $\langle F_j\rangle \cap \langle F_1, F_2, \dots, F_{j-1}\rangle$ is generated by a set of maximal proper subfaces of F_j. Hence, by Definition 3.1 (1), this ordering is a shelling. Thus $vdW(n,1)$ is shellable.

Now suppose $\frac{n}{2} \le k < n$. Since $k \ge \frac{n}{2}$, notice that for $d > 1$, we have

$$1 + dk \ge 1 + 2\frac{n}{2} = 1 + n > n$$

and hence we can have no facet with an increment greater than 1. Hence, we have the facets

$$\begin{array}{rcl} F_1 &=& \{x_1, x_2, x_3, \dots, x_{k-1}, x_k, x_{k+1}\} \\ F_2 &=& \{x_2, x_3, x_4, \dots, x_k, x_{k+1}, x_{k+2}\} \\ F_3 &=& \{x_3, x_4, x_5, \dots, x_{k+1}, x_{k+2}, x_{k+3}\} \\ & \vdots & \\ F_{n-k-2} &=& \{x_{n-k-2}, x_{n-k-1}, x_k, \dots, x_{n-4}, x_{n-3}, x_{n-2}\} \\ F_{n-k-1} &=& \{x_{n-k-1}, x_{n-k}, x_{n-k+1}, \dots, x_{n-3}, x_{n-2}, x_{n-1}\} \\ F_{n-k} &=& \{x_{n-k}, x_{n-k+1}, x_{n-k+2}, \dots, x_{n-2}, x_{n-1}, x_n\}. \end{array}$$

Using this ordering, notice that

$$\begin{array}{rcl} \langle F_2\rangle \cap \langle F_1\rangle &=& \langle\{x_2, x_3, \dots, x_k, x_{k+1}\}\rangle \\ \langle F_3\rangle \cap \langle F_1, F_2\rangle &=& \langle\{x_3, x_4, \dots, x_k, x_{k+2}\}\rangle \\ \langle F_4\rangle \cap \langle F_1, F_2, F_3\rangle &=& \langle\{x_4, x_5, \dots, x_k, x_{k+3}\}\rangle \\ & \vdots & \\ \langle F_{n-k-2}\rangle \cap \langle F_1, F_2, \dots, F_{n-k-3}\rangle &=& \langle\{x_{n-k-2}, x_{n-k-1}, \dots, x_{n-4}, x_{n-3}\}\rangle \\ \langle F_{n-k-1}\rangle \cap \langle F_1, F_2, \dots, F_{n-k-2}\rangle &=& \langle\{x_{n-k-1}, x_{n-k}, \dots, x_{n-3}, x_{n-2}\}\rangle \\ \langle F_{n-k}\rangle \cap \langle F_1, F_2, \dots, F_{n-k-1}\rangle &=& \langle\{x_{n-k}, x_{n-k+1}, \dots, x_{n-2}, x_{n-1}\}\rangle. \end{array}$$

Definition 3.1 (1), this ordering is a shelling and so $vdW(n,k)$ is shellable for $\frac{n}{2} \le k < n$.

□

While the previous theorem covers most of the cases for which the van der Waerden complex is shellable, there are two special cases. We consider them now.

PROPOSITION 4.9. *For $n = 5, 6$, $vdW(n, 2)$ is shellable.*

PROOF. To show $vdW(5, 2)$ and $vdW(6, 2)$ are shellable, we will show that the Alexander duals of their Stanley-Reisner ideals have linear quotients. Using the definition of the van der Waerden complex, we have

$$\begin{aligned} vdW(5,2) &= \langle\{x_1,x_2,x_3\},\{x_2,x_3,x_4\},\{x_3,x_4,x_5\},\{x_1,x_3,x_5\}\rangle, \text{ and} \\ vdW(6,2) &= \langle\{x_1,x_2,x_3\},\{x_2,x_3,x_4\},\{x_3,x_4,x_5\},\{x_4,x_5,x_6\},\{x_1,x_3,x_5\},\{x_2,x_4,x_6\}\rangle. \end{aligned}$$

We label the facets of $vdW(5, 2)$ as F_1 through F_4, and the facets of $vdW(6, 2)$ as G_1 through G_6. Using the order given above, notice

$$\begin{aligned} \langle F_2\rangle \cap \langle F_1\rangle &= \langle\{x_2,x_3\}\rangle \\ \langle F_3\rangle \cap \langle F_1,F_2\rangle &= \langle\{x_3,x_4\}\rangle \\ \langle F_4\rangle \cap \langle F_1,F_2,F_3\rangle &= \langle\{x_1,x_3\},\{x_3,x_5\}\rangle, \end{aligned}$$

and

$$\begin{aligned} \langle G_2\rangle \cap \langle G_1\rangle &= \langle\{x_2,x_3\}\rangle \\ \langle G_3\rangle \cap \langle G_1,G_2\rangle &= \langle\{x_3,x_4\}\rangle \\ \langle G_4\rangle \cap \langle G_1,G_2,G_3\rangle &= \langle\{x_4,x_5\}\rangle \\ \langle G_5\rangle \cap \langle G_1,G_2,G_3,G_4\rangle &= \langle\{x_1,x_3\},\{x_3,x_5\}\rangle \\ \langle G_6\rangle \cap \langle G_1,G_2,G_3,G_4,G_5\rangle &= \langle\{x_2,x_4\},\{x_4,x_6\}\rangle. \end{aligned}$$

Since $\dim(vdW(5, 2)) = \dim(vdW(6, 2)) = 2$, maximal proper subfaces have dimension 1. Since each intersection is generated by a set of faces with dimension 1, we have that $vdW(5, 2)$ and $vdW(6, 2)$ are shellable by Definition 3.1 (1). □

COROLLARY 4.10. *Let $1 \le k < n \le 6$. Then the simplicial complex $vdW(n, k)$ is shellable.*

PROOF. This result follows easily from Theorem 4.8 and Proposition 4.9. For $n < 5$, Theorem 4.8 covers $vdW(n, k)$ for all k. For $n = 5, 6$, Theorem 4.8 gives that $vdW(n, k)$ is shellable for all k except $k = 2$; this final case is then taken care of by Proposition 4.9. □

Since we have thoroughly examined those van der Waerden complexes which are shellable, we now look to those which are not.

THEOREM 4.11. *Let $n > 6$ and $1 < k < \frac{n}{2}$. Then $vdW(n, k)$ is not shellable.*

Proof. Recall that the facets of $vdW(n,k)$ are given by arithmetic progressions of the form $x_i, x_{i+d}, x_{i+2d}, \ldots, x_{i+kd}$. When $n > 6$ and $1 < k < \frac{n}{2}$, we have

$$1 + 2k < 1 + 2\frac{n}{2} \leq 1 + 2\frac{7}{2} = 1 + 7 = 8,$$

i.e., $1 + 2k \leq 7$, and so we will always have facets with increments at least 1 and 2. From Corollary 4.7, we know that given any two facets where one has the largest odd increment and one has any other increment, then the two facets will differ by at least two vertices. Suppose the vertices omitted from F_1 and F_2 are $\{x_p, x_q\}$ and $\{x_r, x_t\}$, respectively. Recall from Corollary 2.15 that the Stanley-Reisner ideal of the Alexander dual is generated by monomials related to these facets by $m_i = \prod_{x \notin F_i} x$. Then the variables corresponding to these vertices will not appear in the monomials corresponding to the opposite facets, i.e., $x_p x_q \nmid m_2$ but $x_p x_q \mid m_1$, and $x_r x_t \nmid m_1$ but $x_r x_t \mid m_2$. As a result, we have that the monomial generators of $I_{vdW(n,k)^\vee}$ which correspond to these facets will also differ by at least two variables.

Using this fact, we now look at the Stanley Reisner ideal of the Alexander dual, $I_{vdW(n,k)^\vee}$. Suppose $I_{vdW(n,k)^\vee} = \langle m_1, m_2, \ldots, m_s \rangle$, where $m_i = \prod_{x \notin F_i} x$ as described above. Now consider any ordering of the monomials, say $m_{j_1}, m_{j_2}, \ldots, m_{j_s}$, and suppose m_{j_p} is the first monomial of the ordering which corresponds to a facet F_{j_p} with the largest odd increment.

First, suppose that $j_p > 1$. Recall from Proposition 2.18 that

$$\langle m_{j_1}, m_{j_2}, \ldots, m_{j_{p-1}} \rangle : \langle m_{j_p} \rangle = \left\langle \frac{m_{j_1}}{gcd(m_{j_p}, m_{j_1})}, \frac{m_{j_2}}{gcd(m_{j_p}, m_{j_2})}, \ldots, \frac{m_{j_{p-1}}}{gcd(m_{j_p}, m_{j_{p-1}})} \right\rangle.$$

Since m_{j_p} is the first occurrence of a monomial corresponding to a facet with the largest odd increment, we know that it differs from each of the preceding monomials by at least two variables. That is, for all $1 \leq i < j_p$, we have that $\frac{m_i}{gcd(m_{j_p}, m_i)}$ is a product of at least two variables. As a result, the colon ideal $\langle m_{j_1}, m_{j_2}, \ldots, m_{j_{p-1}} \rangle : \langle m_{j_p} \rangle$ cannot be generated by a set of variables. Hence, if we have linear quotients, an ordering must begin with such a monomial.

Now suppose $j_p = 1$. That is, the first monomial of the ordering corresponds to a facet with the largest odd increment. Suppose that m_{j_q} is the first monomial which corresponds to a facet with any other increment. In this case, for the same reasons as before, we can see that $\langle m_{j_1}, m_{j_2}, \ldots, m_{j_{q-1}} \rangle : \langle m_{j_q} \rangle$ cannot be generated by a set of variables, and hence will also not give linear quotients.

From above, we see that no ordering which includes a monomial corresponding to the facet with largest odd increment can give linear quotients. Hence $I_{vdW(n,k)^\vee}$ does not have linear quotients for $n > 6$ and $1 < k < \frac{n}{2}$. As a result, we have that $vdW(n,k)$ is not shellable. $\square$

The main theorem follows immediately from Theorems 4.8 and 4.11.

THEOREM 4.12. *Let* $0 < k < n$ *be integers and consider the van der Waerden complex* $vdW(n,k)$ *of dimension* k *on* n *vertices. Then* $vdW(n,k)$ *is shellable if and only if*

- $n \leq 6$, *or*
- $n > 6$ *and* $k = 1$, *or*
- $n > 6$ *and* $\frac{n}{2} \leq k < n$.

We now have a condition which is both necessary and sufficient for the shellability of the van der Waerden complex given any values of n and k, answering Question 1.5 stated in the introduction.

CHAPTER 5

Conclusion

Combining all of the work done through previous chapters, we are finally able to reach a necessary and sufficient condition for the shellability of the van der Waerden complex. In this chapter, we state the main theorem and highlight an important implication it gives as well as some related open questions.

The main theorem of this project is given by combining the results of Theorem 4.8 and Theorem 4.11. We state the theorem again below.

THEOREM 4.12. *Let $0 < k < n$ be integers and consider the van der Waerden complex $vdW(n,k)$ of dimension k on n vertices. Then $vdW(n,k)$ is shellable if and only if*

- $n \leq 6$, *or*
- $n > 6$ *and* $k = 1$, *or*
- $n > 6$ *and* $\frac{n}{2} \leq k < n$.

As the concept of the van der Waerden complex is still very new, there are several different areas in which this research could be expanded. One idea, as mentioned in the introduction as motivation for this research, is to address the implications this result has in terms of these simplicial complexes being Cohen-Macaulay. Vander Meulen, Van Tuyl, and Watt [**9**] studied circulant graphs and determining when they are Cohen-Macaulay using simplicial complexes. Using this approach to study van der Waerden complexes could lead to useful conclusions.

An additional topic of interest, which is related to the concept of shellability, is that of simplicial complexes which are k-shellable. Some simplicial complexes which are not shellable can still have orderings of their facets which are "nice". Rahmati-Asghar [**8**] defined such complexes as follows.

DEFINITION 5.1. Let Δ be a d-dimensional pure simplicial complex on $\{x_1, \dots, x_n\}$, and let $k \in \mathbb{Z}$ with $1 \leq k \leq d+1$. An ordering $F_1, F_2, \dots, F_s$ of the facets of Δ is called a *k-shelling* if for each integer $1 < j \leq s$, $\mathcal{F}_j = \langle F_j \rangle \cap \langle F_1, F_2, \dots, F_{j-1} \rangle$ satisfies the following two conditions:

(1) $\mathcal{F}_j$ is generated by a nonempty set of maximal proper subfaces of F_j, i.e., subfaces of dimension $|F_j| - k - 1 = \dim(F_j) - k$, and
(2) if $\mathcal{F}_j$ has more than one facet then for every two disjoint facets $G, \tilde{G} \in \mathcal{F}_j$, $F_j \subseteq G \cup \tilde{G}$.

A simplicial complex with a k-shelling is called *k-shellable*.

Notice that if a simplicial complex is shellable, it is also 1-shellable, and so this notion is a relaxation of the concept of shellability.

EXAMPLE 5.2. Recall the simplicial complex given in Example 1.1,

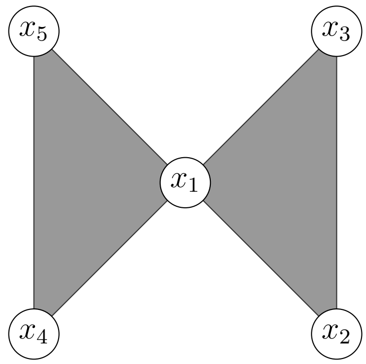

FIGURE 5.1. Simplicial complex $\Delta = \langle \{x_1, x_2, x_3\}, \{x_1, x_4, x_5\} \rangle$

Let $F_1 = \{x_1, x_2, x_3\}$, $F_2 = \{x_1, x_4, x_5\}$. Then, we have $\mathcal{F}_2 = \langle \{x_1\} \rangle$. This is not a shelling, since $\{x_1\}$ is not a maximal proper subface of F_2. However, it is a 2-shelling since $\{x_1\} \subset F_2$ has dimension $\dim(\{x_1\}) = \dim(F_2) - 2$. Hence, this simplicial complex is 2-shellable.

Exploring the k-shellability of the van der Waerden complexes which are not shellable could be an interesting endeavor. Given the results of this research, we know which van der Waerden complexes are 1-shellable. It could be useful to determine the k-shellability of these complexes for varying values of k. We speculate that no van der Waerden complexes are k-shellable, outside of those which have been identified here as having 1-shellings.

CHAPTER 6

Appendix

The table below indicates whether the van der Waerden complex $vdW(n,k)$ is shellable for varying values of n and k. A ✓ indicates that the complex with a n value corresponding to its row and k to its column is shellable, and a ✗ indicates that this complex is not shellable. Entries are shaded in grey if $vdW(n,k)$ does not exist for the given values. This table was filled in as work on this project progressed to help with finding a pattern.

$n \backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	✓															
3	✓	✓														
4	✓	✓	✓													
5	✓	✓	✓	✓												
6	✓	✓	✓	✓	✓											
7	✓	✗	✗	✓	✓	✓										
8	✓	✗	✗	✓	✓	✓	✓									
9	✓	✗	✗	✗	✓	✓	✓	✓								
10	✓	✗	✗	✗	✓	✓	✓	✓	✓							
11	✓	✗	✗	✗	✗	✓	✓	✓	✓	✓						
12	✓	✗	✗	✗	✗	✓	✓	✓	✓	✓	✓					
13	✓	✗	✗	✗	✗	✗	✓	✓	✓	✓	✓	✓				
14	✓	✗	✗	✗	✗	✗	✓	✓	✓	✓	✓	✓	✓			
15	✓	✗	✗	✗	✗	✗	✗	✓	✓	✓	✓	✓	✓	✓		
16	✓	✗	✗	✗	✗	✗	✗	✓	✓	✓	✓	✓	✓	✓	✓	
17	✓	✗	✗	✗	✗	✗	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓

TABLE 1. A table showing which of the van der Waerden complexes are shellable

Code used to generate van der Waerden complexes in Macaulay2 [**5**]. Generating these complexes using software was helpful for quickly determining Alexander duals, Stanley-Reisner ideals, and shellability of certain complexes. This was especially helpful when looking to confirm or reject a hypothesis of which complexes are shellable.

```
VDW = (n,k) -> (
    l = {};
    for i from 1 to n do (
        for d from 1 to floor((n-1)/k) do (
            if (i+d*k <=n) then (
                sublist = {};
                    for j from 0 to k do (
                        sublist = append(sublist,x_(i+d*j))
                    );
                l = append(l, product sublist));
        );
    );
    return simplicialComplex l;
);
```

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