Sets of Points in Multi-Projective Spaces and their Hilbert Function

by

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To my Parents,

who taught me about $\mathbb{N}$.

To my teachers,

who taught me about $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{F}_q$, $\mathbb{Z}/m\mathbb{Z}, \ldots$

To Catherine,

for her love.

The fear of the Lord is the beginning of knowledge

Proverbs 7:1 (NIV)

Abstract

In this thesis we study the Hilbert functions of sets of distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with $k \geq 2$. This thesis extends the work of Giuffrida, Maggioni, and Ragusa (1992) on the Hilbert functions of points in $\mathbb{P}^1 \times \mathbb{P}^1$. The goal of this thesis is to establish the algebraic foundation for this topic. The main results of this thesis are:

(1) We describe the eventual behaviour of the Hilbert function of a set of distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. As a consequence of this result, we show that the Hilbert function of a set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ can be determined by computing the Hilbert function at only a finite number of values. The other values of the Hilbert function will then follow from our description of the eventual behaviour of the Hilbert function. The values at which we need to compute the Hilbert function can be determined from numerical information about the set. Our result motivates us to define the *border* of the Hilbert function of a set of points. This result extends the result that the Hilbert function of a set of points in $\mathbb{P}^n$ stabilizes at the cardinality of the set of points.

(2) We show that H is the Hilbert function of an arithmetically Cohen-Macaulay (ACM) set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ if and only if ΔH, the first difference function of H, is the Hilbert function of an $\mathbb{N}^k$-graded artinian quotient of a polynomial ring. This result generalizes a theorem of Geramita, Maroscia, and Roberts (1983) about points in $\mathbb{P}^n$.

(3) We introduce a new necessary condition on the Hilbert function of a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ by uncovering a link between sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ and $(0,1)$-matrices. By using the Gale-Ryser Theorem, a classical result about $(0,1)$-matrices, we can characterize *all* borders of points in $\mathbb{P}^1 \times \mathbb{P}^1$. We also give a new characterization of the ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ which depends only upon numerical information

describing the set of points. The ACM sets of points were first characterized by Giuffrida, Maggioni, and Ragusa (1992) via different methods.

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Contents

CHAPTER 1

Introduction

The Point is a Being like ourselves, but confined to the non-dimensional Gulf. He is himself his own World, his own Universe; of any other than himself he can form no conception; he knows not Length, nor Breadth, nor Height, for he has no experience of them; he has no cognizance even of the number Two; nor has he a thought of Plurality; for he is himself his One and All, being really Nothing.

– *The Sphere in* Flatland

1. Motivation and Overview

Contrary to the above quotation from Edwin Abbott's novella *Flatland* [**1**], a point in $\mathbb{P}^n = \mathbb{P}^n_{\mathbf{k}}$, the n-dimensional projective space over the field $\mathbf{k}$, or more generally, a set of points in $\mathbb{P}^n$ is anything but "Nothing." Indeed, to provide a complete listing of the literature devoted to the study of sets of points in $\mathbb{P}^n$ would prove to be a Herculean task. One can, however, consult the conference proceedings [**41**] [**20**], especially the survey article by Geramita [**15**], for motivation and for a flavour of the past and present research about points in $\mathbb{P}^n$. The lecture notes of Geramita [**14**] and Robbiano [**44**] provide a gentle introduction to the topic of points. Even though this field has a long and deep history, many fascinating problems remain.

The *Hilbert function* of a set of points in $\mathbb{P}^n$ is the basis for many questions about sets of points. To any set of points, we can associate an algebraic object which we call the *coordinate ring.* The Hilbert function is used to obtain, among other things, algebraic information about the coordinate ring and geometric information about the set of points. The papers [**16**], [**17**], [**19**], [**34**], [**36**], [**37**], and [**49**] are just a partial list of the papers that study the connection between a set of points and its Hilbert function. As a tool for studying sets of points, the Hilbert function is extremely useful due, in part, to a result of

Geramita, Maroscia, and Roberts [**19**] which gives a precise description of which functions can be the Hilbert function of a set of points in $\mathbb{P}^n$.

The goal of this thesis is to study sets of points in a more general ambient setting. Specifically, we wish to extend the study of collections of points in projective space to collections of points in the multi-projective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. This is an area, to our knowledge, that has seen little exploration. The first foray into this territory, that we are aware of in modern times, appears to be a series of papers, authored by Giuffrida, Maggioni, and Ragusa ([**24**],[**25**],[**26**]), on points that lie on the quadric surface $\mathcal{Q} \subseteq \mathbb{P}^3$. Because $\mathcal{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$, some of the results of Giuffrida, *et al.* can be translated into results about points in multi-projective space. However, there seems to be more questions about sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ than there are answers.

To narrow the scope of this thesis, we will focus primarily on the Hilbert functions of sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Because the characterization of Hilbert functions of points in $\mathbb{P}^n$ due to Geramita, *et al.* [**19**] plays such an important rôle in the study of those sets, a generalization of this characterization should be a primary objective. In fact, this problem is the underlying question that guides this thesis. We state this question formally:

Question 1.1.1. *What can be the Hilbert function of a set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$?*

If $k = 1$, then, as already noted, a solution exists. If $k \geq 2$, then the problem remains open. This thesis should be viewed as one attack (of hopefully many) on Question 1.1.1. Although we were not successful in providing a complete solution, we have made some progress. Some of our successes are detailed in the later sections of this chapter.

There are many reasons to study sets of point in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and their Hilbert functions. We give two such reasons. First, the value of the Hilbert function at certain sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ has shown up in connection with other problems. For example, Catalisano, Geramita, and Gimigliano [**10**] have recently shown that a specific value of the Hilbert function of a collection of *fat points* in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is related to a classical problem of algebraic geometry concerning the dimension of certain secant varieties of Segre varieties. Catalisano, *et al.* were able to compute the desired value for only *some* sets of points. A complete understanding of the Hilbert function of a set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ might provide us with some understanding about the Hilbert functions of fat points, and thus, provide us with a complete solution to this problem.

A second motivation for studying the Hilbert function of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is to provide a series of examples of Hilbert functions for multi-graded rings. Multi-graded rings appear throughout algebraic geometry and commutative algebra. Two examples of a multi-graded ring are: (1) the coordinate ring of a blow-up, and (2) a Rees Algebra (see [**11**], [**27**], [**30**], [**50**], [**55**] for these examples and more). However, we are still only beginning to understand the structure of multi-graded rings. As a consequence of this fact, there are many open problems concerning the Hilbert functions of multi-graded rings. Some results concerning the Hilbert function of multi-graded rings have been established, as is evident in [**2**], [**5**], [**7**], [**33**], [**45**], [**52**], [**53**], [**54**]. However, the question of what functions can be the Hilbert function of a multi-graded ring remains an open problem, except in the case of standard graded rings. For the case of standard graded rings, i.e., rings graded in the usual sense, then we have *Macaulay's Theorem* [**35**] which characterizes all functions that can be the Hilbert function of a finitely generated graded $\mathbf{k}$-algebra. By studying the Hilbert functions of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ we can perhaps get an insight into a multi-graded version of Macaulay's Theorem. At the very least, such a study provides a nice stable of examples.

This thesis is divided into five chapters and one appendix. In Chapter 1, we summarize the main results of this work. We will emphasize where we have been successful in answering Question 1.1.1, the underlying question of this thesis. We will also give a series of open problems. These unanswered question provide ample motivation for future work on points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

In Chapter 2, we build the mathematical framework for the thesis. The topics introduced in this chapter are: multi-graded rings, Hilbert functions, points in $\mathbb{P}^n$ and $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, resolutions and projective dimension, and the combinatorics of $(0,1)$-matrices. With the exception of the material on points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, the contents of this chapter are well known. However, for the convenience of the reader, we have attempted to include as many of the proofs as possible.

Our primary goal in Chapter 3 is to generalize a classical result about the eventual behaviour of the Hilbert function of a set of points in $\mathbb{P}^n$ to sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

Our result will motivate us to define the *border* of a Hilbert function of a set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. The border divides the values of the Hilbert function into two sets: those values which need to be computed and those values which rely on our result describing the eventual behaviour of the Hilbert function. We also show how the notion of *points in generic position* generalizes to sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

In Chapter 4 we explore *arithmetically Cohen-Macaulay* sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. One of the striking differences between sets of points in $\mathbb{P}^n$ and $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with $k > 1$ is that the former are *always* arithmetically Cohen-Macaulay, while the latter can fail to have this property. We show that if we restrict to arithmetically Cohen-Macaulay sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, then the characterization of Geramita, *et al.* [**19**] given for the Hilbert functions of points in $\mathbb{P}^n$ can be generalized. We also characterize the Hilbert functions of all bigraded quotients of $\mathbf{k}[x_1, y_1, \ldots, y_m]$ and $\mathbb{N}^k$-graded quotients of $\mathbf{k}[x_1, \ldots, x_k]$. As a consequence, we can completely describe the Hilbert functions of arithmetically Cohen-Macaulay sets of points in $\mathbb{P}^1 \times \mathbb{P}^m$ and $\underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{k}$ for any k. Our results are a generalization of a result about points in $\mathbb{P}^1 \times \mathbb{P}^1$ due to Giuffrida, *et al.* [**26**].

In the final chapter, Chapter 5, we continue the program first begun by Giuffrida, *et al.* ([**24**], [**25**],[**26**]), by restricting our focus to points in $\mathbb{P}^1 \times \mathbb{P}^1$. If $\mathbb{X}$ is a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$, then we show that the border of the Hilbert function of $\mathbb{X}$ depends only upon the combinatorics of $\mathbb{X}$. Moreover, we characterize all possible borders by uncovering a link between sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ and $(0,1)$-matrices. As a consequence, we give a new necessary condition on the Hilbert function of points in $\mathbb{P}^1 \times \mathbb{P}^1$. We also give a combinatorial characterization of arithmetically Cohen-Macaulay points in $\mathbb{P}^1 \times \mathbb{P}^1$. This characterization is a new characterization of arithmetically Cohen-Macaulay sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$. A non-combinatorial characterization of arithmetically Cohen-Macaulay sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ is originally due to Giuffrida, *et al.* [**26**].

Many of the results in this thesis have their genesis in examples. Instrumental in generating these examples was the computational commutative algebra program `CoCoA` [**8**]. In Appendix A the code used to compute the Hilbert function of points in $\mathbb{P}^n \times \mathbb{P}^m$ is provided. We also explain the mathematical underpinnings of the code and give some examples of its use.

The following notation will be used for the remainder of this chapter. We always use $\mathbf{k}$ to denote an algebraically closed field of characteristic zero. We let $\mathbb{P}^n = \mathbb{P}^n_{\mathbf{k}}$ be the n-dimensional projective space over $\mathbf{k}$. Unless otherwise specified, $\mathbb{X}$ denotes a set of distinct points either in $\mathbb{P}^n$ or in the multi-projective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. We induce an $\mathbb{N}^k$-grading on the polynomial ring $R = \mathbf{k}[x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{k,0}, \ldots, x_{k,n_k}]$ by setting $\deg x_{i,j} = e_i$, where e_i is the i^{th} standard basis vector of $\mathbb{N}^k$. If $\mathbb{X}$ is a set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, then we write $I_{\mathbb{X}}$ for the $\mathbb{N}^k$-homogeneous ideal of R that is generated by the $\mathbb{N}^k$-homogeneous elements of R that vanish on $\mathbb{X}$. The Hilbert function of $\mathbb{X}$ is the numerical function $H_{\mathbb{X}} : \mathbb{N}^k \to \mathbb{N}$ defined by $\underline{i} = (i_1, \ldots, i_k) \mapsto \dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{\underline{i}}$. Finally, if $\underline{i} = (i_1, \ldots, i_k), \underline{j} = (j_1, \ldots, j_k) \in \mathbb{N}^k$, then we will write $\underline{i} \leq \underline{j}$ if and only if $i_l \leq j_l$ for $l = 1, \ldots, k$. A detailed account of these definitions is given in Chapter 2. Any definitions or terminology used below which is not explicitly defined can be found in the latter chapters.

2. The Border of the Hilbert Function of a Set of Points

Let $\mathbb{X}$ be a set of distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, and suppose that $H_{\mathbb{X}}$ is the Hilbert function of $\mathbb{X}$. In this section we summarize the main results of Chapters 2 and 3 related to Question 1.1.1.

Our quest to answer Question 1.1.1 begins in Chapter 2 where we place some necessary conditions on the values of $H_{\mathbb{X}}$.

Proposition 1.2.1. *Let $\mathbb{X}$ be a set of distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and suppose that $H_{\mathbb{X}}$ is the Hilbert function of $\mathbb{X}$.*

(*i*) (Proposition 2.2.13) *Then for all $\underline{i} = (i_1, \ldots, i_k) \in \mathbb{N}^k$ we have*

$$H_{\mathbb{X}}(\underline{i}) \leq H_{\mathbb{X}}(\underline{i} + e_j) \quad \text{for all } j = 1, \ldots, k.$$

(*ii*) (Proposition 2.2.14) *Fix an integer $j \in \{1, \ldots, k\}$. If $H_{\mathbb{X}}(\underline{i}) = H_{\mathbb{X}}(\underline{i} + e_j)$, then*

$$H_{\mathbb{X}}(\underline{i} + e_j) = H_{\mathbb{X}}(\underline{i} + 2e_j).$$

It follows from this proposition that a large number of numerical functions $H : \mathbb{N}^k \to \mathbb{N}$ cannot be the Hilbert function of a finite set of points.

If $\mathbb{X} \subseteq \mathbb{P}^n$, then Proposition 1.2.1 (*i*) implies that $H_{\mathbb{X}}(i) \leq H_{\mathbb{X}}(i+1)$ for all $i \in \mathbb{N}$. The following well known proposition shows that $H_{\mathbb{X}}$ is also bounded.

Proposition 1.2.2. (Proposition 2.3.4) *Let $\mathbb{X} \subseteq \mathbb{P}^n$ be a collection of s distinct points. Then*

$$H_{\mathbb{X}}(i) = s \quad \textit{for all } i \geq s-1.$$

This proposition has two consequences that makes it extremely interesting. First, to compute $H_{\mathbb{X}}(i)$ for all $i \in \mathbb{N}$, it is sufficient to compute the value of $H_{\mathbb{X}}$ at only a finite number of values. Second, those values at which we need to compute $H_{\mathbb{X}}$ can be derived from simple numerical information describing $\mathbb{X}$.

Using the case of points in $\mathbb{P}^n$ as our inspiration, we are led to ask if the values of the Hilbert function of $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ are also bounded? If so, does $H_{\mathbb{X}}$ have an analog to Proposition 1.2.2? Moreover, does this analog have the same consequences as Proposition 1.2.2? In Chapter 3, we give an affirmative answer to all three questions.

Because of the complexity of the notation in the general case, we state the result only for sets of points in $\mathbb{P}^n \times \mathbb{P}^m$. A complete discussion can be found in Chapter 3. We let $\pi_1 : \mathbb{P}^n \times \mathbb{P}^m \rightarrow \mathbb{P}^n$ be the projection morphism defined by $P \times Q \mapsto P$. We define $\pi_2 : \mathbb{P}^n \times \mathbb{P}^m \rightarrow \mathbb{P}^m$ to be the other projection morphism. Our first major result is the following generalization of Proposition 1.2.2.

Theorem 1.2.3. (Corollary 3.1.7) *Let $\mathbb{X} \subseteq \mathbb{P}^n \times \mathbb{P}^m$ be a set of s distinct points. Suppose that $t = |\pi_1(\mathbb{X})|$ and $r = |\pi_2(\mathbb{X})|$. Then*

$$H_{\mathbb{X}}(i,j) = \begin{cases} s & \textit{if } (i,j) \geq (t-1, r-1) \\ H_{\mathbb{X}}(t-1,j) & \textit{if } i \geq t-1 \textit{ and } j < r-1 \\ H_{\mathbb{X}}(i,r-1) & \textit{if } j \geq r-1 \textit{ and } i < t-1 \end{cases}.$$

This result has all the desired ingredients. Indeed, the value of the Hilbert function is bounded by $|\mathbb{X}| = s$. From this theorem, we deduce that we need to compute $H_{\mathbb{X}}(i,j)$ for only those $(i,j) \leq (|\pi_1(\mathbb{X})| - 1, |\pi_2(\mathbb{X})| - 1)$ to completely determine all values of $H_{\mathbb{X}}$. Since $|\pi_1(\mathbb{X})|$ (respectively, $|\pi_2(\mathbb{X})|$) is the number of distinct first (respectively, second) coordinates of $\mathbb{X}$, the values at which we need to calculate $H_{\mathbb{X}}$ can be ascertained from numerical information about $\mathbb{X}$. There exists a generalization of this result to points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ as we show in Theorem 3.2.1 and Corollary 3.2.6.

For the present, we continue to consider points $\mathbb{X} \subseteq \mathbb{P}^n \times \mathbb{P}^m$. Suppose that $|\pi_1(\mathbb{X})| = t$ and $|\pi_2(\mathbb{X})| = r$. We can represent $H_{\mathbb{X}}$ as an infinite matrix $(m_{i,j})$ where $m_{i,j} := H_{\mathbb{X}}(i,j)$. In light of Theorem 1.2.3 we have

$$H_{\mathbb{X}} = \begin{bmatrix} & & & \mathbf{m_{0,r-1}} & m_{0,r-1} & \cdots \\ & * & & \mathbf{m_{1,r-1}} & m_{1,r-1} & \cdots \\ & & & \vdots & \vdots & \\ \mathbf{m_{t-1,0}} & \mathbf{m_{t-1,1}} & \cdots & \mathbf{m_{t-1,r-1}} = \mathbf{s} & s & \cdots \\ m_{t-1,0} & m_{t-1,1} & \cdots & s & \ddots & \\ \vdots & \vdots & & \vdots & & \end{bmatrix}.$$

We define $B_C := (m_{t-1,0}, m_{t-1,1}, \ldots, m_{t-1,r-1})$ and $B_R := (m_{0,r-1}, m_{1,r-1}, \ldots, m_{t-1,r-1})$ and set $B_{\mathbb{X}} = (B_C, B_R)$. We call $B_{\mathbb{X}}$ the *border* of the Hilbert function of $\mathbb{X}$. From the matrix representation of $H_{\mathbb{X}}$ given above, the name is appropriate because the border, the bold numbers, separates those values $(*)$ at which we must compute $H_{\mathbb{X}}$, and those values which depend only upon Theorem 1.2.3. Note that if we know $B_{\mathbb{X}}$, then we know $H_{\mathbb{X}}$ at all but a finite number of values. The border of the Hilbert function of a set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is defined similarly (see Definition 3.2.8).

Example 1.2.4. We illustrate some of the above results with the following example. Let $P_i := [1 : i] \in \mathbb{P}^1$ for all $i \in \mathbb{N}$. Similarly, we define $Q_i := [1 : i] \in \mathbb{P}^1$. Let $\mathbb{X}$ be the following collection of points in $\mathbb{P}^1 \times \mathbb{P}^1$:

$$\mathbb{X} := \{P_1 \times Q_1, P_1 \times Q_2, P_1 \times Q_3, P_1 \times Q_4, P_2 \times Q_2, P_2 \times Q_3, P_2 \times Q_4, P_3 \times Q_4, P_4 \times Q_4\}.$$

Then the Hilbert function of $\mathbb{X}$, expressed as a matrix, is

$$H_{\mathbb{X}} = \begin{bmatrix} 1 & 2 & 3 & \mathbf{4} & 4 & \cdots \\ 2 & 4 & 6 & \mathbf{7} & 7 & \cdots \\ 3 & 5 & 7 & \mathbf{8} & 8 & \cdots \\ \mathbf{4} & \mathbf{6} & \mathbf{8} & \mathbf{9} & 9 & \cdots \\ 4 & 6 & 8 & 9 & 9 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We observe, that in accordance with Proposition 1.2.1, that the values in each row (respectively, column) strictly increase until they stabilize. For this example, $B_C = (4, 6, 8, 9)$

and $B_R = (4,7,8,9)$, and so, the border is $B_{\mathbb{X}} = (B_C, B_R)$. We have written the values in the border in bold. Since $|\pi_1(\mathbb{X})| = 4$ and $|\pi_2(\mathbb{X})| = 4$, only the values of $H_{\mathbb{X}}(i,j)$ with $(i,j) \leq (3,3)$ need to be calculated, and the remaining values can be computed by using Theorem 1.2.3.

The fact that the Hilbert function of any set of points $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ has a border places a new restriction on the numerical functions that can be the Hilbert function of a set of points. We weaken Question 1.1.1 to the following question:

Question 1.2.5. (Question 3.2.10) *What can be the border of the Hilbert function of a set of points in* $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$*?*

An answer to the above question would impose a severe restriction on what could be the Hilbert function of a set of points. This question, although weaker, is still difficult. However, we can answer Question 1.2.5 for the case of points in $\mathbb{P}^1 \times \mathbb{P}^1$ (we discuss this Section 4 of this chapter). We also show that there are a number of necessary conditions on the values of the border (for example, Corollary 3.2.4). In general, this weaker question still requires further work.

We can use the fact that every Hilbert function of a set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ has a border to deduce the existence of sets of points in generic position. If $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is a set of s points, then $\mathbb{X}$ is said to be in *generic position* if

$$H_{\mathbb{X}}(j_1,\ldots,j_k) = \min\left\{\binom{n_1+j_1}{j_1}\cdots\binom{n_k+j_k}{j_k}, s\right\} \quad \text{for all } (j_1,\ldots,j_k) \in \mathbb{N}^k.$$

We, in fact, generalize a result of Geramita and Orecchia [**21**] about points in generic position in $\mathbb{P}^n$ to show that "most" sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ are in generic position.

Theorem 1.2.6. (Theorem 3.3.2) *The s-tuples of points of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, $(P_1,\ldots,P_s)$, considered as points of $(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})^s$, which are in generic position form a non-empty open subset of $(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})^s$.*

Because questions about generic sets of points in $\mathbb{P}^n$ command a lot of interest in current research, it would be useful to determine if generic sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ behave like generic sets of points in $\mathbb{P}^n$. For example, one can try to formulate a Minimal Resolution

Conjecture (see Lorenzini [**34**]) for generic sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. We leave that problem for now but will return to it at a future date.

3. Arithmetically Cohen-Macaulay Sets of Points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$

It has already been noted that Question 1.1.1 has a complete answer for sets of points in $\mathbb{P}^n$ (see [**19**]). This result is stated below.

Proposition 1.3.1. (Proposition 2.3.10) *Let $H : \mathbb{N} \to \mathbb{N}$ be a numerical function. Then H is the Hilbert function of a set of distinct points in $\mathbb{P}^n$ if and only if the first difference function $\Delta H : \mathbb{N} \to \mathbb{N}$, where $\Delta H(i) := H(i) - H(i-1)$ for all $i \in \mathbb{N}$, is the Hilbert function of a graded artinian quotient of $\mathbf{k}[x_1, \ldots, x_n]$. ($H(i) = 0$ if $i < 0$.)*

The proof of Proposition 1.3.1 relies, in part, on the fact that the coordinate ring of a finite set of points in $\mathbb{P}^n$ is *always* Cohen-Macaulay. Unfortunately, any attempt to generalize this proof to sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ will be hampered by the fact that the corresponding coordinate ring may fail to be *Cohen-Macaulay.* We call sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with a Cohen-Macaulay coordinate ring an *arithmetically Cohen-Macaulay* (ACM for short) set of points.

In Chapter 4, we study the following weaker version of Question 1.1.1:

Question 1.3.2. *What can be the Hilbert function of an ACM set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$?*

The main result of Chapter 4 is to show that if we restrict to ACM sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, then there is a natural generalization of Proposition 1.3.1.

Theorem 1.3.3. (Theorem 4.3.14) *Let $H : \mathbb{N}^k \to \mathbb{N}$ be a numerical function. Then H is the Hilbert function of an ACM set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ if and only if*

$$\Delta H(i_1, \ldots, i_k) := \sum_{\underline{l}=(l_1,\ldots,l_k) \leq (1,\ldots,1)} (-1)^{|\underline{l}|} H(i_1 - l_1, \ldots, i_k - l_k),$$

where $H(i_1, \ldots, i_k) = 0$ if $(i_1, \ldots, i_k) \not\geq \underline{0}$, is the Hilbert function of some $\mathbb{N}^k$-graded artinian quotient of $S = \mathbf{k}[x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{k,1}, \ldots, x_{k,n_k}]$.

For ACM sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, Theorem 1.3.3 enables us to translate Question 1.3.2 into the following question:

Question 1.3.4. *What can be the Hilbert function of an $\mathbb{N}^k$-graded artinian quotient of $\mathbf{k}[x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{k,1}, \ldots, x_{k,n_k}]$?*

Because there is no known analog of Macaulay's Theorem (see [**35**] or Theorem 2.1.2) for multi-graded rings, Theorem 1.3.3 turns one open problem into another open problem. However, the other main result of Chapter 4 is to show that we can answer Question 1.3.4 if (i) $S = \mathbf{k}[x_1, y_1, \ldots, y_m]$ is bigraded, or if (ii) $S = \mathbf{k}[x_1, x_2, \ldots, x_k]$ is $\mathbb{N}^k$-graded.

For (i), we suppose that $S = \mathbf{k}[x_1, y_1, \ldots, y_m]$ with $\deg x_1 = (1,0)$ and $\deg y_i = (0,1)$. In Chapter 4, we will give a much stronger result characterizing the Hilbert functions of all bigraded quotients of S. As a corollary, we answer Question 1.3.4 for S. To prove (i) we will use some necessary conditions about bigraded rings given by Aramova, Crona, and De Negri [**2**].

To state our result, we recall the notion of an i-binomial expansion of an integer. Let i and a be positive integers. Then the *i-binomial expansion of a* is the unique expression

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$$

where $a_i > a_{i-1} > \cdots > a_j \geq j \geq 1$. The function ${}^{<i>} : \mathbb{N} \to \mathbb{N}$, sometimes called *Macaulay's function*, is defined by

$$a \longmapsto a^{<i>} = \binom{a_i + 1}{i+1} + \binom{a_{i-1}+1}{i} + \cdots + \binom{a_j + 1}{j+1}$$

where $a_i, a_{i-1}, \ldots, a_j$ are as in the i-binomial expansion of a.

Theorem 1.3.5. (Theorem 4.4.11) *Suppose that $S = \mathbf{k}[x_1, y_1 \ldots, y_m]$, where $\deg x_1 = (1,0)$ and $\deg y_i = (0,1)$ for $i = 1, \ldots, m$, and let $H : \mathbb{N}^2 \to \mathbb{N}$ be a numerical function. Then there exists a bihomogeneous ideal $I \subsetneq S = \mathbf{k}[x_1, y_1, \ldots, y_m]$ such that the Hilbert function $H_{S/I} = H$ if and only if*

- (i) $H(0,0) = 1$,
- (ii) $H(0,1) \leq m$,
- (iii) $H(i+1, j) \leq H(i,j)$ *for all* $(i,j) \in \mathbb{N}^2$, *and*
- (iv) $H(i, j+1) \leq H(i,j)^{<j>}$ *for all* $(i,j) \in \mathbb{N}^2$ *with* $j \geq 1$.

As a corollary of this theorem, we can give a complete answer to Question 1.1.1 for the case of ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^m$.

Corollary 1.3.6. (Corollary 4.4.15) *Let $H : \mathbb{N}^2 \to \mathbb{N}$ be a numerical function. Then H is the Hilbert function of an ACM set of points in $\mathbb{P}^1 \times \mathbb{P}^m$ if and only if the numerical function*

$$\Delta H(i,j) := H(i,j) - H(i,j-1) - H(i-1,j) + H(i-1,j-1),$$

where $H(i,j) = 0$ if $(i,j) \not\geq (0,0)$, satisfies:

(*i*) $\Delta H(0,0) = 1$,
(*ii*) $\Delta H(0,1) \leq m$,
(*iii*) $\Delta H(i+1,j) \leq \Delta H(i,j)$ *for all* $(i,j) \in \mathbb{N}^2$,
(*iv*) $\Delta H(i,j+1) \leq \Delta H(i,j)^{<j>}$ *for all* $(i,j) \in \mathbb{N}^2$ *with* $j \geq 1$,
(*v*) *there exists a positive integer t such that $\Delta H(t,0) = 0$, and*
(*vi*) *there exists a positive integer r such that $\Delta H(0,r) = 0$.*

The above corollary generalizes the characterization of Hilbert functions of ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ first given by Giuffrida, Maggioni, and Ragusa [**26**].

We also characterize the Hilbert functions of the $\mathbb{N}^k$-graded quotients of the ring $\mathbf{k}[x_1, \ldots, x_k]$. As a corollary, we can answer Question 1.1.1 for the case of ACM sets of points in $\underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{k}$ for any $k \in \mathbb{N}$.

Theorem 1.3.7. (Theorem 4.4.16) *Let $S = \mathbf{k}[x_1, \ldots, x_k]$ be an $\mathbb{N}^k$-graded ring with $\deg x_i = e_i$, the i^{th} standard basis vector of $\mathbb{N}^k$, and let $H : \mathbb{N}^k \to \mathbb{N}$ be a numerical function. Then there exists a proper ideal $I \subsetneq S$ such that the Hilbert function $H_{S/I} = H$ if and only if*

(*i*) $H(0, \ldots, 0) = 1$,
(*ii*) $H(i_1, \ldots, i_k) = 1$ *or* 0 *if* $(i_1, \ldots, i_k) > (0, \ldots, 0)$, *and*
(*iii*) *if* $H(i_1, \ldots, i_k) = 0$, *then* $H(j_1, \ldots, j_k) = 0$ *for all* $(j_1, \ldots, j_k) \geq (i_1, \ldots, i_k)$.

Corollary 1.3.8. (Corollary 4.4.18) *Let $H : \mathbb{N}^k \to \mathbb{N}$ be a numerical function. Then H is the Hilbert function of an ACM set of distinct points in $\underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{k}$ if and only if the* *numerical function*

$$\Delta H(i_1,\ldots,i_k) := \sum_{\underline{l}=(l_1,\ldots,l_k)\leq(1,\ldots,1)} (-1)^{|\underline{l}|} H(i_1-l_1,\ldots,i_k-l_k),$$

where $H(i_1,\ldots,i_k)=0$ *if* $(i_1,\ldots,i_k)\not\geq \underline{0}$, *satisfies:*

(i) $\Delta H(0,\ldots,0)=1$,
(ii) $\Delta H(i_1,\ldots,i_k)=1$ *or* 0 *if* $(i_1,\ldots,i_k)>(0,\ldots,0)$,
(iii) *if* $\Delta H(i_1,\ldots,i_k)=0$, *then* $H(j_1,\ldots,j_k)=0$ *for all* $(j_1,\ldots,j_k)\geq(i_1,\ldots,i_k)$, *and*
(iv) *for each integer* $1\leq i\leq k$, *there exists an integer* t_i *such that* $\Delta H(t_1,0,\ldots,0)=\Delta H(0,t_2,0,\ldots,0)=\cdots=\Delta H(0,\ldots,0,t_k)=0$.

One question that is not answered within this thesis is whether Theorem 1.3.3 classifies the ACM sets of points in $\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$. That is,

Question 1.3.9. *If* $H_{\mathbb{X}}$ *is the Hilbert function of a set of points* $\mathbb{X}$ *in* $\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$, *and if* $\Delta H_{\mathbb{X}}$ *is the Hilbert function of an* $\mathbb{N}^k$*-graded artinian quotient of* $\mathbf{k}[x_{1,1},\ldots,x_{1,n_1},\ldots,x_{k,1},\ldots,x_{k,n_k}]$, *then is* $\mathbb{X}$ *necessarily an ACM set of points?*

We can give a positive answer to Question 1.3.9 for sets of points in $\mathbb{P}^1\times\mathbb{P}^1$ (see Theorem 5.4.4). This result is expanded upon in the next section.

4. The Hilbert Function of Points in $\mathbb{P}^1\times\mathbb{P}^1$

Sets of points in $\mathbb{P}^1\times\mathbb{P}^1$ have enjoyed more exposure than sets of points in more general multi-projective spaces. This is because $\mathcal{Q}\cong\mathbb{P}^1\times\mathbb{P}^1$, where $\mathcal{Q}$ is the quadric surface in $\mathbb{P}^3$. As already noted, the Hilbert function of points in $\mathcal{Q}\cong\mathbb{P}^1\times\mathbb{P}^1$ was first studied, among other things, by Giuffrida, Maggioni, and Ragusa [**24**], [**25**], [**26**]. There has also been other work on sets of points on $\mathcal{Q}$. For example, Guardo studied fat points in $\mathbb{P}^1\times\mathbb{P}^1$ [**28**], [**29**]; Paxia, Raciti, and Ragusa considered the uniform position property for points on $\mathcal{Q}$ [**42**]; and Ragusa and Zappalà have recently examined, among other things, Gorenstein sets of points on $\mathcal{Q}$ [**43**]. So, unlike sets of points in an arbitrary multi-projective space, much more is understood about points in $\mathbb{P}^1\times\mathbb{P}^1$.

Although Giuffrida, *et al.* introduced a number of necessary conditions on the Hilbert function of points in $\mathbb{P}^1 \times \mathbb{P}^1$ in [**26**], it remains an open problem to give a complete characterization even in this case. Our goal in Chapter 5 is to continue and to extend the program begun by Giuffrida, *et al.* by studying the Hilbert functions of points in $\mathbb{P}^1 \times \mathbb{P}^1$.

Our first contribution to this program is to introduce a new necessary condition on the Hilbert function of points in $\mathbb{P}^1 \times \mathbb{P}^1$ by answering Question 1.2.5.

To answer Question 1.2.5, we demonstrate that the border of a Hilbert function of a set of points $\mathbb{X}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ can be determined from crude numerical information describing $\mathbb{X}$. To state our result, we need to define some appropriate notation and introduce some concepts from combinatorics, specifically, the notion of a *partition* and its *conjugate.*

Suppose that $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is a collection of s distinct points. Let $\pi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection morphism defined by $P \times Q \mapsto P$, and let $\pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the other projection morphism. We associate to $\mathbb{X}$ two tuples, $\alpha_{\mathbb{X}}$ and $\beta_{\mathbb{X}}$, as follows. For each $P_i \in \pi_1(\mathbb{X}) = \{P_1, \ldots, P_t\}$ we set $\alpha_i := |\pi_1^{-1}(P_i)|$. After relabelling the α_i's so that $\alpha_i \geq \alpha_{i+1}$ for $i = 1, \ldots, t-1$, we set $\alpha_{\mathbb{X}} := (\alpha_1, \ldots, \alpha_t)$. Analogously, for every $Q_i \in \pi_2(\mathbb{X}) = \{Q_1, \ldots, Q_r\}$ we set $\beta_i := |\pi_2^{-1}(Q_i)|$. After relabelling the β_i's so that $\beta_i \geq \beta_{i+1}$ for $i = 1, \ldots, r-1$, we let $\beta_{\mathbb{X}}$ be the r-tuple $\beta_{\mathbb{X}} := (\beta_1, \ldots, \beta_r)$.

Definition 1.4.1. (Definition 2.5.1) A tuple $\lambda = (\lambda_1, \ldots, \lambda_r)$ of positive integers is a *partition* of an integer s if $\sum \lambda_i = s$ and $\lambda_i \geq \lambda_{i+1}$ for every i. We write $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash s$. The *conjugate* of λ is the λ_1-tuple $\lambda^* = (\lambda_1^*, \ldots, \lambda_{\lambda_1}^*)$ where $\lambda_i^* = \#\{\lambda_j \in \lambda \mid \lambda_j \geq i\}$. Furthermore, $\lambda^* \vdash s$.

Example 1.4.2. Suppose $\lambda = (4,4,3,1) \vdash 12$. Then the conjugate of λ is $\lambda^* = (4,3,3,2)$. In Section 5 of Chapter 2, we will show how to compute the conjugate of a partition from the *Ferrers diagram* of the partition.

The tuples $\alpha_{\mathbb{X}}$ and $\beta_{\mathbb{X}}$ are both partitions of the integer $s = |\mathbb{X}|$. For any tuple $p = (p_1, p_2, \ldots, p_k)$ we define $\Delta p := (p_1, p_2 - p_1, \ldots, p_k - p_{k-1})$. With this notation we show

Proposition 1.4.3. (Corollary 5.2.3) *Let $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be s distinct points with $\alpha_{\mathbb{X}}$ and $\beta_{\mathbb{X}}$. Suppose that $B_{\mathbb{X}} = (B_C, B_R)$ is the border of the Hilbert function of $\mathbb{X}$. Then*

(i) $\Delta B_C = \alpha_{\mathbb{X}}^*$.

(ii) $\Delta B_R = \beta_{\mathbb{X}}^*$.

Example 1.4.4. Let $\mathbb{X}$ be the set of points from Example 1.2.4. That is, $\mathbb{X}$ is the following collection of points in $\mathbb{P}^1 \times \mathbb{P}^1$:

$$\mathbb{X} := \{P_1 \times Q_1, P_1 \times Q_2, P_1 \times Q_3, P_1 \times Q_4, P_2 \times Q_2, P_2 \times Q_3, P_2 \times Q_4, P_3 \times Q_4, P_4 \times Q_4\},$$

where $P_i = Q_i := [1 : i] \in \mathbb{P}^1$. For this example, $\pi_1(\mathbb{X}) = \{P_1, P_2, P_3, P_4\}$. Furthermore, $|\pi_1^{-1}(P_1)| = 4$, $|\pi_1^{-1}(P_2)| = 3$, $|\pi_1^{-1}(P_3)| = 1$, and $|\pi_1^{-1}(P_4)| = 1$, and hence, $\alpha_{\mathbb{X}} = (4, 3, 1, 1)$. The conjugate of $\alpha_{\mathbb{X}}$ is the tuple $\alpha_{\mathbb{X}}^* = (4, 2, 2, 1)$. Hence, by Proposition 1.4.3, we have $\Delta B_C = (4, 2, 2, 1)$, or equivalently, $B_C = (4, 6, 8, 9)$. We see that this agrees with Example 1.2.4. A similar computation will enable to us to compute B_R directly from $\mathbb{X}$.

We show in Theorem 5.2.8 that there exists a link between sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ and $(0, 1)$-matrices. Then, by using a classical result about $(0, 1)$-matrices due to Gale and Ryser (see Theorem 2.5.6) we can answer Question 1.2.5 for sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$. But first, we need to define majorization.

Definition 1.4.5. (Definition 2.5.4) Let $\lambda = (\lambda_1, \ldots, \lambda_t)$ and $\delta = (\delta_1, \ldots, \delta_r)$ be two partitions of s. If one partition is longer, we add zeros to the shorter one until they have the same length. We say λ *majorizes* δ, written $\lambda \trianglerighteq \delta$, if

$$\lambda_1 + \cdots + \lambda_i \geq \delta_1 + \cdots + \delta_i \qquad \text{for } i = 1, \ldots, \max\{t, r\}.$$

Example 1.4.6. Let $\lambda = (4, 3, 2, 1)$ and $\delta = (4, 2, 2, 1, 1)$. Then $\lambda, \delta \vdash 10$, and $\lambda \trianglerighteq \delta$. Now let $\gamma = (5, 2, 1, 1, 1) \vdash 10$. Then $\gamma \not\trianglerighteq \lambda$ because $4 \leq 5$ but $4 + 3 + 2 \geq 5 + 2 + 1$. It is also immediate that $\lambda \not\trianglerighteq \gamma$.

Theorem 1.4.7. (Corollary 5.2.11) *Suppose $B_C = (b_0, \ldots, b_{r-1})$ and $B_R = (b'_0, \ldots, b'_{t-1})$ are two tuples such that $b_0 = t$, $b'_0 = r$, and $\Delta B_C, \Delta B_R \vdash s$. Then $B = (B_C, B_R)$ is the border of a Hilbert function of a set of s points $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ if and only if $\Delta B_C \trianglerighteq (\Delta B_R)^*$.*

As one application of the above theorem, we can answer Question 1.1.1 for a particular class of points in $\mathbb{P}^1 \times \mathbb{P}^1$. In particular, we have

Theorem 1.4.8. (Theorem 5.3.3) *Let $H : \mathbb{N}^2 \to \mathbb{N}$ be a numerical function. Then H is the Hilbert function of a set of points $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ with $|\pi_1(\mathbb{X})| = 2$ if and only if the following conditions hold:*

(*i*)

$$H = \begin{bmatrix} 1 & 2 & 3 & \cdots & r-1 & \mathbf{r} & r & \cdots \\ \mathbf{2} & \mathbf{m_{1,1}} & \mathbf{m_{1,2}} & \cdots & \mathbf{m_{1,r-2}} & \mathbf{s} & s & \cdots \\ 2 & m_{1,1} & m_{1,2} & \cdots & m_{1,r-2} & s & & \\ \vdots & \vdots & \vdots & & \vdots & & & \ddots \end{bmatrix},$$

(*ii*) $r \leq s$,

(*iii*) $2 \leq m_{1,1} \leq \cdots \leq m_{1,r-2} \leq s$, *and* $m_{1,j} \leq 2(j+1)$ *for* $j = 1, \ldots, r-2,$, *and*

(*iv*) *if* $B_1 = (2, m_{1,1}, \ldots, m_{1,r-2}, s)$ *and* $B_2 = (r, s)$, *then* $\Delta B_1, \Delta B_2 \vdash s$, *and* $\Delta B_1 \trianglerighteq (\Delta B_2)^*$.

Example 1.4.9. Consider the matrix

$$H = \begin{bmatrix} 1 & 2 & \mathbf{3} & 3 & \cdots \\ \mathbf{2} & \mathbf{3} & \mathbf{5} & 5 & \cdots \\ 2 & 3 & 5 & 5 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \end{bmatrix}.$$

The matrix H cannot be the Hilbert function of any set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ because it fails to meet condition (*iv*) of the above theorem. Indeed, from H, we have $B_1 = (2, 3, 5)$, and thus, $\Delta B_1 = (2, 1, 2)$. But this is not a partition of 5, so this cannot be the Hilbert function of a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ with $|\pi_1(\mathbb{X})| = 2$. On the other hand, the matrix

$$H = \begin{bmatrix} 1 & 2 & \mathbf{3} & 3 & \cdots \\ \mathbf{2} & \mathbf{3} & \mathbf{4} & 4 & \cdots \\ 2 & 3 & 4 & 4 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \end{bmatrix}$$

is the Hilbert function of a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$.

The second major result of Chapter 5 is a classification of the arithmetically Cohen-Macaulay sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Giuffrida, *et al.* originally classified ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ via the first difference function ΔH where H is the Hilbert function of a set of points (see [**26**]). We present not only a new proof of this result, but we give a new characterization of ACM sets of points via the numerical information describing $\mathbb{X}$. In particular, we show

Theorem 1.4.10. (Theorem 5.4.4) *Let $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a set of s distinct points, let $\alpha_{\mathbb{X}}$ and $\beta_{\mathbb{X}}$ be constructed as above, and let $H_{\mathbb{X}}$ be the Hilbert function of $\mathbb{X}$. Then the following are equivalent:*

(*i*) $\mathbb{X}$ *is ACM.*
(*ii*) *The function*

$$\Delta H_{\mathbb{X}}(i,j) := H_{\mathbb{X}}(i,j) - H_{\mathbb{X}}(i-1,j) - H_{\mathbb{X}}(i,j-1) + H_{\mathbb{X}}(i-1,j-1)$$

is the Hilbert function of a bigraded artinian quotient of $\mathbf{k}[x_1, y_1]$.
(*iii*) $\alpha^*_{\mathbb{X}} = \beta_{\mathbb{X}}$.

The equivalence of (*i*) and (*ii*) was first demonstrated by Giuffrida, *et al.* (Theorem 4.1 [**26**]).

By using Theorem 1.4.10, we show that ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ are similar, in some respects, to sets of points in $\mathbb{P}^1$. In particular, we show that like a set of points in $\mathbb{P}^1$, the Hilbert function and the graded Betti numbers in the resolution of an ACM set of points depend only upon crude numerical information about $\mathbb{X}$ and not upon the coordinates of the set of points themselves. Our results are given below.

Theorem 1.4.11. (Theorem 5.4.9) *Let $\mathbb{X}$ be an ACM set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ with $\alpha_{\mathbb{X}} = (\alpha_1, \ldots \alpha_t)$. Then*

$$H_{\mathbb{X}}(i,j) = \begin{bmatrix} 1 & 2 & \cdots & \alpha_1 - 1 & \alpha_1 & \alpha_1 & \cdots \\ 1 & 2 & \cdots & \alpha_1 - 1 & \alpha_1 & \alpha_1 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 1 & 2 & \cdots & \alpha_2 - 1 & \alpha_2 & \alpha_2 & \cdots \\ 1 & 2 & \cdots & \alpha_2 - 1 & \alpha_2 & \alpha_2 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 1 & 2 & \cdots & \alpha_3 - 1 & \alpha_3 & \alpha_3 & \cdots \\ 1 & 2 & \cdots & \alpha_3 - 1 & \alpha_3 & \alpha_3 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{bmatrix} + \cdots + \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 1 & 2 & \cdots & \alpha_t - 1 & \alpha_t & \alpha_t & \cdots \\ 1 & 2 & \cdots & \alpha_t - 1 & \alpha_t & \alpha_t & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Theorem 1.4.12. (Theorem 5.4.11) *Suppose that $\mathbb{X}$ is an ACM set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ with $\alpha_{\mathbb{X}} = (\alpha_1, \ldots, \alpha_t)$. Define*

$$C_{\mathbb{X}} := \{(t,0),(0,\alpha_1)\} \cup \{(i-1,\alpha_i) \mid \alpha_i - \alpha_{i-1} < 0\},$$

and

$$V_{\mathbb{X}} := \{(t,\alpha_t)\} \cup \{(i-1,\alpha_{i-1}) \mid \alpha_i - \alpha_{i-1} < 0\}.$$

Then the bigraded minimal free resolution of $I_{\mathbb{X}}$ is given by

$$0 \longrightarrow \bigoplus_{(v_1,v_2)\in V_{\mathbb{X}}} R(-v_1,-v_2) \longrightarrow \bigoplus_{(c_1,c_2)\in C_{\mathbb{X}}} R(-c_1,-c_2) \longrightarrow I_{\mathbb{X}} \longrightarrow 0.$$

CHAPTER 2

Preliminaries

In this chapter we lay the mathematical foundation for the thesis by collecting the definitions, results, and techniques that we require for the later chapters. As a consequence, most of the material in this chapter is well known, the main exception being the contents of Section 2 which introduces points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

The chapter is divided into five sections. In Section 1 we discuss multi-graded rings S with a special emphasis on the case that S is the quotient of a polynomial ring, or more generally, a finitely generated $\mathbf{k}$-algebra. We also extend the definition of the Hilbert function to this context. In Section 2 we introduce the main object of study in this thesis, namely, points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Because we periodically require results about points in $\mathbb{P}^n$, we assemble in Section 3 the needed propositions. Section 4 is a collection of facts concerning resolutions and projective dimension that we use in Chapter 5. The final section introduces some concepts from combinatorics, specifically the notions of a partition and a $(0,1)$-matrix. These results are required in Chapter 5.

Throughout this thesis $\mathbf{k}$ will denote an algebraically closed field of characteristic zero.

1. Multi-graded Rings and Hilbert Functions

In this section we extend the theory of graded rings to the theory of multi-graded rings. Although a more general theory exists, we have elected to only describe multi-graded rings in the case that the ring S is a finitely generated $\mathbf{k}$-algebra. We also define a multi-graded analog of the Hilbert function.

Let $\mathbb{N} := \{0,1,2,\ldots\}$. If $(i_1,\ldots,i_k) \in \mathbb{N}^k$, then we denote $(i_1,\ldots,i_k)$ by $\underline{i}$. We set $|\underline{i}| := \sum_h i_h$. If $\underline{i},\underline{j} \in \mathbb{N}^k$, then $\underline{i}+\underline{j} := (i_1+j_i,\ldots,i_k+j_k)$. We write $\underline{i} \geq \underline{j}$ if $i_h \geq j_h$ for every $h = 1,\ldots,k$. This ordering is a partial ordering on the elements of $\mathbb{N}^k$. We also observe that $\mathbb{N}^k$ is a semi-group generated by $\{e_1, \ldots, e_k\}$ where e_i is the i^{th} standard basis vector of $\mathbb{N}^k$, that is, $e_i := (0, \ldots, 1, \ldots, 0)$ with 1 being in the i^{th} position.

An $\mathbb{N}^k$-*graded ring* (or simply a *multi-graded ring* if k is clear from the context) is a ring R that has a direct sum decomposition $R = \bigoplus_{\underline{i} \in \mathbb{N}^k} R_{\underline{i}}$ such that $R_{\underline{i}} R_{\underline{j}} \subseteq R_{\underline{i}+\underline{j}}$ for all $\underline{i}, \underline{j} \in \mathbb{N}^k$. We sometimes write $R_{(i_1,\ldots,i_k)} := R_{\underline{i}}$ as $R_{i_1,\ldots,i_k}$ to simplify our notation. An element $x \in R$ is said to be $\mathbb{N}^k$-*homogeneous* (or simply *homogeneous* if it is clear that R is $\mathbb{N}^k$-graded) if $x \in R_{\underline{i}}$ for some $\underline{i} \in \mathbb{N}^k$. If x is homogeneous, then $\deg x := \underline{i}$. If $k = 2$, then we sometimes say that R is *bigraded* and x is *bihomogeneous*.

We now will assume that $R = \mathbf{k}[x_{1,0}, \ldots, x_{1,n_1}, x_{2,0}, \ldots, x_{2,n_2}, \ldots, x_{k,0}, \ldots, x_{k,n_k}]$. We induce an $\mathbb{N}^k$-grading on R by setting $\deg x_{i,j} = e_i$. If $k = 2$, then we sometimes write R as $R = \mathbf{k}[x_0, \ldots, x_n, y_0, \ldots, y_m]$ with $\deg x_i = (1, 0)$ and $\deg y_i = (0, 1)$.

If $m \in R$ is a monomial, then

$$m = x_{1,0}^{a_{1,0}} \cdots x_{1,n_1}^{a_{1,n_1}} x_{2,0}^{a_{2,0}} \cdots x_{2,n_2}^{a_{2,n_2}} \cdots x_{k,0}^{a_{k,0}} \cdots x_{k,n_k}^{a_{k,n_k}}.$$

We sometimes denote m by $X_1^{\underline{a}_1} X_2^{\underline{a}_2} \cdots X_k^{\underline{a}_k}$ where $\underline{a}_i \in \mathbb{N}^{n_i+1}$. It follows that $\deg m = (|\underline{a}_1|, |\underline{a}_2|, \ldots, |\underline{a}_k|)$. If $F \in R$, then we can write $F = F_1 + \cdots + F_r$ where each F_i is homogeneous. The F_i's are called the *homogeneous terms* of F.

For every $\underline{i} \in \mathbb{N}^k$, the set $R_{\underline{i}}$ is a finite dimensional vector space over $\mathbf{k}$. A basis for $R_{\underline{i}}$ as a vector space is the set of monomials $\{m = X_1^{\underline{a}_1} X_2^{\underline{a}_2} \cdots X_k^{\underline{a}_k} \in R \mid \deg m = (|\underline{a}_1|, |\underline{a}_2|, \ldots, |\underline{a}_k|) = \underline{i}\}$. It follows that $\dim_{\mathbf{k}} R_{\underline{i}} = \binom{n_1+i_1}{i_1}\binom{n_2+i_2}{i_2} \cdots \binom{n_k+i_k}{i_k}$.

Suppose that $I = (F_1, \ldots, F_r) \subseteq R$ is an ideal. If each F_j is $\mathbb{N}^k$-homogeneous, then we say I is an $\mathbb{N}^k$-*homogeneous ideal* (or simply, a *homogeneous ideal*). It can be shown that I is homogeneous if and only if for every $F \in I$, all of F's homogeneous terms are in I.

If $I \subseteq R$ is any ideal, then we define $I_{\underline{i}} := I \cap R_{\underline{i}}$ for every $\underline{i} \in \mathbb{N}^k$. It follows that each $I_{\underline{i}}$ is a subvector space of $R_{\underline{i}}$. Clearly $I \supseteq \bigoplus_{\underline{i} \in \mathbb{N}^k} I_{\underline{i}}$. If I is $\mathbb{N}^k$-homogeneous, then $I = \bigoplus_{\underline{i} \in \mathbb{N}^k} I_{\underline{i}}$ because the homogeneous terms of F belong to I if $F \in I$.

Let $I \subseteq R$ be a homogeneous ideal and consider the quotient ring $S = R/I$. The ring S inherits an $\mathbb{N}^k$-graded ring structure if we define $S_{\underline{i}} = (R/I)_{\underline{i}} := R_{\underline{i}}/I_{\underline{i}}$, and hence, $S = \bigoplus_{\underline{i} \in \mathbb{N}^k} (R/I)_{\underline{i}}$.

Example 2.1.1. Let $R = \mathbf{k}[x_0, x_1, y_0, y_1]$ with $\deg x_i = (1,0)$ and $\deg y_i = (0,1)$. Then R is $\mathbb{N}^2$-graded, i.e., $R = \bigoplus_{(i,j)\in\mathbb{N}^2} R_{i,j}$. The element $F = x_0^2y_0y_1^2 + x_0x_1y_1^3 \in R_{2,3}$, and hence, F is a bihomogeneous element of R. The degree of F is $\deg F = (2,3)$. The element $G = x_0^3y_0 + x_0x_1y_0y_1$ is not $\mathbb{N}^2$-homogeneous because $x_0^3y_0 \in R_{3,1}$ and $x_0x_1y_0y_1 \in R_{2,2}$. Note, however, that G is a homogeneous element of R of degree 4 if we give R the normal grading.

Suppose that the polynomial ring R is being considered as an $\mathbb{N}^k$-graded ring. For every $i \in \mathbb{N}$, define $R_i := \bigoplus_{\{\underline{j}\in\mathbb{N}^k \mid\ |\underline{j}|=i\}} R_{\underline{j}}$. We can then consider R as an $\mathbb{N}^1$-graded ring as well. Similarly, an $\mathbb{N}^k$-homogeneous ideal I of R is also an $\mathbb{N}^1$-homogeneous ideal of R. Note however, that an $\mathbb{N}^1$-homogeneous ideal need not be an $\mathbb{N}^k$-homogeneous ideal. It follows that the multi-graded quotient $S = R/I$ is also $\mathbb{N}^1$-graded.

For the remainder of this thesis we restrict our focus to multi-graded rings of the form $S = R/I$, where R is the $\mathbb{N}^k$-graded polynomial ring and I is an $\mathbb{N}^k$-homogeneous ideal of R. In the later chapters we restrict our study even further to the case that I is the homogeneous ideal defining a set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. For the remainder of this section we simply assume that I is a homogeneous ideal of R. We now introduce the multi-graded analog of the Hilbert function.

Let $S = R/I$ be an $\mathbb{N}^k$-graded ring. The numerical function $H_S : \mathbb{N}^k \to \mathbb{N}$ defined by

$$H_S(\underline{i}) := \dim_{\mathbf{k}}(R/I)_{\underline{i}} = \dim_{\mathbf{k}} R_{\underline{i}} - \dim_{\mathbf{k}} I_{\underline{i}}$$

is the *Hilbert function of S*. The *Hilbert-Poincaré series of S* is the infinite series

$$HP_S(t_1, \ldots, t_k) = \sum_{\underline{i}\in\mathbb{N}^k} H_S(\underline{i})t^{\underline{i}} \quad \text{where } t^{\underline{i}} := t_1^{i_1}\cdots t_k^{i_k}.$$

If we can compute HP_S, then we know H_S. For a comprehensive account on the computation of HP_S, one can consult Bigatti [**3**].

If $H : \mathbb{N}^k \to \mathbb{N}$ is a numerical function, then we call $\Delta H : \mathbb{N}^k \to \mathbb{N}$ the *first difference function of H* where

$$\Delta H(\underline{i}) := \sum_{\underline{0}\le\underline{l}=(l_1,\ldots,l_k)\le(1,\ldots,1)} (-1)^{|\underline{l}|} H(i_1 - l_1, \ldots, i_k - l_k),$$

where $H(\underline{j}) = 0$ if $\underline{j} \not\geq \underline{0}$. If $k = 1$, then our definition reverts to the classical definition. Indeed,

$$\Delta H(i) = \sum_{0 \leq l \leq 1} (-1)^l H(i-l) = (-1)^0 H(i-0) + (-1)^1 H(i-1) = H(i) - H(i-1).$$

If $k = 2$, then we write ΔH as

$$\Delta H(i,j) = H(i,j) - H(i-1,j) - H(i,j-1) + H(i-1,j-1).$$

We fix, once and for all, a *monomial ordering* $>$ on the monomials of R (see Definition 2.2.1 of Cox, *et al.* [**13**]). If $F \in R$, then the *leading monomial* of F, denoted $\mathrm{Lm}_>(F)$, is the monomial term in the support of F that is maximal with respect to $>$. The coefficient of $\mathrm{Lm}_>(F)$ is 1. The *leading coefficient* of F, denoted $\mathrm{Lc}_>(F)$, is the coefficient of $\mathrm{Lm}_>(F)$ in F. We set $\mathrm{Lt}_>(F) = \mathrm{Lc}_>(F)\,\mathrm{Lm}_>(F)$, and we call $\mathrm{Lt}_>(F)$ the *leading term* of F. If the monomial ordering is clear, then we shall simply write $\mathrm{Lt}(F)$. If $I \subseteq R$ is an ideal, then $\mathrm{Lt}(I) := (\{\mathrm{Lt}(F) \mid F \in I\})$ is the *leading term ideal* of I. If I is any $\mathbb{N}^k$-homogeneous ideal of the multi-graded ring R, then $H_{R/I} = H_{R/\mathrm{Lt}(I)}$ (see Caboara, *et al.* [**7**] and Stanley [**52**]).

The Hilbert functions of finitely generated $\mathbb{N}^1$-graded $\mathbf{k}$-algebras, i.e., rings of the form R/I where I is homogeneous, were originally characterized by Macaulay. To state the result we require some notation. Let i and a be positive integers. Then the *i-binomial expansion of a* is the unique expression

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$$

where $a_i > a_{i-1} > \cdots > a_j \geq j \geq 1$. The function ${}^{<i>} : \mathbb{N} \to \mathbb{N}$, sometimes called *Macaulay's function*, is defined by

$$a \longmapsto a^{<i>} = \binom{a_i+1}{i+1} + \binom{a_{i-1}+1}{i} + \cdots + \binom{a_j+1}{j+1}$$

where $a_i, a_{i-1}, \ldots, a_j$ are as in the i-binomial expansion of a.

Theorem 2.1.2. (Macaulay) *Let $H : \mathbb{N} \to \mathbb{N}$ be a numerical function. Then there exists a homogeneous ideal I in the $\mathbb{N}$-graded ring $R = \mathbf{k}[x_0, \ldots, x_n]$ such that $H_{R/I} = H$ if and only if $H(0) = 1$, $H(1) = n+1$, and $H(i+1) \leq H(i)^{<i>}$ for all $i \geq 1$.*

PROOF. See the paper of Macaulay [**35**] or Chapter 4 of Bruns and Herzog [**6**]. ∎

Remark 2.1.3. It remains an open problem to find an analog of Macaulay's result for $\mathbb{N}^k$-graded rings with $k \geq 2$. Aramova, *et al.* [**2**] give some results in this direction by demonstrating some necessary conditions in the case that $k = 2$. In Chapter 4, we will give a Macaulay-type result for $\mathbb{N}^2$-graded quotients of $\mathbf{k}[x_1, y_1, \ldots, y_m]$ and $\mathbb{N}^k$-graded quotients of $\mathbf{k}[x_1, \ldots, x_k]$.

2. The Multi-Projective Space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and Subsets of Points

The goal of this thesis is to understand sets of distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. In this section we set up the needed algebraic and geometric structures associated to sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. To define these points, we proceed in a manner analogous to the definition of points in $\mathbb{P}^n$. We begin by extending the classical definition of projective space to multi-projective space.

We define the *multi-projective space* $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ to be

$$\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} := \left\{ \begin{array}{l} ((a_{1,0}, \ldots, a_{1,n_1}), \ldots, (a_{k,0}, \ldots, a_{k,n_k})) \in \mathbf{k}^{n_1+1} \times \cdots \times \mathbf{k}^{n_k+1} \\ \text{with no } \underline{a}_i = (a_{i,0}, \ldots, a_{i,n_i}) = \underline{0} \end{array} \right\} \Big/ \sim$$

where $(\underline{a}_1, \ldots, \underline{a}_k) \sim (\underline{b}_1, \ldots, \underline{b}_k)$ if there exists non-zero $\lambda_1, \ldots, \lambda_k \in \mathbf{k}$ such that for all $i = 1, \ldots, k$

$$\underline{b}_i = (b_{i,0}, \ldots, b_{i,n_i}) = (\lambda_i a_{i,0}, \ldots, \lambda_i a_{i,n_i}) \text{ where } \underline{a}_i = (a_{i,0}, \ldots, a_{i,n_i}).$$

An element of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is called a *point*. We sometimes denote the equivalence class of $((a_{1,0}, \ldots, a_{1,n_1}), \ldots, (a_{k,0}, \ldots, a_{k,n_k}))$ by $[a_{1,0} : \cdots : a_{1,n_1}] \times \cdots \times [a_{k,0} : \cdots : a_{k,n_k}]$. It follows that $[a_{i,0} : \cdots : a_{i,n_i}]$ is a point of $\mathbb{P}^{n_i}$ for every i.

We give the polynomial ring $R = \mathbf{k}[x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{k,0}, \ldots, x_{k,n_k}]$ an $\mathbb{N}^k$-grading by setting $\deg x_{i,j} = e_i$, where e_i is the i^{th} standard basis vector of $\mathbb{N}^k$. If $F \in R$ is an $\mathbb{N}^k$-homogeneous element of degree $(d_1, \ldots, d_k)$ and $P = [a_{1,0} : \cdots : a_{1,n_1}] \times \cdots \times [a_{k,0} : \cdots : a_{k,n_k}]$ is a point of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, then

$$F(\lambda_1 a_{1,0}, \ldots, \lambda_2 a_{2,0}, \ldots, \lambda_k a_{k,0}, \ldots) = \lambda_1^{d_1} \lambda_2^{d_2} \cdots \lambda_k^{d_k} F(a_{1,0}, \ldots, a_{2,0}, \ldots, a_{k,0}, \ldots).$$

To say that F vanishes at a point of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is, therefore, a well-defined notion.

If T is any collection of $\mathbb{N}^k$-homogeneous elements of R, then define

$$\mathbf{V}(T) := \{P \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \mid F(P) = 0 \text{ for all } F \in T\}.$$

If I is an $\mathbb{N}^k$-homogeneous ideal of R, then $\mathbf{V}(I) = \mathbf{V}(T)$ where T is the set of all homogeneous elements of I. If $I = (F_1, \ldots, F_r)$, then $\mathbf{V}(I) = \mathbf{V}(F_1, \ldots, F_r)$.

The multi-projective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ can be endowed with a topology by defining the *closed sets* to be all subsets of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ of the form $\mathbf{V}(T)$ where T is a collection of $\mathbb{N}^k$-homogeneous elements of R. If Y is a subset of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ that is closed and irreducible with respect to this topology, then we say Y is a *multi-projective variety*, or simply, a *variety*.

If Y is any subset of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, then we set

$$\mathbf{I}(Y) := \{F \in R \mid F(P) = 0 \text{ for all } P \in Y\}.$$

The set $\mathbf{I}(Y)$ is an $\mathbb{N}^k$-homogeneous ideal of R. We call $\mathbf{I}(Y)$ the *$\mathbb{N}^k$-homogeneous ideal associated to Y*, or simply, the *ideal associated to Y*. If $Y \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, then we set $I_Y := \mathbf{I}(Y)$, and we call R/I_Y the *$\mathbb{N}^k$-homogeneous coordinate ring of Y*, or simply, the *coordinate ring of Y*. If H_{R/I_Y} is the Hilbert function of R/I_Y, then we sometimes write H_Y for H_{R/I_Y}, and we say H_Y is the *Hilbert function of Y*.

By adopting the proofs of the well known homogeneous case, it can be shown that

Proposition 2.2.1.

(*i*) *If $I_1 \subseteq I_2$ are $\mathbb{N}^k$-homogeneous ideals, then $\mathbf{V}(I_1) \supseteq \mathbf{V}(I_2)$.*
(*ii*) *If $Y_1 \subseteq Y_2$ are subsets of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, then $\mathbf{I}(Y_1) \supseteq \mathbf{I}(Y_2)$.*
(*iii*) *For any two subsets Y_1, Y_2 of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, $\mathbf{I}(Y_1 \cup Y_2) = \mathbf{I}(Y_1) \cap \mathbf{I}(Y_2)$.*

The $\mathbb{N}^k$-graded analog of the Nullstellensatz also holds in this context. Again, the proof follows as in the graded case.

Theorem 2.2.2. ($\mathbb{N}^k$-homogeneous Nullstellensatz) *If $I \subseteq R$ is an $\mathbb{N}^k$-homogeneous ideal and $F \in R$ is an $\mathbb{N}^k$-homogeneous polynomial with $\deg F > \underline{0}$ such that $F(P) = 0$ for all $P \in \mathbf{V}(I) \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, then $F^t \in I$ some $t > 0$.*

Set $\mathbf{m}_i := (x_{i,0}, x_{i,1}, \ldots, x_{i,n_i})$ for $i = 1, \ldots, k$. An $\mathbb{N}^k$-homogeneous ideal I of R is called *projectively irrelevant* if $\mathbf{m}_i^a \subseteq I$ for some $i \in \{1, \ldots, k\}$ and some positive integer a. An ideal $I \subseteq R$ is *projectively relevant* if it is not projectively irrelevant. By employing the $\mathbb{N}^k$-homogeneous Nullstellensatz, there is a one-to-one correspondence between the non-empty closed subsets of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and the $\mathbb{N}^k$-homogeneous ideals of R that are radical and projectively relevant. The correspondence is given by $Y \mapsto \mathbf{I}(Y)$ and $I \mapsto \mathbf{V}(I)$. This is analogous to the well known graded case. For the case $k = 2$, this correspondence can be found in Van der Waerden [**53**],[**54**]. Van der Waerden also asserts that for arbitrary k the results are analogous to the case $k = 2$.

Remark 2.2.3. Our construction of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and its subsets follows the classical definition of the projective space $\mathbb{P}^n$ as described, for example, in Section 1.2 of Hartshorne's book [**31**]. The paper of Van der Waerden [**53**] gives a construction similar to the approach we have given above. The multi-projective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ can also be constructed via the modern methods of schemes. For details, see the thesis of Vidal [**55**]. We will not use the language of schemes because we wish to focus on sets of distinct points. In the language of schemes, a set of distinct points is a reduced scheme, and hence, the classical approach is equivalent to the schematic approach.

We now restrict our attention to subsets $\mathbb{X}$ of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ such that $\mathbb{X}$ is a finite collection of distinct points. If $P \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, we define

$$I_P := \mathbf{I}(P) = \{F \in R \mid F(P) = 0\}.$$

If $I_{\mathbb{X}}$ is the ideal associated to $\mathbb{X}$, then the goal of this thesis is to study the Hilbert function of the coordinate ring of $\mathbb{X}$, that is, of the ring $R/I_{\mathbb{X}}$. The remainder of this section is a collection of results concerning the structure of the ring $R/I_{\mathbb{X}}$ that will be used throughout the thesis.

We begin by describing the generators of I_P. To do so, we require some results about Groebner bases. Our primary reference is Cox, Little, and O'Shea [**13**].

Theorem 2.2.4. (Division Algorithm, [**13**] Theorem 2.3.3) *Let* $R = \mathbf{k}[x_1, \ldots, x_n]$. *Fix a monomial ordering* $>$ *and let* $(F_1, \ldots, F_r)$ *be an ordered tuple of polynomials in* R. *Then* *every $F \in R$ can by written as*

$$F = G_1F_1 + \cdots + G_rF_r + H$$

where $G_i, H \in R$ and either $H = 0$, or H is a linear combination, with coefficients in $\mathbf{k}$, of monomials, none of which is divisible by any of $\mathrm{Lt}(F_1), \ldots, \mathrm{Lt}(F_r)$. *We call H a* remainder *of F on division by $(F_1, \ldots, F_r)$.*

Definition 2.2.5. Fix a monomial ordering. A finite subset $G = \{G_1, \ldots, G_r\}$ of an ideal I is said to be a *Groebner basis* if $\langle \mathrm{Lt}(G_1), \ldots, \mathrm{Lt}(G_r)\rangle = \langle \mathrm{Lt}(I)\rangle$.

Theorem 2.2.6. ([**13**] Theorem 2.6.6) *Let I be an ideal of $R = \mathbf{k}[x_1, \ldots, x_n]$. Then a basis $G = \{G_1, \ldots, G_r\}$ for I is a Groebner basis for I if and only if for all pairs $i \neq j$, the remainder on division of*

$$\frac{M}{\mathrm{Lt}(G_i)} \cdot G_i - \frac{M}{\mathrm{Lt}(G_j)} \cdot G_j$$

where $M = \mathrm{LCM}(\mathrm{Lm}(G_i), \mathrm{Lm}(G_j))$, by the tuple $(G_1, \ldots, G_r)$ is zero.

Proposition 2.2.7. *For any point $P \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, let I_P be the ideal associated to the point P. Then*

(*i*) *I_P is a prime ideal.*
(*ii*) *$I_P = (L_{1,1}, \ldots, L_{1,n_1}, L_{2,1}, \ldots, L_{2,n_2}, \ldots, L_{k,1}, \ldots, L_{k,n_k})$ where $\deg L_{i,j} = e_i$.*

PROOF. (*i*) If $FG \in I_P$, then $(FG)(P) = F(P)G(P) = 0$. Hence, either F or G must vanish at P, and thus is an element of I_P.

(*ii*) Suppose that $P = [a_{1,0} : \cdots : a_{1,n_1}] \times \cdots \times [a_{k,0} : \cdots : a_{k,n_k}]$. For each $i \in \{1, \ldots, k\}$ there exists $a_{i,j} \neq 0$. Assume for the moment that $a_{i,n_i} \neq 0$ for all i. We can then assume that $P = [a_{1,0} : \cdots : a_{1,n_1-1} : 1] \times [a_{2,0} : \cdots : a_{2,n_2-1} : 1] \times \cdots \times [a_{k,0} : \cdots : a_{k,n_k-1} : 1]$. Set

$$I := \left(\begin{array}{l} x_{1,0} - a_{1,0}x_{1,n_1},\ x_{1,1} - a_{1,1}x_{1,n_1}, \ldots, x_{1,n_1-1} - a_{1,n_1-1}x_{1,n_1}, \\ x_{2,0} - a_{2,0}x_{2,n_2}, x_{2,1} - a_{2,1}x_{2,n_2}, \ldots, x_{2,n_2-1} - a_{2,n_2-1}x_{2,n_2}, \\ \quad \vdots \\ x_{k,0} - a_{k,0}x_{k,n_k}, x_{k,1} - a_{k,1}x_{k,n_k}, \ldots, x_{k,n_k-1} - a_{k,n_k-1}x_{k,n_k} \end{array} \right).$$

Then $I \subseteq I_P$ because all of the generators of I vanish at P. If we show that $I_P \subseteq I$, then we will be finished because $\deg(x_{i,j} - a_{i,j}x_{i,n_i}) = e_i$. To accomplish this, we need two claims.

Claim 1. The generators of I are a Groebner basis for I.

Proof of the Claim. Let $x_{i,j} - a_{i,j}x_{i,n_i}$ and $x_{i',j'} - a_{i',j'}x_{i',n_{i'}}$ be two distinct generators of I. By Theorem 2.2.6, we need to check that the division of

$$S = \frac{x_{i,j}x_{i',j'}}{x_{i,j}} \cdot (x_{i,j} - a_{i,j}x_{i,n_i}) - \frac{x_{i,j}x_{i',j'}}{x_{i',j'}} \cdot (x_{i',j'} - a_{i',j'}x_{i',n_{i'}})$$

by the generators of I has a remainder of zero. A routine calculation will verify that

$$S = a_{i,j}x_{i',j'}x_{i,n_i} - a_{i',j'}x_{i,j}x_{i',n_{i'}} = a_{i,j}x_{i,n_i}(x_{i',j'} - a_{i',j'}x_{i',n_{i'}}) - a_{i',j'}x_{i',n_{i'}}(x_{i,j} - a_{i,j}x_{i,n_i}).$$

Hence, division of S by the generators of I results in a remainder of zero. □

Claim 2. I is a prime ideal.

Proof of the Claim. Suppose that $F, G \notin I$. Since $F, G \notin I$, the division of F and G by the generators of I yields

$$F = F' + F'' \quad \text{and} \quad G = G' + G''$$

where $F', G' \in I$ and $F'', G'' \notin I$. Furthermore, since the generators of I are a Groebner basis by Claim 1, F'', G'' must be polynomials in the indeterminates $x_{1,n_1}, x_{2,n_2}, \ldots, x_{k,n_k}$ alone. If $FG = F'G' + F''G' + F'G'' + F''G'' \in I$, then this would imply that $F''G'' \in I$. But the leading term of $F''G''$ is a monomial only in the indeterminates $x_{1,n_1}, \ldots, x_{k,n_k}$, and so $\mathrm{Lt}(F''G'') \notin \mathrm{Lt}(I)$. But this contradicts the fact $F''G'' \in I$. So $FG \notin I$ and hence, I is prime. □

We now demonstrate that $I_P \subseteq I$. Let $F \in I_P$. Because $\mathbf{V}(I) = \mathbf{V}(I_P) = P$, the Nullstellensatz (Theorem 2.2.2) implies that $F^t \in I$ for some positive integer t. By Claim 2, we then have $F \in I$, as desired.

To complete the proof of (ii), if $a_{i,n_i} = 0$, then there exists an integer $0 \leq j < n_i$ such that $a_{i,j} \neq 0$. We then repeat the above argument, but use $x_{i,j}$ instead of x_{i,n_i} to form the generators of I, and use a monomial ordering so that $x_{r,s} > x_{i,j}$ if $r > i$ and if $r = i$, then $x_{i,s} > x_{i,j}$ for all $s \in \{0, \ldots, \hat{j}, \ldots, n_i\}$. □

Definition 2.2.8. Let $\wp$ be a prime ideal of a ring S. The *height* of $\wp$, denoted $\mathrm{ht}_S(\wp)$, is the integer t such that we can find prime ideals $\wp_i$ of S such that $\wp = \wp_t \supsetneq \wp_{t-1} \supsetneq \cdots \supsetneq \wp_1 \supsetneq \wp_0$ and no longer such chain can be found. If S is a ring, then the *Krull dimension of* S, denoted $\mathrm{K\text{-}dim}\, S$, is the number $\mathrm{K\text{-}dim}\, S := \sup\{\mathrm{ht}_S(\wp) \mid \wp \text{ a prime ideal of } S\}$.

Proposition 2.2.9. *Let $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a set of s distinct points and suppose that I_{P_i} is the ideal associated to the point P_i. Then*

(*i*) $I_{\mathbb{X}} = I_{P_1} \cap I_{P_2} \cap \cdots \cap I_{P_s}$.
(*ii*) $\text{K-}\dim R/I_{\mathbb{X}} = k$.

Proof. Statement (*i*) is an immediate consequence of statement (*iii*) of Proposition 2.2.1.

For (*ii*), the grading of a ring S does not affect the Krull dimension. We therefore consider the multi-graded ring $R/I_{\mathbb{X}}$ as $\mathbb{N}^1$-graded.

For each $i \in \{1, \ldots, k\}$, it follows from Proposition 2.2.7 that the ideal I_{P_i}, as an $\mathbb{N}^1$-graded ideal in the $\mathbb{N}^1$-graded ring R, is generated by linear polynomials. Furthermore, the polynomials are also linearly independent. Thus, the variety $\mathbf{V}(I_{P_i}) \subseteq \mathbb{P}^{N-1}$ where $N = \sum_{j=1}^{k}(n_j + 1)$, is a linear variety. Moreover,

$$\dim \mathbf{V}(I_{P_i}) = N - 1 - \sum_{j=1}^{k} n_j = k - 1.$$

The ideal $I_{\mathbb{X}}$, as an $\mathbb{N}^1$-homogeneous ideal, corresponds to the variety $\mathbf{V}(I_{\mathbb{X}}) \subseteq \mathbb{P}^{N-1}$ where $\mathbf{V}(I_{\mathbb{X}}) = \bigcup_{j=1}^{s} \mathbf{V}(I_{P_j})$. Thus, $\dim \mathbf{V}(I_{\mathbb{X}}) = \max\left\{\dim \mathbf{V}(I_{P_j})\right\}_{j=1}^{s} = k - 1$. But then

$$\text{K-}\dim(R/I_{\mathbb{X}}) = \dim \mathbf{V}(I_{\mathbb{X}}) + 1 = k.$$

This is the desired result. □

For each $i \in \{1, \ldots, k\}$, we define the projective morphism $\pi_i : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \to \mathbb{P}^{n_i}$ by

$$[a_{1,0} : \cdots : a_{1,n_1}] \times \cdots \times [a_{i,0} : \cdots : a_{i,n_i}] \times \cdots \times [a_{k,0} : \cdots : a_{k,n_k}] \longmapsto [a_{i,0} : \cdots : a_{i,n_i}].$$

If $\mathbb{X}$ is a finite collection of distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, then $\pi_i(\mathbb{X}) \subseteq \mathbb{P}^{n_i}$ is the finite set of distinct i^{th} coordinates that appear in $\mathbb{X}$. The Hilbert function of $\pi_i(\mathbb{X})$ can be read from the Hilbert function of $\mathbb{X}$ as we show below.

Proposition 2.2.10. *Suppose that $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is a finite set of points with Hilbert function $H_{\mathbb{X}} := H_{R/I_{\mathbb{X}}}$. Fix an integer $i \in \{1, \ldots, k\}$. Then the sequence $H = \{h_j\}_{j \in \mathbb{N}}$,* *where* $h_j := H_{\mathbb{X}}(0,\ldots,j,\ldots,0)$ *with* j *in the* i^{th} *position, is the Hilbert function of* $\pi_i(\mathbb{X}) \subseteq \mathbb{P}^{n_i}$.

Proof. We will prove the statement for the case $i = 1$. The other cases follow similarly. Let $I = \mathbf{I}(\pi_1(\mathbb{X})) \subseteq S = \mathbf{k}[x_{1,0},\ldots,x_{1,n_1}]$. We wish to show that $(R/I_{\mathbb{X}})_{j,0\ldots,0} \cong (S/I)_j$ for all $j \in \mathbb{N}$. Since $R_{j,0,\ldots,0} \cong S_j$ for all $j \in \mathbb{N}$, it is enough to show that $(I_{\mathbb{X}})_{j,0,\ldots,0} \cong I_j$ for all $j \in \mathbb{N}$.

If P is a point of $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, then, by Proposition 2.2.7, the ideal associated to P is $I_P = (L_{1,1},\ldots,L_{1,n_1},L_{2,1},\ldots,L_{2,n_2},\ldots,L_{k,1},\ldots,L_{k,n_k})$ where $\deg L_{i,j} = e_i$. Let P' denote $\pi_1(P) \in \mathbb{P}^{n_1}$. Then the ideal associated to P' in S is $I_{P'} = (L_{1,1},\ldots,L_{1,n_1})$ where we consider $L_{1,1},\ldots,L_{1,n_1}$ as $\mathbb{N}^1$-graded elements of S. There is then an isomorphism of vector spaces $(I_P)_{j,0,\ldots,0} = (L_{1,1},\ldots,L_{1,n_1})_{j,0,\ldots,0} \cong (I_{P'})_j$ for each positive integer j.

Thus, if $\mathbb{X} = \{P_1,\ldots,P_s\}$, then $\pi_1(\mathbb{X}) = \{\pi_1(P_1),\ldots,\pi_1(P_s)\}$, and hence

$$(I_{\mathbb{X}})_{j,0,\ldots,0} = \bigcap_{i=1}^{s}(I_{P_i})_{j,0,\ldots,0} \cong \bigcap_{l=1}^{s}\left(I_{\pi_1(P_i)}\right)_j = I_j \quad \text{for all } j \in \mathbb{N}.$$

□

We end this section by giving some necessary conditions on the Hilbert function of a set of points $\mathbb{X}$ in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. We will first require the following lemmas.

Lemma 2.2.11. *Let* V *be a vector space over a field* $\mathbf{k}$ *with* $|\mathbf{k}| = \infty$. *Let* $V_i \subsetneq V$ *be any proper subvector space. Then* $V \neq \bigcup_{i=1}^{n} V_i$ *for any collection of* $n < \infty$ *subvector spaces.*

Proof. We consider the cases $n = 2$ and $n > 2$ separately. Suppose that there exists $V_1, V_2 \subsetneq V$ such that $V = V_1 \cup V_2$. Then there exists elements $x_1 \in V\backslash V_1$ and $x_2 \in V\backslash V_2$. Since $V = V_1 \cup V_2$, $x_2 \in V_1$ and $x_1 \in V_2$. But now consider the element $x_1 + x_2 \in V$. If $x_1 + x_2 \in V_1$, then $x_1 \in V_1$, which is a contradiction. Hence, $x_1 + x_2 \notin V_1$. Similarly, $x_1 + x_2 \notin V_2$. So $x_1 + x_2 \in V\backslash(V_1 \cup V_2)$, a contradiction.

Now suppose there are n subvector spaces $V_1,\ldots,V_n \subsetneq V$, with $2 < n < \infty$, such that $V = \bigcup_{i=1}^{n} V_i$. We assume that n is minimal, that is, there is no $j \in \{1,\ldots,n\}$ such that

$\bigcup_i V_i = \bigcup_{i \neq j} V_i$. Thus, for each i we can find an element $x_i \in V_i \backslash \left(\bigcup_{j \neq i} V_j \right)$. Because V is a vector space, $c_1x_1 + c_2x_2 \in V$ for all $c_1, c_2 \in \mathbf{k}$.

Claim. If $c_2 \neq 0$, then $c_1x_1 + c_2x_2 \notin V_1$. If $c_1 \neq 0$, then $c_1x_1 + c_2x_2 \notin V_2$.

Proof of the Claim. If $c_1x_1 + c_2x_2 \in V_1$, then $c_2x_2 \in V_1$. If $c_2 \neq 0$, then $x_2 \in V_1$ which contradicts our choice of x_2. The second statement is proved similarly. □

Let $X := \{x_1 + dx_2 \mid d \in \mathbf{k}\} \subseteq V$. By the above claim, no element of X can be in either V_1 or V_2. On the other hand, because $\mathbf{k}$ is infinite and $V = \bigcup_{i=1}^{n} V_i$, there exists a subvector space V_i, with $i \geq 3$, such that V_i contains an infinite number of elements of X. Thus, within this V_i there exists $x_1 + d_1x_2$ and $x_1 + d_2x_2$ with $d_1 \neq d_2$. It then follows that

$$\frac{1}{d_1 - d_2}(x_1 + d_1x_2 - x_1 - d_2x_2) = x_2 \in V_i.$$

But this contradicts our choice of x_2, and hence $V \neq \bigcup_{i=1}^{n} V_i$. □

Lemma 2.2.12. *Suppose $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is a finite set of distinct points. Then there exists a form $L \in R$ of degree e_1 such that $\overline{L}$ is a non-zero divisor in $R/I_{\mathbb{X}}$.*

Proof. The primary decomposition of $I_{\mathbb{X}}$ is $I_{\mathbb{X}} = \wp_1 \cap \cdots \cap \wp_s$ where $\wp_i$ is an $\mathbb{N}^k$-homogeneous prime ideal associated to a point of $\mathbb{X}$. The set of zero divisors of $R/I_{\mathbb{X}}$, denoted $\mathbf{Z}(R/I_{\mathbb{X}})$, are precisely the elements of $\mathbf{Z}(R/I_{\mathbb{X}}) = \bigcup_{i=1}^{s} \overline{\wp}_i$. We want to show $\mathbf{Z}(R/I_{\mathbb{X}})_{e_1} \subsetneq (R/I_{\mathbb{X}})_{e_1}$, or equivalently, $\bigcup_{i=1}^{s} (\wp_i)_{e_1} \subsetneq R_{e_1}$. By Proposition 2.2.7 it is clear that $(\wp_i)_{e_1} \subsetneq R_{e_1}$ for each $i = 1, \ldots, s$, and thus, by Lemma 2.2.11, the desired conclusion follows. □

Proposition 2.2.13. *Let $\mathbb{X}$ be a set of distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and suppose that $H_{\mathbb{X}}$ is the Hilbert function of $\mathbb{X}$. Then for all $\underline{i} = (i_1, \ldots, i_k) \in \mathbb{N}^k$ we have*

$$H_{\mathbb{X}}(\underline{i}) \leq H_{\mathbb{X}}(\underline{i} + e_j) \quad \text{for all } j = 1, \ldots, k.$$

Proof. We will only demonstrate that $H_{\mathbb{X}}(\underline{i}) \leq H_{\mathbb{X}}(\underline{i} + e_1) = H_{\mathbb{X}}(i_1 + 1, i_2, \ldots, i_k)$ since the other cases follow similarly. By Lemma 2.2.12 there exists a form $L \in R$ such that

$\deg L = e_1$ and $\overline{L}$ is a non-zero divisor is $R/I_{\mathbb{X}}$. Hence, for any $\underline{i} \in \mathbb{N}^k$, the multiplication map $(R/I_{\mathbb{X}})_{\underline{i}} \xrightarrow{\times \overline{L}} (R/I_{\mathbb{X}})_{\underline{i}+e_1}$ is an injective map of vector spaces. Therefore

$$H_{\mathbb{X}}(\underline{i}) = \dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{\underline{i}} \leq \dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{(i_1+1,i_2,\ldots,i_k)} = H_{\mathbb{X}}(\underline{i}+e_1).$$

□

Proposition 2.2.14. *Let $\mathbb{X}$ be a set of distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and suppose that $H_{\mathbb{X}}$ is the Hilbert function of $\mathbb{X}$. Fix an integer $j \in \{1,\ldots,k\}$. If $H_{\mathbb{X}}(\underline{i}) = H_{\mathbb{X}}(\underline{i}+e_j)$, for some $j \in \{1,\ldots,k\}$, then $H_{\mathbb{X}}(\underline{i}+e_j) = H_{\mathbb{X}}(\underline{i}+2e_j)$.*

PROOF. We will only consider the case that $j = 1$ since the other cases are proved similarly. By Lemma 2.2.12 there exists a form $L \in R$ such that $\deg L = e_1$ and $\overline{L}$ is a non-zero divisor in $R/I_{\mathbb{X}}$. Thus, for each $\underline{i} = (i_1,\ldots,i_k) \in \mathbb{N}^k$, we have the following short exact sequence of vector spaces:

$$0 \longrightarrow (R/I_{\mathbb{X}})_{\underline{i}} \xrightarrow{\times \overline{L}} (R/I_{\mathbb{X}})_{\underline{i}+e_1} \longrightarrow (R/(I_{\mathbb{X}},L))_{\underline{i}+e_1} \longrightarrow 0.$$

If $H_{\mathbb{X}}(\underline{i}) = H_{\mathbb{X}}(\underline{i}+e_1)$, then this implies that the morphism $\times\overline{L}$ is an isomorphism of vector spaces, and thus, $(R/(I_{\mathbb{X}},L))_{\underline{i}+e_1} = 0$. So $(R/(I_{\mathbb{X}},L))_{\underline{i}+2e_1} = 0$ as well. Hence, from the short exact sequence

$$0 \longrightarrow (R/I_{\mathbb{X}})_{\underline{i}+e_1} \xrightarrow{\times \overline{L}} (R/I_{\mathbb{X}})_{\underline{i}+2e_1} \longrightarrow (R/(I_{\mathbb{X}},L))_{\underline{i}+2e_1} \longrightarrow 0$$

we deduce that $(R/I_{\mathbb{X}})_{\underline{i}+e_1} \cong (R/I_{\mathbb{X}})_{\underline{i}+2e_1}$. □

Remark 2.2.15. The above proposition is a generalization of a result for points in $\mathbb{P}^n$ found in Geramita and Maroscia (cf. Proposition 1.1 (2) of [**18**]).

3. Some Results about Points in $\mathbb{P}^n$

Although sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ are our primary object of study, we occasionally need some results about sets of points in $\mathbb{P}^n$. In this section we collect the needed facts. Many of these results, if not all, are well known. However, for the convenience of the reader, we have included most of the proofs.

It follows from Proposition 2.2.7 that if P is a point of $\mathbb{P}^n$, then the ideal associated to P, say I_P, is a prime ideal that is generated by n linear forms. If $\mathbb{X} = \{P_1,\ldots,P_s\}$ is a collection of s distinct points, then the ideal associated to $\mathbb{X}$ is $I_{\mathbb{X}} = I_{P_1} \cap \cdots \cap I_{P_s}$. More properties of the ideal $I_{\mathbb{X}}$ are found in the paper of Geramita and Maroscia [**18**].

Proposition 2.3.1. *Let $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^n$, $R = \mathbf{k}[x_0, \ldots, x_n]$, and $I_{\mathbb{X}} \subseteq R$ the ideal of forms that vanish on $\mathbb{X}$. For any $j \geq 0$, let $\{m_1, \ldots, m_{\binom{n+j}{j}}\}$ be the $\binom{n+j}{j}$ monomials of R of degree j. Set*

$$M_j = \begin{bmatrix} m_1(P_1) & \cdots & m_{\binom{n+j}{j}}(P_1) \\ \vdots & & \vdots \\ m_1(P_s) & \cdots & m_{\binom{n+j}{j}}(P_s) \end{bmatrix}.$$

Then $\operatorname{rk} M_j = H_{\mathbb{X}}(j)$ where $H_{\mathbb{X}}$ is the Hilbert function of $R/I_{\mathbb{X}}$.

PROOF. To compute $H_{\mathbb{X}}(j)$, we need to determine the number of linearly independent forms of degree j that pass through $\mathbb{X}$. A general form of degree j looks like $F = c_1 m_1 + \cdots + c_{\binom{n+j}{j}} m_{\binom{n+j}{j}}$ where $c_i \in \mathbf{k}$. If $F(P_i) = 0$, we get a linear relation among the c_i's, namely

$$c_1 m_1(P_i) + \cdots + c_{\binom{n+j}{j}} m_{\binom{n+j}{j}}(P_i) = 0.$$

The elements of $(I_{\mathbb{X}})_j$ are given by solutions of the system of linear equations $F(P_i) = \cdots = F(P_s) = 0$. The matrix of this system of equations is

$$\begin{bmatrix} m_1(P_1) & \cdots & m_{\binom{n+j}{j}}(P_1) \\ \vdots & & \vdots \\ m_1(P_s) & \cdots & m_{\binom{n+j}{j}}(P_s) \end{bmatrix}$$

which is M_j. Now the number of linearly independent solutions $= \dim_{\mathbf{k}}(I_{\mathbb{X}})_j$. Hence

$$\dim_{\mathbf{k}}(I_{\mathbb{X}})_j \quad = \quad \#\text{columns of } M_j - \operatorname{rk} M_j = \binom{n+j}{j} - \operatorname{rk} M_j.$$

Since $\dim_{\mathbf{k}} R_j = \binom{n+j}{j}$, we have $H_{\mathbb{X}}(j) = \operatorname{rk} M_j$, as desired. □

Remark 2.3.2. This proposition is generalized to points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ in Proposition 3.2.2.

Proposition 2.3.3. *Suppose $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^n$, $R = \mathbf{k}[x_0, \ldots, x_n]$, and $I_{\mathbb{X}} \subseteq R$ is the ideal of forms that vanish on $\mathbb{X}$. Then there exists polynomials $F_1, \ldots, F_s$ of degree $s-1$*

such that $F_i(P_j) = 0$ if $i \neq j$, but $F_i(P_i) \neq 0$. Furthermore, the F_i are linearly independent modulo $I_{\mathbb{X}}$.

PROOF. This result is found in the proof of Theorem 3.4 of Sabourin [**49**]. □

Proposition 2.3.4. *Let $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^n$ be a collection of s distinct points. Let $I_{\mathbb{X}}$ be the ideal in $R = \mathbf{k}[x_0, \ldots, x_n]$ of forms that vanish on $\mathbb{X}$. If $H_{\mathbb{X}}$ is the Hilbert function of $R/I_{\mathbb{X}}$, then $H_{\mathbb{X}}(i) = s$ for all $i \geq s-1$.*

PROOF. Let M_j be the matrix from Proposition 2.3.1. It then follows that $H_{\mathbb{X}}(j) \leq s$ for all j because $\operatorname{rk} M_j \leq s$.

By Proposition 2.3.3 there exists s forms $F_1, \ldots, F_s$ of degree $s-1$ that are linearly independent modulo $I_{\mathbb{X}}$. But this implies that $\dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{s-1} = H_{\mathbb{X}}(s-1) \geq s$. Now let $i \in \mathbb{N}$ be such that $i > s-1$. Then, by Proposition 2.2.13, we have $s \leq H_{\mathbb{X}}(s-1) \leq H_{\mathbb{X}}(i) \leq s$. Hence, the conclusion holds. □

Remark 2.3.5. The main result of Chapter 3 is a generalization of the above result to sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

Proposition 2.3.6. *Let $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^n$, $R = \mathbf{k}[x_0, \ldots, x_n]$, and $I_{\mathbb{X}} \subseteq R$ the ideal of forms vanishing at $\mathbb{X}$. Suppose $H_{\mathbb{X}}(j) = k$. Then we can find a subset $\mathbb{X}' \subseteq \mathbb{X}$ of k elements, say $\mathbb{X}' = \{P_1, \ldots, P_k\}$ (after a possible reordering), such that there exist k forms $G_1, \ldots, G_k$ of degree j with the property that for every $1 \leq l \leq k$, $G_i(P_l) = 0$ if $i \neq l$, and $G_i(P_i) \neq 0$.*

PROOF. Let $\{m_1, \ldots, m_{\binom{n+j}{j}}\}$ be the $\binom{n+j}{j}$ monomials of degree j in R. By Proposition 2.3.1 the matrix

$$M_j = \begin{bmatrix} m_1(P_1) & \cdots & m_{\binom{n+j}{j}}(P_1) \\ \vdots & & \vdots \\ m_1(P_s) & \cdots & m_{\binom{n+j}{j}}(P_s) \end{bmatrix}$$

has rank $\operatorname{rk} M_j = H_{\mathbb{X}}(j) = k$. Without loss of generality, we can assume that the first k rows are linearly independent. So, let $\mathbb{X}' = \{P_1, \ldots, P_k\} \subseteq \mathbb{X}$, and let

$$M'_j = \begin{bmatrix} m_1(P_1) & \cdots & m_{\binom{n+j}{j}}(P_1) \\ \vdots & & \vdots \\ m_1(P_k) & \cdots & m_{\binom{n+j}{j}}(P_k) \end{bmatrix}.$$

Fix an $i \in \{1, \ldots, k\}$ and let $\mathbb{X}'_i = \{P_1, \ldots, \widehat{P}_i, \ldots, P_k\}$. If we remove the i^{th} row of M'_j, then the rank of the resulting matrix decreases by one. Since the rank of the new matrix is equal to the Hilbert function of $\mathbb{X}'_i$, it follows that $\dim_{\mathbf{k}}(I_{\mathbb{X}'})_j + 1 = \dim_{\mathbf{k}}(I_{\mathbb{X}'_i})_j$. Thus, there exists an element $G_i \in (I_{\mathbb{X}'_i})_j$ such that G_i passes through the points of $\mathbb{X}'_i$ but not P_i. We repeat this argument for each $i \in \{1, \ldots, k\}$ to get the desired forms. □

Remark 2.3.7. This result is generalized in Proposition 3.2.3 to points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

Proposition 2.3.8. *Let $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^1$. Then*

$$H_{\mathbb{X}}(i) = \begin{cases} i+1 & 0 \leq i \leq s-1 \\ s & i \geq s \end{cases}.$$

Proof. If $P_i \in \mathbb{X}$, then $I_{P_i} = (L_{P_i}) \subseteq R = \mathbf{k}[x_0, x_1]$ where $\deg L_{P_i} = 1$. Since each I_{P_i} is a principal ideal, $I_{\mathbb{X}} = \bigcap_{i=1}^{s} I_{P_i} = (L_{P_1} \cdots L_{P_s})$. Because $I_{\mathbb{X}}$ is a principal ideal, $R_{j-s} \cong (I_{\mathbb{X}})_j$ via the map $F \mapsto F \cdot (L_{P_1} \cdots L_{P_s})$. But then

$$\dim_{\mathbf{k}}(R/I_{\mathbb{X}})_i = \dim_{\mathbf{k}} R_i - \dim_{\mathbf{k}} R_{i-s} = \begin{cases} i+1 & \text{if } i \leq s-1 \\ i+1-(i-s+1) = s & \text{if } i \geq s \end{cases}.$$

This computes $H_{\mathbb{X}}$ for all i. □

The Hilbert functions of finite sets of distinct points in $\mathbb{P}^n$ have been characterized. To state the result, we require a definition.

Definition 2.3.9. A homogeneous ideal $I \subseteq R = \mathbf{k}[x_0, \ldots, x_n]$ is an *artinian ideal* if any of the following equivalent statements hold:

(*i*) $\text{K-}\dim R/I = 0$.

(*ii*) $\sqrt{I} = (x_0, \ldots, x_n)$.

(*iii*) $x_j^t \in I$ for some positive integer t and all $0 \leq j \leq n$.

(*iv*) $H_{R/I}(i) = 0$ for all $i \gg 0$.

A ring $S = R/I$ is a *graded artinian quotient* if the homogeneous ideal I is an artinian ideal.

Proposition 2.3.10. *Let $H : \mathbb{N} \to \mathbb{N}$ be a numerical function. Then H is the Hilbert function of a set of distinct points in $\mathbb{P}^n$ if and only if the first difference function $\Delta H : \mathbb{N} \to \mathbb{N}$, where $\Delta H(i) := H(i) - H(i-1)$ for all $i \in \mathbb{N}$, is the Hilbert function of a graded artinian quotient of $\mathbf{k}[x_1, \ldots, x_n]$. ($H(i) = 0$ if $i < 0$.)*

PROOF. See Geramita, Maroscia, and Roberts [**19**], or Corollary 2.5 of Geramita, Gregory, and Roberts [**16**]. □

Remark 2.3.11. The result of Geramita, Maroscia, and Roberts [**19**] is a generalization of earlier results due to Maroscia [**37**] and Roberts [**47**]. The original formulation of Proposition 2.3.10 in [**19**] makes no reference to artinian quotients, but instead classifies the Hilbert function of points via the properties of ΔH. The connection between Hilbert functions of points and artinian quotients appears to be first made in Geramita, Gregory, and Roberts [**16**].

4. Resolutions and Projective Dimension

For this section we assume that $R = \mathbf{k}[x_0, \ldots, x_n]$ is an $\mathbb{N}^1$-graded ring. In Chapter 5 we will require some results about the resolution and projective dimension of an R-module. In this section we will recall the necessary results and definitions. Cox, Little, and O'Shea [**12**], Geramita and Small [**22**], and Weibel [**56**] are our main references for this material.

Definition 2.4.1. An R-module M is a *graded R-module* if (i) the module M has a direct sum decomposition $M = \bigoplus_{i \in \mathbb{Z}} M_i$ where each M_i is an additive abelian group, and (ii) the decomposition of M in (i) is compatible with the multiplication of R in the sense that $R_i M_j \subseteq M_{i+j}$ for all $i \in \mathbb{N}$ and all $j \in \mathbb{Z}$.

If I is a homogeneous ideal of R, then I can be viewed as a graded R-module if we take $I_i = 0$ for $i < 0$. Similarly, for any homogeneous ideal $I \subseteq R$, the ring R/I is a graded R-module. If M is any R-graded module, and d is any integer, we let $M(d)$ denote the direct sum $M(d) = \bigoplus_{i \in \mathbb{Z}} M_{d+i}$. Then $M(d)$ is also a graded R-module, and it is sometimes referred to as the *twisted graded module.*

Definition 2.4.2. Let M and N be graded R-modules. A homomorphism $\varphi : M \to N$ is said to be a *graded homomorphism of degree d* if $\varphi(M_i) \subseteq N_{i+d}$ for all $i \in \mathbb{Z}$.

Definition 2.4.3. If M is a graded R-module, then a *graded free resolution of M* is an exact sequence of the form

$$\cdots \longrightarrow \mathcal{F}_2 \xrightarrow{\varphi_2} \mathcal{F}_1 \xrightarrow{\varphi_1} \mathcal{F}_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

where each $\mathcal{F}_i = R(-d_{i,1}) \oplus \cdots \oplus R(-d_{i,p_i})$ for some integers $d_{i,1}, \ldots, d_{i,p_i}$, and each φ_i is a graded homomorphism of degree zero. If there exists an l such that $\mathcal{F}_l \neq 0$, but $\mathcal{F}_{l+i} = 0$ for all $i \geq 1$, then we say the resolution is *finite* of *length l.*

If M is a finitely generated graded R-module, then a classical theorem of Hilbert, specifically, the *Hilbert Syzygy Theorem* (see Theorem 6.3.8 of Cox, *et al.* [**12**]), says that there exists some graded free resolution of M of length at most $n+1$, the number of indeterminates of R. We give a name to the minimal length in the next definition.

Definition 2.4.4. Let M be a finitely generated graded R-module. We say that the *projective dimension* of M is d, and we write $\operatorname{proj.dim}_R M = d$ if (i) there is a graded free resolution of M

$$0 \longrightarrow \mathcal{F}_d \xrightarrow{\varphi_d} \cdots \xrightarrow{\varphi_3} \mathcal{F}_2 \xrightarrow{\varphi_2} \mathcal{F}_1 \xrightarrow{\varphi_1} \mathcal{F}_0 \xrightarrow{\varphi_0} M \longrightarrow 0,$$

and (ii) there is no shorter graded free resolution. If R is clear, then we may simply write $\operatorname{proj.dim} M = d$.

Example 2.4.5. Let $d \in \mathbb{Z}$ and let M be the graded R-module $M = R(d)$. Then $\operatorname{proj.dim}_R M = 0$ because we have the graded free resolution $0 \to R(d) \to M \to 0$.

Suppose that $0 \to M_1 \xrightarrow{\phi} M_2 \to M_3 \to 0$ is an exact sequence of graded R-modules. Furthermore, suppose that we know the graded free resolutions of M_1 and M_2, i.e.,

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \mathcal{F}_2 & \xrightarrow{\varphi_2} & \mathcal{F}_1 & \xrightarrow{\varphi_1} & \mathcal{F}_0 & \xrightarrow{\varphi_0} & M_1 & \longrightarrow & 0 \\ \cdots & \longrightarrow & \mathcal{G}_2 & \xrightarrow{\psi_2} & \mathcal{G}_1 & \xrightarrow{\psi_1} & \mathcal{G}_0 & \xrightarrow{\psi_0} & M_2 & \longrightarrow & 0. \end{array}$$

The *mapping cone construction* enables us to build a graded free resolution of M_3 from the graded free resolutions of M_1 and M_2. The main idea behind this construction is as follows: for each $i \in \mathbb{N}$, define $\mathcal{H}_i = \mathcal{F}_{i-1} \oplus \mathcal{G}_i$ where $F_{-1} = 0$. Then, from the maps $\phi : M_1 \to M_2$, φ_{i-1}, and ψ_i we can construct a map $\delta_i : \mathcal{H}_i \to \mathcal{H}_{i-1}$. (We omit the details behind the construction of the maps δ_i since we do not require the maps.) Then the sequence

$$\cdots \xrightarrow{\delta_3} \mathcal{H}_2 \xrightarrow{\delta_2} \mathcal{H}_1 \xrightarrow{\delta_1} \mathcal{H}_0 \xrightarrow{\delta_0} M_3 \longrightarrow 0$$

is a graded free resolution of M_3. See Section 1.5 of Weibel [**56**] for more details.

The following proposition gives some well known properties about the projective dimension of a finitely generated graded R-module that we will require in this thesis.

Proposition 2.4.6.

(*i*) *If I is a homogeneous ideal of R, then* $\operatorname{proj.dim}_R(R/I) = \operatorname{proj.dim}_R I + 1$

(*ii*) *If $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of graded R-modules with degree zero maps, and if* $\operatorname{proj.dim}_R M_1 < \operatorname{proj.dim}_R M_2$, *then* $\operatorname{proj.dim}_R M_2 = \operatorname{proj.dim}_R M_3$.

(*iii*) *If M and N are graded R-modules, then*

$$\operatorname{proj.dim}_R(M \oplus N) = \max\{\operatorname{proj.dim}_R(M), \operatorname{proj.dim}_R(N)\}.$$

Proof. To prove (*i*), one needs to consider the cases that I is free and I is not free separately. The case that I is free is shown in Example 17.9(2) of Geramita and Small [**22**]. The other case is a consequence of Theorem 18.1 of [**22**]. Statement (*ii*) is one part of Theorem 18.1 of [**22**]. Statement (*iii*) is a standard exercise of most homological algebra texts. See, for example, Exercise 4.1.3 of Weibel [**56**]. □

5. Some Combinatorics

We end this chapter by introducing some definitions and results from combinatorics. These facts are used in Chapter 5. Our main reference is Ryser [**48**].

Definition 2.5.1. A tuple $\lambda = (\lambda_1, \ldots, \lambda_r)$ of positive integers is a *partition* of an integer s if $\sum \lambda_i = s$ and $\lambda_i \geq \lambda_{i+1}$ for every i. We write $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash s$. The *conjugate* of λ is the tuple $\lambda^* = (\lambda_1^*, \ldots, \lambda_{\lambda_1}^*)$ where $\lambda_i^* = \#\{\lambda_j \in \lambda \mid \lambda_j \geq i\}$. Furthermore, $\lambda^* \vdash s$.

Definition 2.5.2. To any partition $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash s$ we can associate the following diagram: on an $r \times \lambda_1$ grid, place λ_1 points on the first line, λ_2 points on the second, and so on. The resulting diagram is called the *Ferrers diagram* of λ.

Example 2.5.3. Suppose $\lambda = (4, 4, 3, 1) \vdash 12$. Then the Ferrers diagram is

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & & & \end{array}$$

The conjugate of λ can be read off the Ferrers diagram by counting the number of dots in each column as opposed to each row. In this example $\lambda^* = (4, 3, 3, 2)$.

Definition 2.5.4. Let $\lambda = (\lambda_1, \ldots, \lambda_t)$ and $\delta = (\delta_1, \ldots, \delta_r)$ be two partitions of s. If one partition is longer, we add zeros to the shorter one until they have the same length. We say λ *majorizes* δ, written $\lambda \trianglerighteq \delta$, if

$$\lambda_1 + \cdots + \lambda_i \geq \delta_1 + \cdots + \delta_i \text{ for } i = 1, \ldots, \max\{t, r\}.$$

Majorization induces a partial ordering on the set of all partitions of s.

Definition 2.5.5. A matrix A of size $m \times n$ is a $(0,1)$-*matrix* if all of its entries are either zero or one. The sum of the entries in column j will be denoted by α_j, and the sum of the entries of row i will be denoted by β_i. We call the vector $\alpha_A = (\alpha_1, \ldots, \alpha_n)$ the *column sum vector* and the vector $\beta_A = (\beta_1, \ldots, \beta_m)$ the *row sum vector*.

Given a $(0,1)$-matrix, we can rearrange the rows and columns so that α_A (respectively, β_A) has the property $\alpha_i \geq \alpha_{i+1}$ (respectively $\beta_i \geq \beta_{i+1}$) for every i. Observe that α_A and β_A are partitions of the number of 1's in A. Unless otherwise specified, we assume that any $(0,1)$-matrix has been rearranged into this form.

If α and β are any two partitions of s, then we define

$$\mathcal{M}(\alpha,\beta) := \{(0,1)\text{-matrices } A \mid \alpha_A = \alpha, \beta_A = \beta\}.$$

It is not evident that such a set is nonempty. The following result is a classical result, due to Gale and Ryser, that gives us a criterion to determine if $\mathcal{M}(\alpha,\beta) = \emptyset$.

Theorem 2.5.6. (Gale-Ryser Theorem) *Let α and β be two partitions of s. The class $\mathcal{M}(\alpha,\beta)$ is nonempty if and only if $\alpha^* \trianglerighteq \beta$.*

PROOF. See Theorem 1.1 in Chapter 6 of Ryser's book [**48**]. □

The proof given by Ryser to demonstrate that $\alpha^* \trianglerighteq \beta$ implies $\mathcal{M}(\alpha,\beta)$ is nonempty is a constructive proof. We illustrate this construction with an example.

Example 2.5.7. Let $\alpha = (3,3,2,1)$ and $\beta = (3,3,1,1,1)$. A routine check will show that $\alpha^* = (4,3,2) \trianglerighteq (3,3,1,1,1) = \beta$. We construct a $(0,1)$-matrix with column sum vector α and row sum vector β. Let M be an empty $|\beta| \times |\alpha| = 5 \times 4$ matrix. On top of the j^{th} column place the integer α_j. Beside the i^{th} row, place β_i 1's. For our example we have

$$\begin{array}{r} \\ 1\,1\,1 \\ 1\,1\,1 \\ 1 \\ 1 \\ 1 \end{array} \begin{array}{c} \begin{matrix} 3 & 3 & 2 & 1 \end{matrix} \\ \left(\begin{matrix} & & & \\ \\ \\ \\ \\ \end{matrix} \right) \end{array}.$$

Starting with the rightmost column, we see that this column needs one 1. Move a 1 from the row with the largest number of 1's to this column and fill the remainder of the column with zeroes. If two rows have the same number of ones, we take the first such row. So, after one step,

$$\begin{array}{ll} & \begin{array}{cccc} 3 & 3 & 2 & 1 \end{array} \\ \begin{array}{l} 1\ 1\ \not{1} \\ 1\ 1\ 1 \\ 1 \\ 1 \\ 1 \end{array} & \begin{pmatrix} & & & 1 \\ & & & 0 \\ & & & 0 \\ & & & 0 \\ & & & 0 \end{pmatrix} \end{array}.$$

We now repeat the above procedure on the next to last column. We place two 1's in the third column, taking our 1's from the rows that contain the largest number of ones. Thus, our example becomes

$$\begin{array}{ll} & \begin{array}{cccc} 3 & 3 & 2 & 1 \end{array} \\ \begin{array}{l} 1\ \not{1}\ \not{1} \\ 1\ 1\ \not{1} \\ 1 \\ 1 \\ 1 \end{array} & \begin{pmatrix} & & 1 & 1 \\ & & 1 & 0 \\ & & 0 & 0 \\ & & 0 & 0 \\ & & 0 & 0 \end{pmatrix} \end{array}.$$

We continue the above method for the remaining columns to get

$$\begin{array}{ll} & \begin{array}{cccc} 3 & 3 & 2 & 1 \end{array} \\ \begin{array}{l} \not{1}\ \not{1}\ \not{1} \\ \not{1}\ \not{1}\ \not{1} \\ \not{1} \\ \not{1} \\ \not{1} \end{array} & \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

It follows immediately that our matrix is an element of $\mathcal{M}(\alpha, \beta)$. The proof of the Gale-Ryser Theorem shows that if $\alpha^* \trianglerighteq \beta$, then this algorithm always works.

CHAPTER 3

The Border of a Hilbert Function of a Set of Points

The goal of this chapter is to generalize the following result for sets of points in $\mathbb{P}^n$ to sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

Proposition 3.0.1. *Let $\mathbb{X} \subseteq \mathbb{P}^n$ be a collection of s distinct points. Let $I_{\mathbb{X}}$ be the homogeneous ideal in $R = \mathbf{k}[x_0, \ldots, x_n]$ of forms that vanish on $\mathbb{X}$. If $H_{\mathbb{X}}$ is the Hilbert function of $R/I_{\mathbb{X}}$, then $H_{\mathbb{X}}(i) = s$ for all $i \geq s-1$.*

This proposition was proved in Chapter 2 (cf. Proposition 2.3.4). We observe that the above proposition has two consequences for the Hilbert function of a set of points in $\mathbb{P}^n$. First, to calculate $H_{\mathbb{X}}(i)$ for all $i \in \mathbb{N}$, we need to calculate $H_{\mathbb{X}}(i)$ for only a finite number of i. Second, numerical information about $\mathbb{X}$, in this case the cardinality of $\mathbb{X}$, tells us for which i we need to compute $H_{\mathbb{X}}(i)$ in order to determine the Hilbert function for all $i \in \mathbb{N}$.

The generalization for a set of distinct points $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ that we present in this chapter will also have analogous consequences. Specifically, we demonstrate that to compute $H_{\mathbb{X}}(\underline{i})$ for all $\underline{i} \in \mathbb{N}^k$, we need to compute $H_{\mathbb{X}}(\underline{i})$ for only a finite number of $\underline{i} \in \mathbb{N}^k$. The other values of $H_{\mathbb{X}}(\underline{i})$ are then easily determined from our generalization of Proposition 3.0.1. Moreover, the $\underline{i}$ for which we need to compute $H_{\mathbb{X}}(\underline{i})$ can be determined from the combinatorial properties of $\mathbb{X}$.

The proof of the generalization of Proposition 3.0.1 for sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, while similar to the proof for points in $\mathbb{P}^n$, is more complicated notationally. Hence, to prevent the reader from drowning in notation, we have decided to consider the case of points in $\mathbb{P}^n \times \mathbb{P}^m$ separately so that the reader can follow the idea of the proof. Then, for completeness, we give a proof for sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. In both cases, the proof is a variation on the original proof for Proposition 3.0.1.

In this chapter we also define the *border* of the Hilbert function for a set of points. The border of the Hilbert function divides the values of the Hilbert function into two sets. The first set, which consists of an infinite number of elements, contains the values of the Hilbert function which depend only upon our description of the eventual growth of the Hilbert function. The second set, which is finite, is the set of values at which the Hilbert function has not attained this eventual growth, and therefore, must be calculated.

In the final section we introduce sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ in *generic position*. Since the Hilbert function of a set of points has a border, we deduce that there are only a finite number of distinct Hilbert functions for sets of s points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. This leads us to calculate an expected Hilbert function. Proceeding as in the case of points in $\mathbb{P}^n$, we say that those points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ which satisfy this expected Hilbert function are in generic position.

1. The Border of the Hilbert Function for Points in $\mathbb{P}^n \times \mathbb{P}^m$

Let $\mathbb{X} \subseteq \mathbb{P}^n \times \mathbb{P}^m$ be a collection of s distinct points. Let $I_{\mathbb{X}}$ be the bihomogeneous ideal associated to $\mathbb{X}$ in the bigraded ring $R = \mathbf{k}[x_0, \ldots, x_n, y_0, \ldots, y_m]$ where $\deg x_i = (1,0)$ and $\deg y_i = (0,1)$.

If $\pi_1 : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^n$ is the projection morphism, then $\pi_1(\mathbb{X}) \subseteq \mathbb{P}^n$ is a collection of $t \leq s$ points. The set $\pi_1(\mathbb{X})$ is the set of distinct first coordinates that appear in $\mathbb{X}$. For every $P_i \in \pi_1(\mathbb{X})$, we have

$$\pi_1^{-1}(P_i) = \left\{P_i \times Q_{i_1}, \ldots, P_i \times Q_{i_{\alpha_i}}\right\} \subseteq \mathbb{X}.$$

That is, $\pi_1^{-1}(P_i)$ is the subset of $\mathbb{X}$ consisting of all the points which have P_i as its first coordinate. Observe that $\alpha_i := |\pi_1^{-1}(P_i)| \geq 1$ for all $P_i \in \pi_1(\mathbb{X})$. We also note that the sets $\pi_1^{-1}(P_i)$ partition $\mathbb{X}$, specifically,

$$\mathbb{X} = \bigcup_{P_i \in \pi_1(\mathbb{X})} \pi_1^{-1}(P_i).$$

Let $\pi_2 : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ be the other projection map. For each $P_i \in \pi_1(\mathbb{X})$, the set

$$Q_{P_i} := \pi_2(\pi_1^{-1}(P_i)) = \left\{Q_{i_j} \,\middle|\, P_i \times Q_{i_j} \in \pi_1^{-1}(P_i)\right\}$$

is a collection of α_i distinct points in $\mathbb{P}^m$. With the above notation, we have

Proposition 3.1.1. *Let $\mathbb{X} \subseteq \mathbb{P}^n \times \mathbb{P}^m$ be a set of s distinct points and suppose that $\pi_1(\mathbb{X}) = \{P_1, \ldots, P_t\}$ is the set of $t \leq s$ distinct first coordinates in $\mathbb{X}$. Fix any integer*

$j \geq 0$. Then, for all integers $l \geq t-1 = |\pi_1(\mathbb{X})|-1$,

$$\dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{l,j} = \sum_{P_i \in \pi_1(\mathbb{X})} H_{Q_{P_i}}(j)$$

where $H_{Q_{P_i}}$ is the Hilbert function of the set of points $Q_{P_i} = \pi_2(\pi_1^{-1}(P_i)) \subseteq \mathbb{P}^m$.

PROOF. Fix a $j \in \mathbb{N}$ and set

$$(*) = \sum_{P_i \in \pi_1(\mathbb{X})} H_{Q_{P_i}}(j).$$

We will first show that $\dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{l,j} \leq (*)$ for all $l \in \mathbb{N}$. Let $\{X_1, \ldots, X_{\binom{n+l}{l}}\}$ be all the monomials of degree $(l,0)$ in R and let $\{Y_1, \ldots, Y_{\binom{m+j}{j}}\}$ be the $\binom{m+j}{j}$ monomials of degree $(0,j)$ in R. For any $l \in \mathbb{N}$, an element $L \in R_{l,j}$ has the form

$$\begin{aligned} L &= \left(c_{1,1}X_1 + \cdots + c_{1,\binom{n+l}{l}}X_{\binom{n+l}{l}}\right)Y_1 + \left(c_{2,1}X_1 + \cdots + c_{2,\binom{n+l}{l}}X_{\binom{n+l}{l}}\right)Y_2 + \cdots \\ &\quad + \left(c_{\binom{m+j}{j},1}X_1 + \cdots + c_{\binom{m+j}{j},\binom{n+l}{l}}X_{\binom{n+l}{l}}\right)Y_{\binom{m+j}{j}} \end{aligned}$$

with coefficients $c_{i_1,i_2} \in \mathbf{k}$. By setting $A_i := c_{i,1}X_1 + \cdots + c_{i,\binom{n+l}{l}}X_{\binom{n+l}{l}}$ for $i = 1, \ldots, \binom{m+j}{j}$, we can rewrite L as $L = A_1Y_1 + A_2Y_2 + \cdots + A_{\binom{m+j}{j}}Y_{\binom{m+j}{j}}$.

Claim. Each subset $\pi_1^{-1}(P_i) \subseteq \mathbb{X}$ puts at most $H_{Q_{P_i}}(j)$ linear restrictions on the forms of $R_{l,j}$ that pass through $\mathbb{X}$.

Proof of the Claim. Suppose $\pi_1^{-1}(P_i) = \{P_i \times Q_{i_1}, \ldots, P_i \times Q_{i_{\alpha_i}}\} \subseteq \mathbb{X}$, and thus the set $Q_{P_i} = \{Q_{i_1}, \ldots, Q_{i_{\alpha_i}}\} \subseteq \mathbb{P}^m$. If $L \in R_{l,j}$ vanishes at the s points of $\mathbb{X}$, then it vanishes on $\pi_1^{-1}(P_i)$, and thus

$$\begin{aligned} L(P_i \times Q_{i_1}) &= A_1(P_i)Y_1(Q_{i_1}) + \cdots + A_{\binom{m+j}{j}}(P_i)Y_{\binom{m+j}{j}}(Q_{i_1}) = 0 \\ &\vdots \\ L(P_i \times Q_{i_{\alpha_i}}) &= A_1(P_i)Y_1(Q_{i_{\alpha_i}}) + \cdots + A_{\binom{m+j}{j}}(P_i)Y_{\binom{m+j}{j}}(Q_{i_{\alpha_i}}) = 0. \end{aligned}$$

We can rewrite this system of equations as

$$\begin{bmatrix} Y_1(Q_{i_1}) & \cdots & Y_{\binom{m+j}{j}}(Q_{i_1}) \\ \vdots & & \vdots \\ Y_1(Q_{i_{\alpha_i}}) & \cdots & Y_{\binom{m+j}{j}}(Q_{i_{\alpha_i}}) \end{bmatrix} \begin{bmatrix} A_1(P_i) \\ \vdots \\ A_{\binom{m+j}{j}}(P_i) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The maximum number of linear restrictions $\pi_1^{-1}(P_i)$ can place on the elements of $R_{l,j}$ that pass through $\mathbb{X}$ is simply the rank of the matrix on the left. But by Proposition 2.3.1, the rank of this matrix is equal to $H_{Q_{P_i}}(j)$. This gives the desired result. □

By the claim, for each $P_i \in \pi_1(\mathbb{X})$, the set $\pi_1^{-1}(P_i)$ imposes at most $H_{Q_{P_i}}(j)$ linear restrictions on the forms of $R_{l,j}$ that pass through $\mathbb{X}$. Hence, the set $\mathbb{X}$ imposes at most $\sum_{P_i \in \pi_1(\mathbb{X})} H_{Q_{P_i}}(j)$ linear restrictions. It then follows that

$$\dim_{\mathbf{k}}(I_{\mathbb{X}})_{l,j} \geq \dim_{\mathbf{k}} R_{l,j} - (*),$$

or equivalently, $\dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{l,j} \leq (*)$ for all integers l.

We now show that if $l = t-1$, then the bound $(*)$ is attained. The set $\pi_1(\mathbb{X}) = \{P_1, \ldots, P_t\}$ is a subset of $\mathbb{P}^n$. By Proposition 2.3.3, there exist t forms $F_{P_1}, \ldots, F_{P_t}$ of degree $t-1$ in $\mathbf{k}[x_0, \ldots, x_n]$ such that $F_{P_i}(P_i) \neq 0$ and $F_{P_i}(P_j) = 0$ if $i \neq j$. Under the natural inclusion $\mathbf{k}[x_0, \ldots, x_n] \hookrightarrow \mathbf{k}[x_0, \ldots, x_n, y_0, \ldots, y_m]$ we can consider the forms $F_{P_1}, \ldots, F_{P_t}$ as forms of degree $(t-1, 0)$.

For our fixed j, we partition the points of $\pi_1(\mathbb{X})$ as follows:

$$S_k := \left\{ P_i \in \pi_1(\mathbb{X}) \;\middle|\; H_{Q_{P_i}}(j) = k \right\} \qquad \text{for } k = 1, \ldots, \binom{m+j}{j}.$$

Pick a point $P_i \in \pi_1(\mathbb{X})$ and suppose that $P_i \in S_k$. Furthermore, suppose that $Q_{P_i} = \{Q_{i,1}, \ldots, Q_{i,\alpha_i}\} \subseteq \mathbb{P}^m$. Proposition 2.3.6 implies the existence of a subset $Q \subseteq Q_{P_i}$ of k elements, say $Q = \{Q_{i,1}, \ldots, Q_{i,k}\}$ after a possible reordering, such that for every $Q_{i,d} \in Q$ we can find a form $G_{Q_{i,d}} \in \mathbf{k}[y_0, \ldots, y_m]$ of degree j such that $G_{Q_{i,d}}(Q_{i,d}) \neq 0$ but $G_{Q_{i,d}}(Q_{i,e}) = 0$ if $Q_{i,e} \neq Q_{i,d}$ and $Q_{i,e} \in Q$. Under the natural inclusion $\mathbf{k}[y_0, \ldots, y_m] \hookrightarrow \mathbf{k}[x_0, \ldots, x_n, y_0, \ldots, y_m]$ we consider each $G_{Q_{i,d}}$ as an element of R of degree $(0, j)$. With this P_i and subset $Q \subseteq Q_{P_i}$ we construct the set of forms

$$\mathcal{B}_{P_i} := \left\{ F_{P_i} G_{Q_{i,1}}, F_{P_i} G_{Q_{i,2}}, \ldots, F_{P_i} G_{Q_{i,k}} \right\}.$$

We observe that each $F_{P_i} G_{Q_{i,d}} \notin I_{\mathbb{X}}$ for $d = 1, \ldots, k$ because it fails to vanish at $P_i \times Q_{i,d}$. Moreover, each element of $\mathcal{B}_{P_i}$ has degree $(t-1, j)$ and $|\mathcal{B}_{P_i}| = H_{Q_{P_i}}(j) = k$.

We repeat the above construction for every $P_i \in \pi_1(\mathbb{X})$ and let

$$\mathcal{B} = \bigcup_{P_i \in \pi_1(\mathbb{X})} \mathcal{B}_{P_i}.$$

Claim. The elements of $\mathcal{B}$ are linearly independent modulo $I_{\mathbb{X}}$.

Proof of the Claim. It is enough to show that for each $F_{P_i}Q_{i,l} \in \mathcal{B}$, the point $P_i \times Q_{i,l}$ does not vanish at $F_{P_i}Q_{i,l}$ but vanishes at all the other elements $F_{P_{i'}}Q_{i',l'} \in \mathcal{B}$. But this follows immediately from our construction of $\mathcal{B}$. □

By this claim, the elements of $\mathcal{B}$ are linearly independent modulo $I_{\mathbb{X}}$ of degree $(t-1, j)$, and hence, $\dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{t-1,j} \geq |\mathcal{B}|$. Because

$$|\mathcal{B}| = \sum_{P_i \in \pi_1(\mathbb{X})} |\mathcal{B}_{P_i}| = \sum_{P_i \in \pi_1(\mathbb{X})} H_{Q_{P_i}}(j) = (*)$$

we have $\dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{t-1,j} \geq (*)$. Combining this inequality with our earlier inequality gives $\dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{t-1,j} = (*)$.

To complete the proof we note that we can always pick a form L in R of degree $(1,0)$ that does not vanish at any $P_i \in \pi_1(\mathbb{X})$. Then for any $l > t-1$, the set

$$L^{l-t-1}\mathcal{B} = \{L^{l-t-1}B \mid B \in \mathcal{B}\}$$

is a set of $(*)$ elements of degree (l, j) that is linearly independent modulo $I_{\mathbb{X}}$. □

Remark 3.1.2. Fix an integer $j \geq 0$, let $\pi_1(\mathbb{X}) = \{P_1, \ldots, P_t\}$, and let $(*) = \sum_{P_i \in \pi_1(\mathbb{X})} H_{Q_{P_i}}(j)$. It is sometimes useful to note that $(*)$ is equal to

$$(*) = \#\left\{P_i \in \pi_1(\mathbb{X}) \;\middle|\; H_{Q_{P_i}}(j) = 1\right\} + \cdots + \binom{m+j}{j} \#\left\{P_i \in \pi_1(\mathbb{X}) \;\middle|\; H_{Q_{P_i}}(j) = \binom{m+j}{j}\right\},$$

and that $(*)$ is also equal to

$$\begin{aligned} \#\left\{P_i \in \pi_1(\mathbb{X}) \;\middle|\; H_{Q_{P_i}}(j) \geq 1\right\} &+ \#\left\{P_i \in \pi_1(\mathbb{X}) \;\middle|\; H_{Q_{P_i}}(j) \geq 2\right\} + \cdots \\ &+ \#\left\{P_i \in \pi_1(\mathbb{X}) \;\middle|\; H_{Q_{P_i}}(j) \geq \binom{m+j}{j}\right\}. \end{aligned}$$

Corollary 3.1.3. *Let $\mathbb{X} \subseteq \mathbb{P}^n \times \mathbb{P}^m$ be a set of s distinct points and suppose that $\pi_1(\mathbb{X}) = \{P_1, \ldots, P_t\}$ is the set of $t \leq s$ distinct first coordinates in $\mathbb{X}$. Then*

(i) for all integers $l \geq t-1$, $H_{\mathbb{X}}(l, 0) = t$.
(ii) for $j \gg 0$ and $l \geq t-1$, $H_{\mathbb{X}}(l, j) = s$.
(iii) $H_{\mathbb{X}}(t-1, 1) - H_{\mathbb{X}}(t-1, 0) \geq \#\left\{P_i \in \pi_1(\mathbb{X}) \;\middle|\; \alpha_i = |\pi_1^{-1}(P_i)| \geq 2\right\}$.
(iv) $\Delta H_{\mathbb{X}}(i, j) = 0$ if $i \geq t$.

PROOF. For (i) it is sufficient to note that for every $P_i \in \pi_1(\mathbb{X})$, $H_{Q_{P_i}}(0) = 1$. To prove (ii) we observe that for every Q_{P_i}, the Hilbert function $H_{Q_{P_i}}(j) = |Q_{P_i}| = \alpha_i$ for $j \gg 0$. Since $\sum_{i=1}^{t} \alpha_i = s$, the result follows by applying Proposition 3.1.1.

To prove (iii) we use Remark 3.1.2 to show that

$$H_{\mathbb{X}}(t-1,1) - H_{\mathbb{X}}(t-1,0) \geq \#\{P_i \in \pi_1(\mathbb{X}) \mid H_{Q_{P_i}}(1) \geq 2\}.$$

The only $P_i \in \pi_1(\mathbb{X})$ that are not counted in the set on the right are those in the set $\{P_i \in \pi_1(\mathbb{X}) \mid H_{Q_{P_i}}(1) = 1\}$. Since $H_{Q_{P_i}}(1) = 1$ if and only if $\alpha_i = |\pi_1^{-1}(P_i)| = 1$, the result now follows.

For (iv), we recall that

$$\Delta H_{\mathbb{X}}(i,j) = H_{\mathbb{X}}(i,j) - H_{\mathbb{X}}(i,j-1) - H_{\mathbb{X}}(i-1,j) + H_{\mathbb{X}}(i-1,j-1).$$

If $i \geq t$, then $H_{\mathbb{X}}(i,j) = H_{\mathbb{X}}(i-1,j)$ and $H_{\mathbb{X}}(i,j-1) = H_{\mathbb{X}}(i-1,j-1)$. A simple calculation will then show that $\Delta H_{\mathbb{X}}(i,j) = 0$. □

If we partition the set of points $\mathbb{X} \subseteq \mathbb{P}^n \times \mathbb{P}^m$ with respect to the second coordinates rather than the first coordinates, then we can derive a result identical to Proposition 3.1.1. Indeed, let $\pi_2(\mathbb{X}) = \{Q_1, \ldots, Q_r\}$ be the $r \leq s$ distinct second coordinates of $\mathbb{X}$. For every $Q_i \in \pi_2(\mathbb{X})$, the subset

$$\pi_2^{-1}(Q_i) = \left\{P_{i_1} \times Q_i, \ldots, P_{i_{\beta_i}} \times Q_i\right\} \subseteq \mathbb{X}$$

contains the $\beta_i := |\pi_2^{-1}(Q_i)|$ points of $\mathbb{X}$ whose second coordinate is Q_i. Define P_{Q_i} to be the set of points $P_{Q_i} := \pi_1(\pi_2^{-1}(Q_i)) = \{P_{i_1}, \ldots, P_{i_{\beta_i}}\} \subseteq \mathbb{P}^n$. With this notation we have

Proposition 3.1.4. *Let $\mathbb{X} \subseteq \mathbb{P}^n \times \mathbb{P}^m$ be a set of s distinct points, and suppose that $\pi_2(\mathbb{X}) = \{Q_1, \ldots, Q_r\}$ is the set of $r \leq s$ distinct second coordinates in $\mathbb{X}$. Fix any integer $i \geq 0$. Then, for all integers $l \geq r - 1 = |\pi_2(\mathbb{X})| - 1$,*

$$\dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{i,l} = \sum_{Q_k \in \pi_2(\mathbb{X})} H_{P_{Q_k}}(i)$$

where $H_{P_{Q_k}}$ is the Hilbert function of the set of points $P_{Q_k} \subseteq \mathbb{P}^n$.

In this context Corollary 3.1.3 becomes

Corollary 3.1.5. *Let $\mathbb{X} \subseteq \mathbb{P}^n \times \mathbb{P}^m$ be a set of s distinct points. Let $\pi_2(\mathbb{X}) = \{Q_1, \ldots, Q_r\}$ be the set of $r \leq s$ distinct second coordinates in $\mathbb{X}$. Then*

(i) *for all integers $l \geq r-1$, $H_{\mathbb{X}}(0,l) = r$.*
(ii) *for $i \gg 0$ and $l \geq r-1$, $H_{\mathbb{X}}(i,l) = s$.*
(iii) $H_{\mathbb{X}}(1,r-1) - H_{\mathbb{X}}(0,r-1) \geq \#\left\{Q_i \in \pi_2(\mathbb{X}) \;\middle|\; \beta_i = |\pi_2^{-1}(Q_i)| \geq 2\right\}$.
(iv) $\Delta H_{\mathbb{X}}(i,j) = 0$ *if* $j \geq r$.

Remark 3.1.6. Corollary 3.1.3 (*ii*) and Corollary 3.1.5 (*ii*) can be combined to show that $H_{\mathbb{X}}(i,j) = s$ for all $(i,j) \geq (t-1, r-1)$.

By combining Propositions 3.1.1 and 3.1.4 we derive a generalization of Proposition 2.3.4 for sets of points in $\mathbb{P}^n \times \mathbb{P}^m$. We state this generalization formally as a corollary.

Corollary 3.1.7. *Let $\mathbb{X} \subseteq \mathbb{P}^n \times \mathbb{P}^m$ be a set of s distinct points, and set $t = |\pi_1(\mathbb{X})|$ and $r = |\pi_2(\mathbb{X})|$. Then*

$$H_{\mathbb{X}}(i,j) = \begin{cases} s & \text{if } (i,j) \geq (t-1, r-1) \\ H_{\mathbb{X}}(t-1,j) & \text{if } i \geq t-1 \text{ and } j < r-1 \\ H_{\mathbb{X}}(i,r-1) & \text{if } j \geq r-1 \text{ and } i < t-1 \end{cases}.$$

Remark 3.1.8. This corollary has the two desired properties that we wanted our generalization to have. First, to compute $H_{\mathbb{X}}(i,j)$ for all (i,j) we need to compute the Hilbert function for only a finite number of $(i,j) \in \mathbb{N}^2$, specifically, those $(i,j) \leq (t-1, r-1)$. Second, since $t = |\pi_1(\mathbb{X})|$ and $r = |\pi_2(\mathbb{X})|$, the values for which we need to compute the Hilbert function can be determined solely from numerical information about $\mathbb{X}$.

The above corollary implies that if we know $H_{\mathbb{X}}(t-1,j)$ for $j = 0, \ldots, r-1$ and $H_{\mathbb{X}}(i, r-1)$ for $i = 0, \ldots, t-1$, then we know the Hilbert function for all but a finite number of $(i,j) \in \mathbb{N}^2$. This observation motivates the next definition.

Definition 3.1.9. Suppose $\mathbb{X} \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is a set of s distinct points and let $t = |\pi_1(\mathbb{X})|$ and $r = |\pi_2(\mathbb{X})|$. Suppose that $H_{\mathbb{X}}$ is the Hilbert function of $\mathbb{X}$. We call the tuples

$$B_C := (H_{\mathbb{X}}(t-1,0), H_{\mathbb{X}}(t-1,1), \ldots, H_{\mathbb{X}}(t-1,r-1))$$

and

$$B_R := (H_{\mathbb{X}}(0, r-1), H_{\mathbb{X}}(1, r-1), \ldots, H_{\mathbb{X}}(t-1, r-1))$$

the *eventual column vector* and *eventual row vector* respectively. Let $B_{\mathbb{X}} := (B_C, B_R)$. We call $B_{\mathbb{X}}$ the *border* of the Hilbert function of $\mathbb{X} \subseteq \mathbb{P}^n \times \mathbb{P}^m$.

The term border is inspired by the "picture" of $H_{\mathbb{X}}$ if we visualize $H_{\mathbb{X}}$ as an infinite matrix $(m_{i,j})$ where $m_{i,j} = H_{\mathbb{X}}(i,j)$. Indeed, if $\mathbb{X} \subseteq \mathbb{P}^n \times \mathbb{P}^m$ with $|\pi_1(\mathbb{X})| = t$ and $|\pi_2(\mathbb{X})| = r$, then

$$H_{\mathbb{X}} = \begin{bmatrix} & & & \mathbf{m_{0,r-1}} & m_{0,r-1} & \cdots \\ & * & & \mathbf{m_{1,r-1}} & m_{1,r-1} & \cdots \\ & & & \vdots & \vdots & \\ \mathbf{m_{t-1,0}} & \mathbf{m_{t-1,1}} & \cdots & \mathbf{m_{t-1,r-1} = s} & s & \cdots \\ m_{t-1,0} & m_{t-1,1} & \cdots & s & s & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \ddots \end{bmatrix}.$$

The bold numbers form the border $B_{\mathbb{X}}$. The entries $m_{i,j}$ with $(i,j) \leq (t-1, r-1)$ are either "inside" the border or entries of the border, and need to be determined. Entries with $(i,j) \geq (t,0)$ or $(i,j) \geq (0,r)$ are "outside" the border. These values depend only on values in the border $B_{\mathbb{X}}$.

The term eventual column vector is given to $B_C = (m_{t-1,0}, \ldots, m_{t-1,r-1})$ because the i^{th} entry of B_C is the value at which the $(i-1)^{th}$ column stabilizes (because our indexing starts at zero). We christen B_R the eventual row vector to capture a similar result about the rows. From Corollaries 3.1.3 and 3.1.5 we always have

$$B_C = (t, m_{t-1,1}, \ldots, m_{t-1,r-2}, s) \quad \text{and} \quad B_R = (r, m_{1,r-1}, \ldots, m_{t-2,r-1}, s).$$

Moreover, part (iii) of Corollaries 3.1.3 and 3.1.5 also impose a necessary condition on $m_{t-1,1}$ and $m_{1,r-1}$.

A natural question about the entries in the border arises:

Question 3.1.10. *What tuples can be the border of a set of points in $\mathbb{P}^n \times \mathbb{P}^m$?*

We would like to classify those tuples that arise as the border of a set of points in $\mathbb{P}^n \times \mathbb{P}^m$. If we can answer the above question, then we will have a new necessary condition on the Hilbert functions of sets of points in $\mathbb{P}^n \times \mathbb{P}^m$. In Chapter 5 we answer Question 3.1.10

for sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Our answer depends on forging a link between sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ and the theory of $(0,1)$-matrices.

2. The Border of the Hilbert Function for Points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$

In this section we demonstrate a generalization of Proposition 3.0.1 for sets of points $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with $k \geq 2$. The proof is similar to the proof of Proposition 3.1.1. We begin by introducing some suitable notation.

Suppose $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is a collection of s distinct points. Let $I_{\mathbb{X}}$ be the $\mathbb{N}^k$-homogeneous ideal associated to $\mathbb{X}$ in the $\mathbb{N}^k$-graded ring $R = \mathbf{k}[x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{k,0}, \ldots, x_{k,n_k}]$ where $\deg x_{i,j} = e_i$, the i^{th} standard basis vector of $\mathbb{N}^k$.

Let $\pi_1 : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \to \mathbb{P}^{n_1}$ be the projection morphism. The image of $\pi_1(\mathbb{X})$ in $\mathbb{P}^{n_1}$ is a collection of $t_1 := |\pi_1(\mathbb{X})| \leq s$ points. The set of points $\pi_1(\mathbb{X})$ is the set of distinct first coordinates that appear in $\mathbb{X}$. For every $P_i \in \pi_1(\mathbb{X})$, we have

$$\pi_1^{-1}(P_i) = \left\{P_i \times Q_{i_1}, \ldots, P_i \times Q_{i_{\alpha_i}}\right\} \subseteq \mathbb{X}$$

where $Q_{i_j} \in \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_k}$. Set $\alpha_i := |\pi_1^{-1}(P_i)| \geq 1$ for all $P_i \in \pi_1(\mathbb{X})$. Note that the sets $\pi_1^{-1}(P_i)$ partition $\mathbb{X}$. Let $\pi_{2,\ldots,k} : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \to \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_k}$ be the projection morphism. For each $P_i \in \pi_1(\mathbb{X})$, the set

$$Q_{P_i} := \pi_{2,\ldots,k}(\pi_1^{-1}(P_i)) = \left\{Q_{i_j} \,\middle|\, P_i \times Q_{i_j} \in \pi_1^{-1}(P_i)\right\}$$

is a collection of α_i distinct points in $\mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_k}$.

If $\underline{j} = (j_1, j_2, \ldots, j_k) \in \mathbb{N}^k$, then we sometimes write $\underline{j}$ as $(j_1, \underline{j}^{'})$ where $\underline{j}^{'} = (j_2, \ldots, j_k) \in \mathbb{N}^{k-1}$. Also, recall that we write $R_{j_1,\ldots,j_k}$ for $R_{(j_1,\ldots,j_k)}$. If $\underline{j} = (j_1, \underline{j}^{'})$, then we denote $R_{(j_1,\underline{j}^{'})} = R_{\underline{j}}$ by $R_{j_1,\underline{j}^{'}}$. With the above notation, we have

Proposition 3.2.1. *Let $\mathbb{X}$ be a set of s distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with $k \geq 2$, and suppose that $\pi_1(\mathbb{X}) = \{P_1, \ldots, P_{t_1}\}$ is the set of $t_1 \leq s$ distinct first coordinates in $\mathbb{X}$. Fix any tuple $\underline{j} = (j_2, \ldots, j_k) \in \mathbb{N}^{k-1}$. Set $N = N(\underline{j}) := \binom{n_2+j_2}{j_2}\binom{n_3+j_3}{j_3} \cdots \binom{n_k+j_k}{j_k}$. Then, for all integers $l \geq t_1 - 1 = |\pi_1(\mathbb{X})| - 1$,*

$$\dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{l,\underline{j}} \quad = \quad \sum_{P_i \in \pi_1(\mathbb{X})} H_{Q_{P_i}}(\underline{j})$$

where $H_{Q_{P_i}}$ is the Hilbert function of the set of points $Q_{P_i} \subseteq \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_k}$.

To prove this proposition we require the following two results.

Proposition 3.2.2. *Let $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a set of s distinct points. For any $\underline{j} = (j_1, \ldots, j_k) \in \mathbb{N}^k$, let $\{m_1, \ldots, m_N\}$ be the $N = \binom{n_1+j_1}{j_1}\binom{n_2+j_2}{j_2}\cdots\binom{n_k+j_k}{j_k}$ monomials of R of degree $\underline{j}$. Set*

$$M_{\underline{j}} = \begin{bmatrix} m_1(P_1) & \cdots & m_N(P_1) \\ \vdots & & \vdots \\ m_1(P_s) & \cdots & m_N(P_s) \end{bmatrix}.$$

Then $\operatorname{rk} M_{\underline{j}} = H_{\mathbb{X}}(\underline{j})$ where $H_{\mathbb{X}}$ is the Hilbert function of $R/I_{\mathbb{X}}$.

PROOF. To compute $H_{\mathbb{X}}(\underline{j})$, we need to determine the number of linearly independent forms of degree $\underline{j}$ that pass through $\mathbb{X}$. An element of R of degree $\underline{j}$ has the form $F = c_1m_1 + \cdots + c_Nm_N$ where $c_i \in \mathbf{k}$. If $F(P_i) = 0$, we get a linear relation among the c_i's, namely, $c_1m_1(P_i) + \cdots + c_Nm_N(P_i) = 0$. The elements of $(I_{\mathbb{X}})_{\underline{j}}$ are given by the solutions of the system of linear equations $F(P_i) = \cdots = F(P_s) = 0$. We can rewrite this system of equations as

$$\begin{bmatrix} m_1(P_1) & \cdots & m_N(P_1) \\ \vdots & & \vdots \\ m_1(P_s) & \cdots & m_N(P_s) \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The matrix on the left is $M_{\underline{j}}$. Now the number of linearly independent solutions is equal to $\dim_{\mathbf{k}}(I_{\mathbb{X}})_{\underline{j}}$, and hence,

$$\dim_{\mathbf{k}}(I_{\mathbb{X}})_{\underline{j}} = \#\text{columns of } M_{\underline{j}} - \operatorname{rk} M_{\underline{j}} = N - \operatorname{rk} M_{\underline{j}}.$$

Since $\dim_{\mathbf{k}} R_{\underline{j}} = N$, we have $H_{\mathbb{X}}(\underline{j}) = \operatorname{rk} M_{\underline{j}}$, as desired. □

Proposition 3.2.3. *Let $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and suppose that $H_{\mathbb{X}}(\underline{j}) = h$. Then we can find a subset $\mathbb{X}' \subseteq \mathbb{X}$ of h elements, say $\mathbb{X}' = \{P_1, \ldots, P_h\}$ (after a possible reordering), such that there exists h forms $G_1, \ldots, G_h$ of degree $\underline{j}$ with the property that for every $1 \leq l \leq h$, $G_i(P_l) = 0$ if $i \neq l$, and $G_i(P_i) \neq 0$.*

Proof. Let $\{m_1, \ldots, m_N\}$ be the $N = \binom{n_1+j_1}{j_1} \cdots \binom{n_k+j_k}{j_k}$ monomials of degree $\underline{j}$ in R. By Proposition 3.2.2 the matrix

$$M_{\underline{j}} = \begin{bmatrix} m_1(P_1) & \cdots & m_N(P_1) \\ \vdots & & \vdots \\ m_1(P_s) & \cdots & m_N(P_s) \end{bmatrix}$$

has rank $\operatorname{rk} M_{\underline{j}} = H_{\mathbb{X}}(\underline{j}) = h$. Without loss of generality, we can assume that the first h rows are linearly independent. So, let $\mathbb{X}' = \{P_1, \ldots, P_h\} \subseteq \mathbb{X}$, and let

$$M'_{\underline{j}} = \begin{bmatrix} m_1(P_1) & \cdots & m_N(P_1) \\ \vdots & & \vdots \\ m_1(P_h) & \cdots & m_N(P_h) \end{bmatrix}.$$

Fix an $i \in \{1, \ldots, h\}$ and let $\mathbb{X}'_i = \{P_1, \ldots, \widehat{P}_i, \ldots, P_h\}$. If we remove the i^{th} row of $M'_{\underline{j}}$, then the rank of the resulting matrix decreases by one. Since the rank of the new matrix is equal to the Hilbert function of $\mathbb{X}'_i$, it follows that $\dim_{\mathbf{k}}(I_{\mathbb{X}'})_{\underline{j}} + 1 = \dim_{\mathbf{k}}(I_{\mathbb{X}'_i})_{\underline{j}}$. Thus, there exists an element $G_i \in (I_{\mathbb{X}'_i})_{\underline{j}}$ such that G_i passes through the points of $\mathbb{X}'_i$ but not through P_i. We repeat this argument for each $i \in \{1, \ldots, h\}$ to get the desired forms. □

Proof. (of Proposition 3.2.1) Fix a $\underline{j} = (j_2, \ldots, j_k) \in \mathbb{N}^{k-1}$, let $N = N(\underline{j})$, and set

$$(*) \quad = \sum_{P_i \in \pi_1(\mathbb{X})} H_{Q_{P_i}}(\underline{j})$$

We will first show that $\dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{l,\underline{j}} \leq (*)$ for all $l \in \mathbb{N}$. Let $\{X_1, \ldots, X_{\binom{n_1+l}{l}}\}$ be all the monomials of degree $(l, \underline{0})$ in R and let $\{Y_1, \ldots, Y_N\}$ be the N monomials of degree $(0, \underline{j})$ in R. For any $l \in \mathbb{N}$, a general form $L \in R_{l,\underline{j}}$ looks like

$$\begin{aligned} L \quad = \quad & \left(c_{1,1}X_1 + \cdots + c_{1,\binom{n_1+l}{l}}X_{\binom{n_1+l}{l}}\right)Y_1 + \left(c_{2,1}X_1 + \cdots + c_{2,\binom{n_1+l}{l}}X_{\binom{n_1+l}{l}}\right)Y_2 + \cdots \\ & + \left(c_{N,1}X_1 + \cdots + c_{N,\binom{n_1+l}{l}}X_{\binom{n_1+l}{l}}\right)Y_N \end{aligned}$$

with coefficients $c_{i,j} \in \mathbf{k}$. By setting $A_i := c_{i,1}X_1 + \cdots + c_{i,\binom{n_1+l}{l}}X_{\binom{n_1+l}{l}}$ for $i = 1, \ldots, N$, we can rewrite L as $L = A_1Y_1 + A_2Y_2 + \cdots + A_NY_N$.

Claim. Each subset $\pi_1^{-1}(P_i) \subseteq \mathbb{X}$ puts at most $H_{Q_{P_i}}(\underline{j})$ linear restrictions on the forms of $R_{l,\underline{j}}$ that pass through $\mathbb{X}$.

Proof of the Claim. Suppose $\pi_1^{-1}(P_i) = \{P_i \times Q_{i_1}, \ldots, P_i \times Q_{i_{\alpha_i}}\} \subseteq \mathbb{X}$, and hence, the set $Q_{P_i} = \{Q_{i_1}, \ldots, Q_{i_{\alpha_i}}\} \subseteq \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_k}$. If $L \in R_{l,\underline{j}}$ vanishes at the s points of $\mathbb{X}$, then it vanishes on $\pi_1^{-1}(P_i)$, and thus

$$\begin{aligned} L(P_i \times Q_{i_1}) &= A_1(P_i)Y_1(Q_{i_1}) + \cdots + A_N(P_i)Y_N(Q_{i_1}) = 0 \\ &\vdots \\ L(P_i \times Q_{i_{\alpha_i}}) &= A_1(P_i)Y_1(Q_{i_{\alpha_i}}) + \cdots + A_N(P_i)Y_N(Q_{i_{\alpha_i}}) = 0. \end{aligned}$$

We can rewrite this system of equations as

$$\begin{bmatrix} Y_1(Q_{i_1}) & \cdots & Y_N(Q_{i_1}) \\ \vdots & & \vdots \\ Y_1(Q_{i_{\alpha_i}}) & \cdots & Y_N(Q_{i_{\alpha_i}}) \end{bmatrix} \begin{bmatrix} A_1(P_i) \\ \vdots \\ A_N(P_i) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The maximum number of linear restrictions $\pi_1^{-1}(P_i)$ can place on the forms of $R_{l,\underline{j}}$ that pass through $\mathbb{X}$ is simply the rank of the matrix on the left. By Proposition 3.2.2 the rank of this matrix is equal to $H_{Q_{P_i}}(\underline{j})$. □

By the claim, for each $P_i \in \pi_1(\mathbb{X})$, the set $\pi_1^{-1}(P_i)$ imposes at most $H_{Q_{P_i}}(\underline{j})$ linear restrictions on the elements of $R_{l,\underline{j}}$ that pass through $\mathbb{X}$. Hence the set $\mathbb{X}$ imposes at most $\sum_{P-i \in \pi_1(\mathbb{X})} H_{Q_{P_i}}(\underline{j})$ linear restrictions. By summing over all $P_i \in \pi_1(\mathbb{X})$, we have

$$\dim_{\mathbf{k}}(I_{\mathbb{X}})_{l,\underline{j}} \geq \dim_{\mathbf{k}} R_{l,\underline{j}} - (*),$$

or equivalently, $\dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{l,\underline{j}} \leq (*)$ for all integers l.

We will now show that if $l = t_1 - 1$, then the bound $(*)$ is attained. The set $\pi_1(\mathbb{X}) = \{P_1, \ldots, P_{t_1}\}$ is a subset of $\mathbb{P}^{n_1}$. By Proposition 2.3.3, there exist t_1 forms $F_{P_1}, \ldots, F_{P_{t_1}}$ of degree $t_1 - 1$ in $\mathbf{k}[x_0, \ldots, x_{n_1}]$ such that $F_{P_i}(P_i) \neq 0$ and $F_{P_i}(P_j) = 0$ if $i \neq j$. Under the natural inclusion $\mathbf{k}[x_0, \ldots, x_{n_1}] \hookrightarrow R$ we can consider the forms $F_{P_1}, \ldots, F_{P_{t_1}}$ as forms of R of degree $(t_1, \underline{0})$.

For our fixed $\underline{j}$, we partition the points of $\pi_1(\mathbb{X})$ as follows:

$$S_h := \left\{ P_i \in \pi_1(\mathbb{X}) \,\middle|\, H_{Q_{P_i}}(\underline{j}) = h \right\} \qquad \text{for } h = 1, \ldots, N.$$

Pick a point $P_i \in \pi_1(\mathbb{X})$ and suppose that $P_i \in S_h$ and suppose that $Q_{P_i} = \{Q_{i,1}, \ldots, Q_{i,\alpha_i}\} \subseteq \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_k}$. By using Proposition 3.2.3, there exists a subset $Q \subseteq Q_{P_i}$ of h elements, say $Q = \{Q_{i,1}, \ldots, Q_{i,h}\}$ after a possible reordering, such that for every $Q_{i,d} \in$

Q there exists a form $G_{Q_{i,d}} \in \mathbf{k}[x_{2,0},\ldots,x_{2,n_2},\ldots,x_{k,0},\ldots,x_{k,n_k}]$ of degree $\underline{j}$ such that $G_{Q_{i,d}}(Q_{i,d}) \neq 0$ but $G_{Q_{i,d}}(Q_{i,e}) = 0$ if $Q_{i,e} \neq Q_{i,d}$ and $Q_{i,e} \in Q$. Under the natural inclusion $\mathbf{k}[x_{2,0},\ldots,x_{2,n_2},\ldots,x_{k,0},\ldots,x_{k,n_k}] \hookrightarrow R$ we can consider each $G_{Q_{i,d}}$ as an element of R of degree $(0,\underline{j})$. From this P_i and subset $Q \subseteq Q_{P_i}$ we construct the set of forms

$$\mathcal{B}_{P_i} := \left\{F_{P_i}G_{Q_{i,1}},\ldots,F_{P_i}G_{Q_{i,h}}\right\}.$$

We observe that $F_{P_i}G_{Q_{i,d}} \notin I_{\mathbb{X}}$ for $d = 1,\ldots,h$ because it fails to vanish at $P_i \times Q_{i,d}$. Moreover, each element of $\mathcal{B}_{P_i}$ has degree $(t_1,\underline{j})$ and $|\mathcal{B}_{P_i}| = H_{Q_{P_i}}(\underline{j}) = h$.

We repeat the above construction for every $P_i \in \pi_1(\mathbb{X})$ and let

$$\mathcal{B} := \bigcup_{P_i\in\pi_1(\mathbb{X})} \mathcal{B}_{P_i}.$$

Claim. The elements of $\mathcal{B}$ are linearly independent modulo $I_{\mathbb{X}}$.

Proof of the Claim. It is enough to show that for each $F_{P_i}Q_{i,l} \in \mathcal{B}$, the point $P_i \times Q_{i,l}$ does not vanish at $F_{P_i}Q_{i,l}$ but vanishes at all the other elements $F_{P_{i'}}Q_{i',l'} \in \mathcal{B}$. But this follows immediately from our construction of the elements of $\mathcal{B}$. □

Because the elements of $\mathcal{B}$ are linearly independent elements modulo $I_{\mathbb{X}}$ of degree $(t_1-1,\underline{j})$, it follows that $\dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{t_1-1,\underline{j}} \geq |\mathcal{B}|$. But since

$$|\mathcal{B}| = \sum_{P_i\in\pi_1(\mathbb{X})} |\mathcal{B}_{P_i}| = \sum_{P_i\in\pi_1(\mathbb{X})} H_{Q_{P_i}}(\underline{j}) = (*),$$

the claim implies that $\dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{t_1-1,\underline{j}} \geq (*)$. Combining this inequality with the previous inequality gives $\dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{t_1-1,\underline{j}} = (*)$.

To complete the proof we note that we can always pick a form L of degree $(1,\underline{0})$, where $\underline{0} \in \mathbb{N}^{k-1}$, such that L does not vanish at any $P_i \in \pi_1(\mathbb{X})$. Then for any $l > t_1 - 1$, the set

$$L^{l-t_1-1}\mathcal{B} = \{L^{l-t_1-1}B \mid B \in \mathcal{B}\}$$

is a set of $(*)$ forms of degree $(l,\underline{j})$ that is linearly independent modulo $I_{\mathbb{X}}$. □

For $i = 1,\ldots,k$ we let $\pi_i : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \to \mathbb{P}^{n_i}$ be the projection morphism. Set $t_i := |\pi_i(\mathbb{X})|$. If we partition $\mathbb{X}$ with respect to any of the other $(k-1)$ coordinates, then a result identical to Proposition 3.2.1 holds. Indeed, if $\underline{j} = (j_1,\ldots,j_k) \in \mathbb{N}^k$, and if we fix all but the i^{th} coordinate of $\underline{j}$, then for all integers $l \geq t_i - 1$

$$H_{\mathbb{X}}(j_1,\ldots,j_{i-1},l,j_{i+1},\ldots,j_k) = H_{\mathbb{X}}(j_1,\ldots,j_{i-1},t_i-1,j_{i+1},\ldots,j_k).$$

Corollary 3.2.4. *Let $\mathbb{X}$ be a set of s distinct points in $\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$. Fix an $i \in \{1,\ldots,k\}$. Let $\pi_i(\mathbb{X}) = \{P_1,\ldots,P_{t_i}\}$ be the set of $t_i \leq s$ distinct i^{th} coordinates in $\mathbb{X}$. Then*

(*i*) *for all integers $l \geq t_i - 1$, $H_{\mathbb{X}}(le_i) = t_i$.*
(*ii*) *if $j_h \gg 0$ for all $h \neq i$ and $j_i \geq t_i - 1$, then $H_{\mathbb{X}}(j_1,\ldots,j_i,\ldots,j_k) = s$.*
(*iii*) *$\Delta H_{\mathbb{X}}(j_1,\ldots,j_k) = 0$ if $j_i \geq t_i$.*

Proof. To prove statements (*i*)-(*iii*), we consider only the case that $i = 1$. The other cases will follow similarly.

Set $Q_{P_i} := \pi_{2,\ldots,k}(\pi_1^{-1}(P_i)) \subseteq \mathbb{P}^{n_2}\times\cdots\times\mathbb{P}^{n_k}$ for every $P_i \in \pi_1(\mathbb{X})$, and let $\alpha_i = |Q_{P_i}|$. For all sets Q_{P_i}, $H_{Q_{P_i}}(\underline{0}) = 1$. The conclusion of (*i*) will follow if we use Proposition 3.2.1 to compute $H_{\mathbb{X}}(le_1)$.

To prove (*ii*) we observe that by induction on k and Proposition 3.2.1, if $j_h \gg 0$ for $h \neq 1$, then $H_{Q_{P_i}}(j_2,,\ldots,j_k) = |Q_{P_i}| = \alpha_i$ for every $P_i \in \pi_1(\mathbb{X})$. Since $\sum_{i=1}^{t_1}\alpha_i = s$, the result is now a consequence of Proposition 3.2.1.

For (*iii*) we recall that

$$\Delta H(\underline{j}) = \sum_{(l_1,\ldots,l_k)\leq(1,\ldots,1)} (-1)^{(\sum_j l_j)} H_{\mathbb{X}}(j_1-l_1,\ldots,j_i-l_i,\ldots,j_k-l_k).$$

Let $\mathcal{L} := \{(l_1,\ldots,l_k) \in \mathbb{N}^k \mid (l_1,\ldots,l_k) \leq (1,\ldots,1)\}$ We partition $\mathcal{L}$ into the two sets $\mathcal{L}_0 := \{(l_1,\ldots,l_k) \in \mathcal{L} \mid l_1 = 0\}$ and $\mathcal{L}_1 := \{(l_1,\ldots,l_k) \in \mathcal{L} \mid l_1 = 1\}$ There is an obvious bijection $\varphi : \mathcal{L}_0 \to \mathcal{L}_1$ given by map

$$(0,l_2,\ldots,l_k) \longmapsto (1,l_2,\ldots,l_k).$$

Let $\underline{l}_1 \in \mathcal{L}_0$ and let $\underline{l}_2 = \varphi(\underline{l}_1) \in \mathcal{L}_1$. Now $(-1)^{(\sum_{j\neq 1} l_j)+0}H_{\mathbb{X}}(j_1-0,j_2-l_2,\ldots,j_k-l_k)$ is the term of $\Delta H_{\mathbb{X}}(\underline{j})$ corresponding to $\underline{l}_1 \in \mathcal{L}$ and $(-1)^{(\sum_{j\neq 1} l_j)+1}H_{\mathbb{X}}(j_1-1,,j_2-l_2,\ldots,j_k-l_k)$ is the term of $\Delta H_{\mathbb{X}}(\underline{j})$ corresponding to $\underline{l}_2 \in \mathcal{L}$. If $j_1 \geq t_1$, then Proposition 3.2.1 implies

$$\begin{aligned} H_{\mathbb{X}}(j_1-0,j_2-l_2,\ldots,j_k-l_k) &= H_{\mathbb{X}}(t_1-1,j_2-l_2,\ldots,j_k-l_k) \\ &= H_{\mathbb{X}}(j_1-1,j_2-l_2,\ldots,j_k-l_k). \end{aligned}$$

Since one of $(-1)^{(\sum_{j\neq 1} l_j)+0}$ and $(-1)^{(\sum_{j\neq 1} l_j)+1}$ is -1 and the other is 1, the two terms cancel each other out. But then every term of $\Delta H_{\mathbb{X}}(\underline{j})$ corresponding to $\underline{l}_1 \in \mathcal{L}_0$ is killed by the term $\varphi(\underline{l}_1) \in \mathcal{L}_1$. Because φ is a bijection, it then follows that $\Delta H_{\mathbb{X}}(\underline{j}) = 0$. □

Remark 3.2.5. By Corollary 3.2.4 (ii) we have $H_{\mathbb{X}}(\underline{j}) = s$ for all $\underline{j} \geq (t_1 - 1, \ldots, t_k - 1)$.

If $\underline{j} = (j_1, \ldots, j_k) \in \mathbb{N}^k$, then we denote the vector $(j_1, \ldots, j_{i-1}, \hat{j}_i, j_{i+1}, \ldots, j_k) \in \mathbb{N}^{k-1}$ by $\underline{j}_i$. Using this notation, we have the following consequence of Proposition 3.2.1.

Corollary 3.2.6. *Let $\mathbb{X}$ be a set of s distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and let $t_i = |\pi_i(\mathbb{X})|$ for $1 \leq i \leq k$. Define $l_i := (t_1 - 1, \ldots, \widehat{t_i - 1}, \ldots, t_k - 1)$ for $i = 1, \ldots, k$. Then*

$$H_{\mathbb{X}}(\underline{j}) = \begin{cases} s & (j_1, \ldots, j_k) \geq (t_1 - 1, t_2 - 1, \ldots, t_k - 1) \\ H_{\mathbb{X}}(t_1 - 1, j_2, \ldots, j_k) & \text{if } j_1 \geq t_1 - 1 \text{ and } \underline{j}_1 \ngeq l_1 \\ \vdots & \vdots \\ H_{\mathbb{X}}(j_1, \ldots, j_{i-1}, t_i - 1, j_{i+1}, \ldots, j_k) & \text{if } j_i \geq t_i - 1 \text{ and } \underline{j}_i \ngeq l_i \\ \vdots & \vdots \\ H_{\mathbb{X}}(j_1, \ldots, j_{k-1}, t_k - 1) & \text{if } j_k \geq t_k - 1 \text{ and } \underline{j}_k \ngeq l_k \end{cases}.$$

Remark 3.2.7. Suppose $\mathbb{X}$ is a set of distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Let $\underline{j} = (j_1, \ldots, j_k) \in \mathbb{N}^k$ and suppose that $j_1 \geq t_1 - 1 = |\pi_1(\mathbb{X})|$, and $j_2 \geq t_2 - 1 = |\pi_2(\mathbb{X})| - 1$. Then Corollary 3.2.6 implies that

$$H_{\mathbb{X}}(j_1, j_2, \ldots, j_k) = H_{\mathbb{X}}(t_1 - 1, j_2, \ldots, j_k) = H_{\mathbb{X}}(t_1 - 1, t_2 - 1, j_3, \ldots, j_k).$$

More generally, to compute $H_{\mathbb{X}}(\underline{j})$, the above corollary implies that if $j_i \geq t_i - 1$, then we can replace j_i with $t_i - 1$ and compute the Hilbert function at the resulting tuple. Therefore, to completely determine $H_{\mathbb{X}}$ for all $\underline{j} \in \mathbb{N}^k$, we need to compute the Hilbert function only for $\underline{j} \leq (t_1 - 1, \ldots, t_k - 1)$. Since there are only $\left(\prod_{i=1}^{k} t_i\right)$ k-tuples in $\mathbb{N}^k$ that have this property, we therefore need to compute only a finite number of values. Furthermore, since $t_i = |\pi_i(\mathbb{X})|$, the k-tuples of $\mathbb{N}^k$ for which we need to compute the Hilbert function is determined from crude numerical information about $\mathbb{X}$, namely the sizes of the sets $\pi_i(\mathbb{X})$. Hence, Corollary 3.2.6 is the desired generalization of Proposition 3.0.1 to points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

By the previous remark it follows that if we know the values of $H_{\mathbb{X}}(t_1-1, j_2, \ldots, j_k)$ for all $(j_2, \ldots, j_k) \leq (t_2-1, \ldots, t_k-1)$, $H_{\mathbb{X}}(j_1, t_2-1, \ldots, j_k)$ for all $(j_1, j_3, \ldots, j_k) \leq (t_1-1, t_3-1, \ldots, t_k-1), \ldots,$ and $H_{\mathbb{X}}(j_1, \ldots, j_{k-1}, t_k-1)$ for all $(j_1, \ldots, j_{k-1}) \leq (t_1-1, \ldots, t_{k-1}-1)$, then we know the Hilbert function of $\mathbb{X}$ for all but a finite number of $\underline{j} \in \mathbb{N}^k$. From this observation we can extend the definition of a border of a Hilbert function to sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

Definition 3.2.8. Let $\mathbb{X}$ be a set of s distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, and let $t_i = |\pi_i(\mathbb{X})|$ for $i = 1, \ldots, k$. Suppose that $H_{\mathbb{X}}$ is the Hilbert function of $\mathbb{X}$. For each $1 \leq i \leq k$, let $B_i := \left(b_{j_1,\ldots,\hat{j_i},\ldots,j_k}\right)$ be the $(k-1)$-dimensional array of size $t_1 \times \cdots \times \hat{t_i} \times \cdots t_k$ where

$$b_{j_1,\ldots,\hat{j_i},\ldots,j_k} := H_{\mathbb{X}}(j_1, \ldots, j_{i-1}, t_i - 1, j_{i+1}, \ldots, j_k) \text{ with } 0 \leq j_h \leq t_h - 1.$$

We call B_i the i^{th} *border array* of the Hilbert function of $\mathbb{X}$. We define $B_{\mathbb{X}} := (B_1, \ldots, B_k)$ to be the *border* of the Hilbert function of $\mathbb{X}$.

Remark 3.2.9. If $k = 2$, then B_1 and B_2 are 1-dimensional arrays, i.e., vectors. It is a simple exercise to verify that B_1 is equal to the eventual column vector B_C, and $B_2 = B_R$, the eventual row vector, as defined in Definition 3.1.9.

We end this section by extending Question 3.1.10 to this setting.

Question 3.2.10. *Suppose $B = (B_1, \ldots, B_k)$ is a tuple where each B_i is a $(k-1)$-dimensional array. Under what conditions is B the border of the Hilbert function of a set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$?*

3. Generic Sets of Points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$

In this section we wish to extend the notion of generic sets of points to $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Our discussion follows that of Geramita and Orecchia [**21**] and Geramita [**14**].

If $\mathbb{X}$ is any set of distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, then we denote its Hilbert function by $H_{\mathbb{X}}$. For every integer s we define

$$\mathcal{H}_s := \{H_{\mathbb{X}} \mid \mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \text{ and } |\mathbb{X}| = s\}.$$

If $H_{\mathbb{X}} \in \mathcal{H}_s$, then, by Lemma 2.2.13, for any $\underline{j} = (j_1, \ldots, j_k) \in \mathbb{N}^k$ we have $H_{\mathbb{X}}(\underline{j}) \leq H_{\mathbb{X}}(\underline{j}+e_1)$, $H_{\mathbb{X}}(\underline{j}) \leq H_{\mathbb{X}}(\underline{j}+e_2)$, etc. By Corollary 3.2.6, if $\underline{j} \not\leq (\pi_1(\mathbb{X})-1, \ldots, \pi_k(\mathbb{X})-1) \leq (s-1, \ldots, s-1)$, then $H_{\mathbb{X}}(\underline{j})$ is determined by the border. It follows that $\mathcal{H}_s$ is a finite set.

Since the number of possible Hilbert functions for s points is finite, but the number of sets $\mathbb{X}$ in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with $|\mathbb{X}| = s$ is infinite, we can ask if there exists an expected Hilbert function for s points. We first fix some notation.

Let $R = \mathbf{k}[x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{k,0}, \ldots, x_{k,n_k}]$ with $\deg x_{i,j} = e_i$ be an $\mathbb{N}^k$-graded ring. For every $\underline{j} \in \mathbb{N}^k$, we define

$$N_{\underline{j}} := \binom{n_1 + j_1}{j_1}\binom{n_2 + j_2}{j_2} \cdots \binom{n_k + j_k}{j_k}.$$

Note that $\dim_{\mathbf{k}} R_{\underline{j}} = N_{\underline{j}}$. Let $\{m_1, \ldots, m_{N_{\underline{j}}}\}$ be the $N_{\underline{j}}$ monomials of degree $\underline{j}$ in R. If $F \in R_{\underline{j}}$, then we can write F as

$$F = \sum_{i=1}^{N_{\underline{j}}} c_i m_i \quad \text{where } c_i \in \mathbf{k}.$$

Suppose that $P = [a_{1,0} : \cdots : a_{1,n_1}] \times \cdots \times [a_{k,0} : \cdots : a_{k,n_k}] \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. For $F \in R_{\underline{j}}$ to vanish at P we require

$$F(P) = \sum_{i=1}^{N_{\underline{j}}} c_i m_i(P) = 0.$$

If we consider the c_i's as unknowns, the above equation gives us one linear condition. Suppose that $\mathbb{X} = \{P_1, \ldots, P_s\}$. For $F \in R_{\underline{j}}$ to vanish on $\mathbb{X}$ we therefore require that $F(P_1) = \cdots = F(P_s) = 0$. We then have a linear system of equations represented as

$$\begin{bmatrix} m_1(P_1) & \cdots & m_{N_{\underline{j}}}(P_1) \\ \vdots & & \vdots \\ m_1(P_s) & \cdots & m_{N_{\underline{j}}}(P_s) \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_{N_{\underline{j}}} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The number of linearly independent solutions is the rank of the matrix on the left. For a general enough set of points, we expect this matrix to have rank as large as possible. By Proposition 3.2.2, the rank of this matrix is equal to $H_{\mathbb{X}}(\underline{j})$. Hence, we expect a general enough set of s points $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ to have the Hilbert function

$$H_{\mathbb{X}}(\underline{j}) = \min\left\{N_{\underline{j}}, s\right\} \quad \text{for all } \underline{j} \in \mathbb{N}^k.$$

Proceeding analogously as in the case of points in $\mathbb{P}^n$, we make the following definition.

Definition 3.3.1. Let $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and suppose that $H_{\mathbb{X}}$ is the Hilbert function of $\mathbb{X}$. If

$$H_{\mathbb{X}}(\underline{j}) = \min\left\{N_{\underline{j}}, s\right\} \quad \text{for all } \underline{j} \in \mathbb{N}^k,$$

then the Hilbert function is called *maximal*. If $\mathbb{X}$ has a maximal Hilbert function, then $\mathbb{X}$ is said to be in *generic position*.

We have yet to show the existence of a set of points in generic position in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. As the next theorem demonstrates, "most" sets of s points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ are in generic position. The proof is adapted from the case of points in $\mathbb{P}^n$ due to Geramita and Orecchia [**21**]. We first set some notation. Define

$$D_{\geq s} := \left\{\underline{j} \in \mathbb{N}^k \;\middle|\; N_{\underline{j}} \geq s\right\} \text{ and } D_{>s} := \left\{\underline{j} \in \mathbb{N}^k \;\middle|\; N_{\underline{j}} > s\right\}.$$

Set

$$\operatorname{Min} D_{\geq s} := \min\left\{\underline{j} \mid \underline{j} \in D_{\geq s}\right\} \text{ and } \operatorname{Min} D_{>s} := \min\left\{\underline{j} \mid \underline{j} \in D_{>s}\right\}$$

with respect to our partial ordering $\geq$ on $\mathbb{N}^k$. Note that both $\operatorname{Min} D_{\geq s}$ and $\operatorname{Min} D_{>s}$ are finite sets. We will also denote $(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}) \times \cdots \times (\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})$ (s times) by $(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})^s$.

Theorem 3.3.2. *The s-tuples of points of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, $(P_1, \dots, P_s)$, considered as points of $(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})^s$, which are in generic position form a non-empty open subset of $(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})^s$.*

PROOF. Let $R = \mathbf{k}[x_{1,0}, \dots, x_{1,n_1}, \dots, x_{k,0}, \dots, x_{k,n_k}]$ be an $\mathbb{N}^k$-graded ring. If $\underline{j} \in \mathbb{N}^k$, then let $\{m_1, \dots, m_{N_{\underline{j}}}\}$ be the $N_{\underline{j}}$ monomials of degree $\underline{j}$ of R. We have a morphism $\nu_{\underline{j}} : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \to \mathbb{P}^{N_{\underline{j}}-1}$ defined by

$$[x_{1,0} : \cdots : x_{1,n_1}] \times \cdots \times [x_{k,0} : \cdots : x_{k,n_k}] \longmapsto [m_1 : \cdots : m_{N_{\underline{j}}}].$$

This induces a morphism

$$\varphi_{\underline{j}} = (\nu_{\underline{j}} : \cdots : \nu_{\underline{j}}) : (\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})^s \longrightarrow \left(\mathbb{P}^{N_{\underline{j}}-1}\right)^s = V_{\underline{j}}.$$

We can view an element of $V_{\underline{j}}$ as an $s \times N_{\underline{j}}$ matrix. Notice that the s-tuple $(P_1, \ldots, P_s) \in (\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})^s$ is sent to the "matrix"

$$\varphi_{\underline{j}}(P_1, \ldots, P_s) = M_{\underline{j}}(P_1, \ldots, P_s) := \begin{bmatrix} m_1(P_1) & \cdots & m_{N_{\underline{j}}}(P_1) \\ \vdots & & \vdots \\ m_1(P_s) & \cdots & m_{N_{\underline{j}}}(P_s) \end{bmatrix}.$$

Let $T_{\underline{j}}$ be the collection of equations describing the situation that every maximal minor of an $s \times N_{\underline{j}}$ matrix is zero. Every element of $T_{\underline{j}}$ is $\mathbb{N}^s$-homogeneous, that is, the elements are homogeneous in each set of variables corresponding to a factor of $V_{\underline{j}}$. It follows that $C_{\underline{j}} := \mathbf{V}(T_{\underline{j}})$ is a closed subset of $V_{\underline{j}}$, and hence, $\varphi_{\underline{j}}^{-1}(V_{\underline{j}} \backslash C_{\underline{j}})$ is an open subset of $(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})^s$. Set

$$U := \bigcap_{\underline{j} \in \mathbb{N}^k} \varphi_{\underline{j}}^{-1}(V_{\underline{j}} \backslash C_{\underline{j}}).$$

If $(P_1, \ldots, P_s) \in U$, then for all $\underline{j} \in \mathbb{N}^k$ the matrix $M_{\underline{j}}(P_1, \ldots, P_s)$ has maximal rank, i.e., $\operatorname{rk} M_{\underline{j}}(P_1, \ldots, P_s) = \min\left\{N_{\underline{j}}, s\right\}$. By Proposition 3.2.2, the Hilbert function of the tuple $(P_1, \ldots, P_s)$, considered as a subset of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, is maximal.

To finish the proof we need to show that U is open and non-empty. To show that U is open we require the following claim.

Claim 1. Suppose that $\underline{j} \in \mathbb{N}^k$. If there exists $\underline{j}' \in \operatorname{Min} D_{>s}$ such that $\underline{j} > \underline{j}'$, then

$$\varphi_{\underline{j}'}^{-1}(V_{\underline{j}'} \backslash C_{\underline{j}'}) \subseteq \varphi_{\underline{j}}^{-1}(V_{\underline{j}} \backslash C_{\underline{j}}).$$

Proof of the Claim. Let $(P_1, \ldots, P_s) \in \varphi_{\underline{j}'}^{-1}(V_{\underline{j}'} \backslash C_{\underline{j}'})$. After a suitable change of coordinates we can assume that each $P_i \in \{P_1, \ldots, P_s\}$ can be written as

$$P_i := [1 : a_{i,1,1} : \cdots : a_{i,1,n_1}] \times [1 : a_{i,2,1} : \cdots : a_{i,2,n_2}] \times \cdots \times [1 : a_{i,k,1} : \cdots : a_{i,k,n_k}].$$

If $\{m_1, \ldots, m_{N_{\underline{j}'}}\}$ are the $N_{\underline{j}'}$ monomials of degree $\underline{j}'$, then the matrix

$$M_{\underline{j}'}(P_1, \ldots, P_s) := \begin{bmatrix} m_1(P_1) & \cdots & m_{N_{\underline{j}'}}(P_1) \\ \vdots & & \vdots \\ m_1(P_s) & \cdots & m_{N_{\underline{j}'}}(P_s) \end{bmatrix}$$

has rank $= \min\left\{N_{\underline{j}'}, s\right\} = s$. Since $\underline{j} > \underline{j}'$, we have $(\underline{j} - \underline{j}') > \underline{0}$. Let m be the monomial $m = x_{1,0}^{a_1} x_{2,0}^{a_2} \cdots x_{k,0}^{a_k}$ of degree $(\underline{j} - \underline{j}') = (a_0, \ldots, a_k)$. The monomial m does not vanish at any of $P_1, \ldots, P_s$. It follows that

$$\begin{bmatrix} m(P_1)m_1(P_1) & \cdots & m(P_1)m_{N_{\underline{j}'}}(P_1) \\ \vdots & & \vdots \\ m(P_s)m_1(P_s) & \cdots & m(P_s)m_{N_{\underline{j}'}}(P_s) \end{bmatrix} = \begin{bmatrix} m_1(P_1) & \cdots & m_{N_{\underline{j}'}}(P_1) \\ \vdots & & \vdots \\ m_1(P_s) & \cdots & m_{N_{\underline{j}'}}(P_s) \end{bmatrix}$$

is a sub-matrix of $M_{\underline{j}}(P_1, \ldots, P_s)$. Therefore, the tuple $(P_1, \ldots, P_s) \in \varphi_{\underline{j}}^{-1}(V_{\underline{j}} \backslash C_{\underline{j}})$ because the maximal minors describing this sub-matrix fail to vanish at this tuple. $\square$

In light of the claim, we have

$$U := \bigcap_{\underline{j} \in \mathbb{N}^k} \varphi_{\underline{j}}^{-1}(V_{\underline{j}} \backslash C_{\underline{j}}) = \bigcap_{\left\{ \underline{j} \in \mathbb{N}^k \;\middle|\; \substack{\underline{j} \leq \underline{j}' \text{ for some} \\ \underline{j}' \in \operatorname{Min} D_{>s}} \right\}} \varphi_{\underline{j}}^{-1}(V_{\underline{j}} \backslash C_{\underline{j}}).$$

The set $\{\underline{j} \in \mathbb{N}^k \mid \underline{j} \leq \underline{j}' \text{ for some } \underline{j}' \in \operatorname{Min} D_{>s}\}$ is a finite set, and so U is open in $(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})^s$.

To show that U is non-empty, we will show that there exists a tuple $(P_1, \ldots, P_s) \in (\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})^s$ such that $(P_1, \ldots, P_s) \in \varphi_{\underline{j}}^{-1}(V_{\underline{j}} \backslash C_{\underline{j}})$ for all $\underline{j} \in \mathbb{N}^k$. We proceed by induction on s; the case $s = 1$ is trivial.

So, let $P_1, \ldots, P_{s-1}$ be $(s-1)$ generic points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Furthermore, suppose that $\underline{j}_1, \ldots, \underline{j}_l$ is a complete list of elements in $\operatorname{Min} D_{\geq s}$. For each $\underline{j}_i \in \{\underline{j}_1, \ldots, \underline{j}_l\}$ we define a morphism $\nu_{\underline{j}_i} : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \to \mathbb{P}^{N_{\underline{j}_i} - 1}$ by

$$[x_{1,0} : \cdots : x_{1,n_1}] \times \cdots \times [x_{k,0} : \cdots : x_{k,n_k}] \longrightarrow [m_1 : \cdots : m_{N_{\underline{j}_i}}]$$

where $\{m_1, \ldots, m_{N_{\underline{j}_i}}\}$ are the $N_{\underline{j}_i}$ monomials of degree $\underline{j}_i$. Let $L_{\underline{j}_i}$ be the linear sub-variety spanned by $\nu_{\underline{j}_i}(P_1), \ldots, \nu_{\underline{j}_i}(P_{s-1})$. Because $\nu_{\underline{j}_i}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})$ is not contained in any linear sub-variety of $\mathbb{P}^{N_{\underline{j}_i} - 1}$, the set $\nu_{\underline{j}_i}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}) \backslash \{\nu_{\underline{j}_i}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}) \cap L_{\underline{j}_i}\}$ is a non-empty set.

From the above morphism we obtain a morphism

$$\begin{array}{rcl} \phi_{\underline{j}_1, \ldots, \underline{j}_l} : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} & \longrightarrow & \mathbb{P}^{N_{\underline{j}_1} - 1} \times \cdots \times \mathbb{P}^{N_{\underline{j}_l} - 1} \\ P & \longmapsto & \nu_{\underline{j}_1}(P) \times \cdots \times \nu_{\underline{j}_l}(P). \end{array}$$

Let $L = L_{\underline{j}_1} \times \cdots \times L_{\underline{j}_l} \subseteq \mathbb{P}^{N_{\underline{j}_1}-1} \times \cdots \times \mathbb{P}^{N_{\underline{j}_l}-1}$. From our construction of L, it follows that $Z := \phi_{\underline{j}_1,\ldots,\underline{j}_l}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}) \not\subseteq L$. Hence, the set $Y := Z \backslash \{Z \cap L\}$ is non-empty. Pick any $P_s \in \phi^{-1}_{\underline{j}_1,\ldots,\underline{j}_l}(Y)$. Note that for our choice of P_s, $\nu_{\underline{j}_i}(P_s) \notin L_{\underline{j}_i}$ for any $\underline{j}_i \in \{\underline{j}_1, \ldots, \underline{j}_l\}$.

We claim that the set $\{P_1, \ldots, P_s\}$ is in generic position. It is sufficient to check that $\operatorname{rk} M_{\underline{j}}(P_1, \ldots, P_s)$ is maximal for all $\underline{j} \in \mathbb{N}^k$.

Case 1. $\underline{j}$ is such that $N_{\underline{j}} < s$.

The matrix $M_{\underline{j}}(P_1, \ldots, P_s)$ can have at most rank $N_{\underline{j}}$. The sub-matrix $M_{\underline{j}}(P_1, \ldots, P_{s-1})$ has this rank by induction. This completes this case. □

Case 2. $\underline{j}$ is such that $N_{\underline{j}} \geq s$, i.e., $\underline{j} \in D_{\geq s}$.

Because $\underline{j} \in D_{\geq s}$, there exists $\underline{j}' \in \operatorname{Min} D_{\geq s}$ such that $\underline{j} \geq \underline{j}'$. It follows from the proof of Claim 1 that if $\operatorname{rk} M_{\underline{j}'}(P_1, \ldots, P_s) = s$, then $\operatorname{rk} M_{\underline{j}}(P_1, \ldots, P_s) = s$. But by our choice of P_s, $\nu_{\underline{j}'}(P_s) \notin L_{\underline{j}'}$. But this is equivalent to saying that P_s is not in the linear span of $\nu_{\underline{j}'}(P_1), \ldots, \nu_{\underline{j}'}(P_{s-1})$, and hence, $\operatorname{rk} M_{\underline{j}'}(P_1, \ldots, P_s) = \operatorname{rk} M_{\underline{j}'}(P_1, \ldots, P_{s-1}) + 1 = s$. □

Corollary 3.3.3. *Let s and k be positive integers. For each $i \in \{1, \ldots, k\}$, let B_i be a $(k-1)$-dimensional array of size $\underbrace{s \times \cdots \times s}_{k-1}$ such that every entry of B_i is s. Then $B = (B_1, \ldots, B_k)$ is the border of a set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.*

PROOF. A set $\mathbb{X}$ of s points in generic position has border $B_{\mathbb{X}} = B$. □

Remark 3.3.4. For "most" sets $\mathbb{X}$ of s points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, Theorem 3.3.2 shows that the Hilbert function of $\mathbb{X}$ is simply a function of $n_1, \ldots, n_k$, and s. The Hilbert function, however, provides us with very coarse information about $\mathbb{X}$. A wealth of information about $\mathbb{X}$ is contained within the minimal free resolution of $I_{\mathbb{X}}$. This leads us to ask if there is an expected resolution for a set of s points in generic position in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. For points in $\mathbb{P}^n$ this question is known as the Minimal Resolution Conjecture which was first formulated by Lorenzini [**34**]. An interesting problem is to determine a minimal resolution conjecture for points in generic position in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. As we will see in the next chapter, points in generic position in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ fail to be arithmetically Cohen-Macaulay. This fact prevents us from passing to the artinian case and formulating the conjecture in this setting as was done in [**34**]. However, because we know the Hilbert function of a set of points in generic position, we may be able to make a conjecture about the generators of the defining ideal. In other words, we can make a conjecture about some of the graded Betti numbers in the resolution. We are currently considering this problem.

Remark 3.3.5. Because $\#\mathcal{H}_s < \infty$, we can ask if it is possible to explicitly determine $\#\mathcal{H}_s$ for all $s \in \mathbb{N}$. This turns out to be a difficult problem. L. Roberts posed exactly this problem for points in $\mathbb{P}^n$ in [**46**]. Little progress, however, has been made on this question. Carlini, Hà, and Van Tuyl [**9**] point out that by using the n-type vectors of Geramita, Harima, and Shin [**17**] then the problem of computing $\#\mathcal{H}_s$ for s points in $\mathbb{P}^2$ is equivalent to computing the number of sequences of strictly increasing integers that sum to s. We give a lower bound for $\#\mathcal{H}_s$ for points in $\mathbb{P}^1 \times \mathbb{P}^1$ (cf. Proposition 5.3.1).

CHAPTER 4

The Hilbert Functions of Arithmetically Cohen-Macaulay Sets of Points

Cohen-Macaulay rings are the "workhorse of commutative algebra" (page 57 of [**6**]). If $\mathbb{X}$ is any collection of points in $\mathbb{P}^n$, then the graded ring $R/I_{\mathbb{X}}$ is *always* Cohen-Macaulay. This fact is used, either directly or indirectly, to prove many results that describe the properties of points in $\mathbb{P}^n$. Unfortunately, when we extend our study to sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with $k > 1$, we encounter the unpleasant fact that the multi-graded ring $R/I_{\mathbb{X}}$ may fail to be Cohen-Macaulay. The following example, which is found in Giuffrida, Maggioni, and Ragusa [**26**], and which is generalized in Lemma 4.2.4, illustrates that even the coordinate ring of a very simple set of points can fail to be Cohen-Macaulay.

Example 4.0.1. Let $R = \mathbf{k}[x_0, x_1, y_0, y_1]$ with $\deg x_i = (1,0)$ and $\deg y_i = (0,1)$, and let

$$\mathbb{X} = \{[0:1] \times [0:1], [1:0] \times [1:0]\} \subseteq \mathbb{P}^1 \times \mathbb{P}^1.$$

Then $I_{\mathbb{X}} = (x_0, y_0) \cap (x_1, y_1) = (x_0x_1, x_0y_1, x_1y_0, y_0y_1) \subseteq R$. The element $\overline{x_0 + x_1}$ is a non-zero divisor in $R/I_{\mathbb{X}}$ because the form $x_0 + x_1$ does not vanish at either point in $\mathbb{X}$. The non-zero elements $\overline{g}$ of $R/(I_{\mathbb{X}}, x_0 + x_1)$ are either $g = h(y_0, y_1)$ where $\deg h = (0, d)$ and $d > 0$, or $\overline{g} = \overline{x}_0$. Both types of elements are annihilated by $\overline{x}_0$. Hence, $\operatorname{depth} R/I_{\mathbb{X}} = 1 < 2 = \operatorname{K-dim} R/I_{\mathbb{X}}$. It then follows from Definition 4.1.6 that $R/I_{\mathbb{X}}$ is not Cohen-Macaulay. (An alternative proof is to observe that the ideal $I_{\mathbb{X}}$, as a homogeneous ideal of R, is the defining ideal of two skew lines in $\mathbb{P}^3$, and hence, $R/I_{\mathbb{X}}$ is not Cohen-Macaulay. (cf. [**23**]))

A set of points whose coordinate ring is Cohen-Macaulay will be called an *arithmetically Cohen-Macaulay* (ACM for short) set of points. Because of the importance of Cohen-Macaulay rings in commutative algebra and algebraic geometry, it is natural to ask the following variation on Question 1.1.1:

Question 4.0.2. *What can be the Hilbert function of an ACM set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$?*

If $k = 1$, then this question is equivalent to Question 1.1.1 because all sets of points in $\mathbb{P}^n$ are ACM sets of points. Hence, Question 4.0.2 has been answered by Geramita, Maroscia, and Roberts [**19**] for the case $k = 1$. In light of the above example, if $k > 1$, then Question 4.0.2 is weaker than Question 1.1.1.

The main result of this chapter is the following theorem:

Theorem 4.0.3. (Theorem 4.3.14) *Let $H : \mathbb{N}^k \to \mathbb{N}$ be a numerical function. Then H is the Hilbert function of an ACM set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ if and only if*

$$\Delta H(i_1, \ldots, i_k) = \sum_{\underline{0} \leq \underline{l} = (l_1, \ldots, l_k) \leq (1, \ldots, 1)} (-1)^{|\underline{l}|} H(i_1 - l_1, \ldots, i_k - l_k),$$

where $H(i_1, \ldots, i_k) = 0$ if $(i_1, \ldots, i_k) \not\geq \underline{0}$, is the Hilbert function of some multi-graded artinian quotient of $S = \mathbf{k}[x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{k,1}, \ldots, x_{k,n_k}]$.

As a consequence of this result, the answer to Question 4.0.2 is equivalent to the answer of the following question:

Question 4.0.4. *What can be the Hilbert function of a multi-graded artinian quotient of $\mathbf{k}[x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{k,1}, \ldots, x_{k,n_k}]$?*

In this chapter we will also give an answer to Question 4.0.4 for the cases: (i) $S = \mathbf{k}[x_{1,1}, x_{2,1}, \ldots, x_{2,m}]$ and (ii) $S = \mathbf{k}[x_{1,1}, x_{2,1}, \ldots, x_{k,1}]$. As a corollary, we give a complete answer to Question 4.0.2 for ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^m$ and $\underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{k}$. In each case, our result is a generalization of an earlier result about ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ due to Giuffrida, Maggioni, and Ragusa [**26**].

This chapter is structured as follows. In the first section, we recall the definition of a Cohen-Macaulay ring and describe some of its properties. In the second section we make some general remarks about the depth of $R/I_{\mathbb{X}}$ where $R/I_{\mathbb{X}}$ is the coordinate ring of a set $\mathbb{X}$ of distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. In the third section, we restrict our focus to ACM sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. The main result of this section (Theorem 4.3.14) demonstrates that the numerical function $H : \mathbb{N}^k \to \mathbb{N}$ is the Hilbert function of an ACM set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ if and only if ΔH, the first difference function of H, is the Hilbert function of an $\mathbb{N}^k$-graded artinian quotient of $\mathbf{k}[x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{k,1}, \ldots, x_{k,n_k}]$. This characterization is similar to the characterization of Hilbert functions of points in $\mathbb{P}^n$ given by Geramita, Maroscia, and Roberts [**19**] (also see Proposition 2.3.10). In the fourth section we characterize the Hilbert functions of bigraded quotients of $\mathbf{k}[x_1, y_1, \ldots, y_m]$. As a corollary, we have a precise description of the Hilbert functions of ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^m$. We also answer Question 4.0.4 for the $\mathbb{N}^k$-graded ring $S = \mathbf{k}[x_{1,1}, x_{2,1}, \ldots, x_{k,1}]$. In the last section we give the proof of a technical lemma used in the proof of Theorem 4.3.14.

1. Cohen-Macaulay Rings

In this section we define Cohen-Macaulay (CM for short) rings and collect the facts we need in the later sections. A general theory of CM rings is developed in the wonderful book of Bruns and Herzog [**6**]. We use [**6**], Balcerzyk and Józefiak [**4**], and Matsumura [**38**] as our primary references for the material of this section.

Unless stated otherwise, we assume that $R = \mathbf{k}[x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{k,0}, \ldots, x_{k,n_k}]$. We induce an $\mathbb{N}^k$-grading on R by setting $\deg x_{i,j} = e_i$ where e_i is the i^{th} standard basis vector in $\mathbb{N}^k$. We define $\mathbf{m}$ to be the ideal $\mathbf{m} := \bigoplus_{0 \neq \underline{j} \in \mathbb{N}^k} R_{\underline{j}}$. We recall that R is Noetherian. We will use A to denote an arbitrary Noetherian ring. We recall Definition 2.2.8.

Definition 4.1.1. Let $\wp$ be a prime ideal of A. The *height* of $\wp$ is the integer t such that we can find prime ideals $\wp_i$ of A such that $\wp = \wp_t \supsetneq \wp_{t-1} \supsetneq \cdots \supsetneq \wp_1 \supsetneq \wp_0$ and no longer such chain can be found. We write $\operatorname{ht}_A(\wp) = t$. If I is any ideal of A, then $\operatorname{ht}_A(I)$ is defined to be the number $\operatorname{ht}_A(I) := \inf\{\operatorname{ht}_A(\wp) \mid \wp \supseteq I\}$. If the ring A is clear from the context, then we shall omit the subscript A and simply write $\operatorname{ht}(I)$. The *Krull dimension of* A, denoted $\operatorname{K-dim} A$, is the number $\operatorname{K-dim} A := \sup\{\operatorname{ht}(\wp) \mid \wp \text{ a prime ideal of } A\}$.

Definition 4.1.2. Let $F_1, F_2, \ldots, F_r$ be a sequence of non-constant elements of R and let I be an $\mathbb{N}^k$-homogeneous ideal. Then we say $F_1, \ldots, F_r$ is a *regular sequence modulo* I or *give rise to a regular sequence in* R/I if and only if

(*i*) $(I, F_1, \ldots, F_r) \subseteq \mathbf{m}$,
(*ii*) $\overline{F}_1$ is not a zero divisor in R/I, and

(*iii*) $\overline{F}_i$ is not a zero divisor in $R/(I, F_1, \ldots, F_{i-1})$ for $1 < i \leq r$.

The sequence $F_1, \ldots, F_r$ is called a *maximal regular sequence modulo* I if $F_1, \ldots, F_r$ is a regular sequence which cannot be made longer.

Remark 4.1.3. A more general notion of a regular sequence exists for R-modules M. See, for example, Definition 1.1.1 of [**6**]. Since we do not require this generality, we omit it. For an arbitrary ring it is not true that all maximal regular sequences have the same length. However, since we shall only consider $\mathbb{N}^k$-homogeneous ideals of R, the following theorem applies.

Theorem 4.1.4. ([**6**] Theorem 1.2.5) *Suppose that* $I \subseteq \mathbf{m}$ *is an* $\mathbb{N}^k$*-homogeneous ideal of the Noetherian ring* R*. Then all maximal regular sequences modulo* I *have the same length.*

Because all maximal regular sequences modulo I have the same length, we give a name to this common value.

Definition 4.1.5. Let $I \subseteq \mathbf{m}$ be an $\mathbb{N}^k$-homogeneous ideal of R. Then the *depth of* R/I, written $\operatorname{depth} R/I$, is the length of a maximal regular sequence modulo I.

One can show, using *Krull's Principal Ideal Theorem* (see Theorem 15.2 of Sharp [**51**]), that $\operatorname{depth} R/I \leq \text{K-}\dim R/I$ always holds. If equality occurs, then we give the ring R/I a special name.

Definition 4.1.6. Let $I \subseteq \mathbf{m}$ be an $\mathbb{N}^k$-homogeneous ideal of R. Then the ring R/I is called *Cohen-Macaulay* (or CM for short) if $\operatorname{depth} R/I = \text{K-}\dim R/I$.

Example 4.1.7. The polynomial ring $R = \mathbf{k}[x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{k,0}, \ldots, x_{k,n_k}]$ is a Cohen-Macaulay ring because the indeterminates $x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{k,0}, \ldots, x_{k,n_k}$ give rise to a regular sequence in R of length $\displaystyle\sum_{i=1}^{k}(n_i + 1) = \text{K-}\dim R$.

Definition 4.1.8. Let M be a module over the commutative Noetherian ring A, and let $\wp$ be a prime ideal of A. Then we say $\wp$ is an *associated prime ideal of* M precisely when there exists an element $m \in M$ with $(0 : m) = \wp$. The set of associated prime ideals of M is denoted by $\mathrm{Ass}_A(M)$.

Remark 4.1.9. Suppose that I is a proper ideal of a commutative Noetherian ring A and suppose that $I = Q_1 \cap \cdots \cap Q_r$ is the primary decomposition of I. Set $\wp_i = \sqrt{Q_i}$. Then $\mathrm{Ass}_A(A/I) = \{\wp_1, \ldots, \wp_r\}$. See, for example, Remark 9.33 (i) of Sharp [**51**].

Definition 4.1.10. Let A be a commutative Noetherian ring, I an ideal of A, and suppose that $\mathrm{Ass}_A(A/I) = \{\wp_1, \ldots, \wp_r\}$. We say that I is *unmixed* if $\mathrm{ht}_A(\wp_i) = \mathrm{ht}_A(I)$ for all i.

As the next theorem shows, if I is a homogeneous ideal of a graded polynomial ring $k[x_0, \ldots, x_n]$ with the property that the quotient ring $k[x_0, \ldots, x_n]/I$ is CM, then the associated primes of I all have the same height.

Theorem 4.1.11. *Let I be a homogeneous ideal of $R = \mathbf{k}[x_0, \ldots, x_n]$ and suppose that $I \subseteq \mathbf{m} := (x_0, \ldots, x_n)$. Then*

(i) $\mathrm{ht}_R(I) + \mathrm{K\text{-}dim}\, R/I = \mathrm{K\text{-}dim}\, R$.
(ii) *If R/I is a CM ring, then the ideal I is unmixed.*

This result appears to be well known. However, we could find no reference for the graded version that we stated above. For completeness, we will prove Theorem 4.1.11. To prove this result, we will require the following results about Cohen-Macaulay local rings. We will only give a reference to their proofs.

Lemma 4.1.12. ([**4**] Property 10, page 122) *If (A, m) is a local CM ring, then for any ideal $I \subsetneq A$, we have* $\mathrm{ht}_A(I) + \mathrm{K\text{-}dim}\, A/I = \mathrm{K\text{-}dim}\, A$.

Lemma 4.1.13. ([**4**] Property 1, page 118) *If A is a CM ring, and if S is any multiplicatively closed subset S, then $S^{-1}A$ is also CM. In other words, the CM property is preserved under localization.*

Lemma 4.1.14. ([**38**] Lemma 7.C, page 50) *Let S be a multiplicative subset of A, and let M be a finitely generated A-module. Put $A' = S^{-1}A$ and $M' = S^{-1}M$. Then there exists a 1-1 correspondence between the sets*

$$\mathrm{Ass}_A(M) \cap \{\wp \subseteq A \mid \wp \text{ prime},\ \wp \cap S = \emptyset\} \overset{1-1}{\longleftrightarrow} \mathrm{Ass}_{A'}(M')$$

via the map $\wp \mapsto \wp S^{-1}A$.

Proposition 4.1.15. ([**38**] Theorem 30, page 104) *Let* (A, m) *be a Noetherian local ring, and let* $M \neq 0$ *be a finitely generated CM A-module. Then*

$$\operatorname{depth} M = \text{K-}\dim A/\wp \quad \textit{for every } \wp \in \operatorname{Ass}_A(M).$$

Proof. (of Theorem 4.1.11) (i) Let $(R_{\mathbf{m}}, \mathbf{m}R_{\mathbf{m}})$ denote the local ring formed by localizing R at the maximal ideal $\mathbf{m}$. By Lemma 4.1.13, the ring $(R_{\mathbf{m}}, \mathbf{m}R_{\mathbf{m}})$ is CM because R is a CM ring. Since $I \subseteq \mathbf{m}$, it follows that $\operatorname{ht}_{R_{\mathbf{m}}}(IR_{\mathbf{m}}) = \operatorname{ht}_R(I)$ and $\text{K-}\dim R_{\mathbf{m}} = \text{K-}\dim R$. Furthermore, since $(R_{\mathbf{m}}/IR_{\mathbf{m}}) \cong (R/I)_{\overline{\mathbf{m}}}$, $\text{K-}\dim R_{\mathbf{m}}/IR_{\mathbf{m}} = \text{K-}\dim R/I$. By applying Lemma 4.1.12, we thus have

$$\begin{aligned} \operatorname{ht}_R(I) + \text{K-}\dim R/I &= \operatorname{ht}_{R_{\mathbf{m}}}(IR_{\mathbf{m}}) + \text{K-}\dim R_{\mathbf{m}}/IR_{\mathbf{m}} \\ &= \text{K-}\dim R_{\mathbf{m}} = \text{K-}\dim R. \end{aligned}$$

(ii) Let $\wp \in \operatorname{Ass}_R(R/I)$. We need to show that $\operatorname{ht}_R(\wp) = \operatorname{ht}_R(I)$. Because $I \subseteq \mathbf{m}$, $\wp$ is homogeneous, i.e., $\wp \subseteq \mathbf{m}$, and hence $\wp \in \{\wp \subseteq R \mid \wp \text{ prime}, \ \wp \cap (R \backslash \mathbf{m}) = \emptyset\}$. Thus, by Lemma 4.1.14, $\wp R_{\mathbf{m}} \in \operatorname{Ass}_{R_{\mathbf{m}}}(R_{\mathbf{m}}/IR_{\mathbf{m}})$.

The ring $R_{\mathbf{m}}/IR_{\mathbf{m}} \cong (R/I)_{\overline{\mathbf{m}}}$ is CM by Lemma 4.1.13. So, by Theorem 4.1.15 we deduce that $\text{K-}\dim R_{\mathbf{m}}/\wp R_{\mathbf{m}} = \operatorname{depth} R_{\mathbf{m}}/IR_{\mathbf{m}}$. But since the ring $R_{\mathbf{m}}/IR_{\mathbf{m}}$ is CM, we in fact have

$$\text{K-}\dim R_{\mathbf{m}}/\wp R_{\mathbf{m}} = \text{K-}\dim R_{\mathbf{m}}/IR_{\mathbf{m}} = \text{K-}\dim R_{\mathbf{m}} - \operatorname{ht}_{R_{\mathbf{m}}}(IR_{\mathbf{m}}) = \text{K-}\dim R - \operatorname{ht}_R(I).$$

On the other hand, we use Lemma 4.1.12 to compute $\text{K-}\dim R_{\mathbf{m}}/\wp R_{\mathbf{m}}$:

$$\text{K-}\dim R_{\mathbf{m}}/\wp R_{\mathbf{m}} = \text{K-}\dim R_{\mathbf{m}} - \operatorname{ht}_{R_{\mathbf{m}}}(\wp R_{\mathbf{m}}) = \text{K-}\dim R - \operatorname{ht}_R(\wp).$$

If we substitute this value for $\text{K-}\dim R_{\mathbf{m}}/\wp R_{\mathbf{m}}$ into our previous expression and simplify the resulting expression, then we find that $\operatorname{ht}_R(\wp) = \operatorname{ht}_R(I)$. Because this is true for any $\wp \in \operatorname{Ass}_R(R/I)$, the ideal I is unmixed as desired. □

Definition 4.1.16. A variety $\mathbb{X} \subseteq \mathbb{P}^n$ is *arithmetically Cohen-Macaulay* (ACM for short) if the graded coordinate ring $R/I_{\mathbb{X}}$ is CM. More generally, a variety $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is ACM if the multi-graded coordinate ring $R/I_{\mathbb{X}}$ is CM.

Remark 4.1.17. Suppose $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is a variety. The multi-homogeneous ideal $I_{\mathbb{X}}$ in the $\mathbb{N}^k$-graded polynomial ring R corresponding to $\mathbb{X}$ is also a homogeneous ideal in the normal sense. We let $\widetilde{\mathbb{X}}$ denote the variety in $\mathbb{P}^N$, where $N = \left(\sum_{i=1}^{k}(n_i + 1)\right) - 1$, defined by $I_{\mathbb{X}}$. The condition of being CM is a condition on the depth of $R/I_{\mathbb{X}}$. Because the grading of a ring does not influence the depth of a ring,

$$\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \text{ is ACM } \Leftrightarrow \widetilde{\mathbb{X}} \subseteq \mathbb{P}^N \text{ is ACM.}$$

Note that the dimension of the variety $\widetilde{\mathbb{X}}$ is bigger than $\dim \mathbb{X}$. Specifically, if $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is a variety, then $\dim \widetilde{\mathbb{X}} = \dim \mathbb{X} + k$.

The following results about CM rings will be required in the later sections.

Lemma 4.1.18. ([**4**] Property 4, page 119) *If A is a CM ring and x is a non-zero divisor in A, then the ring $A/(x)$ is also CM. Moreover,* $\text{K-dim}\, A/(x) = \text{K-}\dim A - 1$.

Lemma 4.1.19. *Let $J = (F_1, \ldots, F_r) \subseteq \mathbf{m} \subseteq R$ be an $\mathbb{N}^k$-homogeneous ideal. Suppose that $F_1, \ldots, F_r$ give rise to a regular sequence in R. Then R/J is CM.*

Proof. By Theorem 4.1.4, the regular sequence $F_1, \ldots, F_r$ can be extended to a maximal regular sequence, say $F_1, \ldots, F_r, G_{r+1}, \ldots, G_t$, in R. Because R is Cohen-Macaulay, $t = \sum_{i=1}^{k}(n_i + 1) = \text{K-}\dim R$. From Lemma 4.1.18 we have

$$\text{K-}\dim R/J = \text{K-}\dim R - r = t - r = \operatorname{depth} R/J.$$

The conclusion now follows. □

Definition 4.1.20. Suppose that $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is a variety. If the $\mathbb{N}^k$-homogeneous ideal $I_{\mathbb{X}}$ is generated by a regular sequence in R, then we say $\mathbb{X}$ is a *complete intersection.* By Lemma 4.1.19, a complete intersection is always ACM.

We have defined Cohen-Macaulay rings in terms of the depth of the ring. Alternatively, Cohen-Macaulay rings can be characterized via the projective dimension (see Definition 2.4.4) of the ring. To demonstrate this characterization, we will require the following special case of the Auslander-Buchsbaum formula.

Theorem 4.1.21. (Auslander-Buchsbaum) *Let I be a homogeneous ideal in the $\mathbb{N}^1$-graded ring $R = \mathbf{k}[x_0, \ldots, x_n]$. Then*

$$\text{proj.}\dim_R R/I + \operatorname{depth} R/I = \text{K-}\dim R.$$

PROOF. This is a very special case of the Auslander-Buchsbaum formula. See Theorem 4.4.15 of Weibel [**56**] or Theorem 1.3.3 in Bruns and Herzog [**6**]. □

Theorem 4.1.22. *Let I be a homogeneous ideal in the $\mathbb{N}^1$-graded ring $R = \mathbf{k}[x_0, \ldots, x_n]$. Then R/I is Cohen-Macaulay if and only if* $\text{proj.}\dim_R R/I = n + 1 - \text{K-}\dim R/I$.

PROOF. The ring R/I is Cohen-Macaulay if and only if $\operatorname{depth} R/I = \text{K-}\dim R/I$. Hence, by the Auslander- Buchsbaum formula we have

$$\text{proj.}\dim_R R/I + \text{K-}\dim R/I = \text{proj.}\dim_R R/I + \operatorname{depth} R/I = n + 1.$$

□

Remark 4.1.23. The above result will be used in Chapter 5.

2. The Depth of the Coordinate Ring Associated to a Set of Points

Let $R = \mathbf{k}[x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{k,0}, \ldots, x_{k,n_k}]$ where $\deg x_{i,j} = e_i$ where e_i is the i^{th} standard basis vector of $\mathbb{N}^k$. In this section we study the depth of $R/I_{\mathbb{X}}$, where $\mathbb{X}$ is a set of distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. From the next lemma, it follows that the depth of $R/I_{\mathbb{X}}$ is always at least one.

Lemma 4.2.1. (Lemma 2.2.12) *Suppose $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is a finite set of distinct points. Then there exists a form $L \in R$ of degree e_1 such that $\overline{L}$ is a non-zero divisor in $R/I_{\mathbb{X}}$.*

Corollary 4.2.2. $1 \leq \operatorname{depth} R/I_{\mathbb{X}} \leq k$.

PROOF. The result follows from the fact that $\operatorname{depth} R/I_{\mathbb{X}} \leq \text{K-}\dim R/I_{\mathbb{X}} = k$. □

Remark 4.2.3. Recall from Proposition 2.2.9 that if $\mathbb{X}$ is a set of distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, then $\text{K-dim}\, R/I_{\mathbb{X}} = k$. So, suppose that $\mathbb{X}$ is a set of points in $\mathbb{P}^n$. Then Corollary 4.2.2 implies that $1 \leq \text{depth } R/I_{\mathbb{X}} \leq 1 = \text{K-dim } R/I_{\mathbb{X}}$. Thus, sets of points in $\mathbb{P}^n$ are *always* ACM. However, if $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with $k \geq 2$, then $\mathbb{X}$ may fail to be ACM. In fact, as we show below, for every integer $l \in \{1, \ldots, k\}$ there exists a set of points $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with $\text{depth}\, R/I_{\mathbb{X}} = l$. We begin with a lemma that generalizes Example 4.0.1.

Lemma 4.2.4. *Fix a positive integer k. We denote by X_1 and X_2 the two points*

$$X_1 := [1:0] \times [1:0] \times \cdots \times [1:0], \text{ and } X_2 := [0:1] \times [0:1] \times \cdots \times [0:1],$$

in $\underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{k}$. Set $\mathbb{X} := \{X_1, X_2\}$. Then $\text{depth}\, R/I_{\mathbb{X}} = 1$.

Proof. The defining ideal of $\mathbb{X}$ is

$$I_{\mathbb{X}} = I_{X_1} \cap I_{X_2} = (\{x_{a,0}x_{b,1} \mid 1 \leq a \leq k, 1 \leq b \leq k\})$$

in the $\mathbb{N}^k$-graded ring $R = \mathbf{k}[x_{1,0}, x_{1,1}, x_{2,0}, x_{2,1}, \ldots, x_{k,0}, x_{k,1}]$. The element $\overline{x_{1,0} + x_{1,1}} \in R/I_{\mathbb{X}}$ is a non-zero divisor because $x_{1,0} + x_{1,1}$ does not vanish at either point of $\mathbb{X}$. Thus $\text{depth}\, R/I_{\mathbb{X}} \geq 1$. To complete the proof, it suffices to show that every non-zero element of $R/(I_{\mathbb{X}}, x_{1,0} + x_{1,1})$ is a zero divisor.

So, set $J = (I_{\mathbb{X}}, x_{1,0} + x_{1,1})$ and suppose that $\overline{F}$ is a non-zero element of R/J. Without loss of generality, we can take F to be $\mathbb{N}^k$-homogeneous. We write F as

$$F = F_0(x_{1,1}, x_{2,0}, \ldots, x_{k,1}) + F_1(x_{1,1}, x_{2,0}, \ldots, x_{k,1})x_{1,0} + F_2(x_{1,1}, x_{2,0}, \ldots, x_{k,1})x_{1,0}^2 + \cdots$$

Since $x_{1,0}x_{1,1} \in I_{\mathbb{X}}$, it follows that $x_{1,0}^2 = x_{1,0}(x_{1,0} + x_{1,1}) - x_{1,0}x_{1,1} \in J$. Hence, we can assume that $F = F_0 + F_1x_{1,0}$. The element $x_{1,0} \notin J$. For each integer $1 \leq b \leq k$, $x_{1,0}x_{b,1} \in I_{\mathbb{X}} \subseteq J$. Furthermore, for each integer $1 \leq a \leq k$, the element $x_{1,0}x_{a,0} = x_{a,0}\,(x_{1,0} + x_{1,1}) - x_{a,0}x_{1,1} \in J$. Hence, each term of $F_0x_{1,0}$ is in J, so $F_0x_{1,0} \in J$. Moreover, since $x_{1,0}^2 \in J$, we therefore have $Fx_{1,0} = F_0x_{1,0} + F_1x_{1,0}^2 \in J$. So, every non-zero element of R/J is a zero divisor because it is annihilated by the non-zero element $\overline{x}_{1,0}$. □

Proposition 4.2.5. *Fix a positive integer k, and let $n_1, \ldots, n_k$ be any positive integers. Then for every integer $l \in \{1, \ldots, k\}$ there exists a set $\mathbb{X}$ of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ such that* $\operatorname{depth} R/I_{\mathbb{X}} = l$.

PROOF. For every integer $l \in \{1, \ldots, k\}$ we will show how to construct a set $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with the desired depth. Define $P_i := [1 : 0 : \cdots : 0] \in \mathbb{P}^{n_i}$ for $1 \leq i \leq k$ and $Q_i := [0 : 1 : 0 \cdots : 0] \in \mathbb{P}^{n_i}$ for $1 \leq i \leq k$. Let l be an integer in $\{1, \ldots, k\}$ and let X_1 and X_2 be the following two points of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$:

$$X_1 := P_1 \times P_2 \times \cdots \times P_k \text{ and } X_2 := P_1 \times P_2 \times \cdots \times P_{l-1} \times Q_l \times \ldots \times Q_k.$$

If we let $\mathbb{X}_l := \{X_1, X_2\}$, then we will show that $\operatorname{depth} R/I_{\mathbb{X}_l} = l$.

The defining ideal of $\mathbb{X}_l$ is

$$I_{\mathbb{X}_l} = \left(\begin{array}{l} x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{l-1,1}, \ldots, x_{l-1,n_{l-1}}, x_{l,2}, \ldots, x_{l,n_l}, \ldots, x_{k,2}, \ldots, x_{k,n_k}, \\ \{x_{a,0}x_{b,1} \mid l \leq a \leq k,\ l \leq b \leq k\} \end{array} \right).$$

in the $\mathbb{N}^k$-graded ring $R = \mathbf{k}[x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{k,0}, \ldots, x_{k,n_k}]$. It then follows that $R/I_{\mathbb{X}_l} \cong S/J$ where

$$S/J = \frac{\mathbf{k}[x_{1,0}, x_{2,0}, x_{3,0}, \ldots, x_{l-1,0}, x_{l,0}, x_{l,1}, x_{l+1,0}, x_{l+1,1}, \ldots, x_{k,0}, x_{k,1}]}{(\{x_{a,0}x_{b,1} \mid l \leq a \leq k, l \leq b \leq k\})}.$$

The indeterminates $x_{1,0}, x_{2,0}, \ldots, x_{l-1,0}$ give rise to a regular sequence in S/J. Thus, $\operatorname{depth} R/I_{\mathbb{X}_l} = \operatorname{depth} S/J \geq l - 1$. Set $K = (J, x_{1,0}, \ldots, x_{l-1,0})$. Then

$$S/K \cong \frac{\mathbf{k}[x_{l,0}, x_{l,1}, x_{l+1,0}, x_{l+1,1}, \ldots, x_{k,0}, x_{k,1}]}{(\{x_{a,0}x_{b,1} \mid l \leq a \leq k, l \leq b \leq k\})}.$$

The ring S/K is then isomorphic to the $\mathbb{N}^{k-l+1}$-graded coordinate ring of the set of points

$$\{[1:0] \times [1:0] \times \cdots \times [1:0], [0:1] \times [0:1] \times \cdots \times [0:1]\}$$

in $\underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{k-l+1}$. It therefore follows from Lemma 4.2.4 that $\operatorname{depth} S/K = 1$, and hence,

$$\operatorname{depth} R/I_{\mathbb{X}_l} = l - 1 + 1 = l.$$

□

The final result of this section calculates the depth of a set of points in generic position. The proof will require the following combinatorial lemma.

Lemma 4.2.6. *Let $n, l \geq 1$ be integers. Then $\binom{n+l+1}{l+1} \leq \binom{n+l}{l}(n+1)$.*

PROOF. By definition, $\binom{n+l+1}{l+1} = \frac{(n+l+1)(n+l)\cdots(l+2)}{n!}$ and $\binom{n+l}{l} = \frac{(n+l)(n+l-1)\cdots(l+1)}{n!}$. It then follows that $\binom{n+l+1}{l+1} = \binom{n+l}{l}\cdot\frac{(n+l+1)}{(l+1)} = \binom{n+l}{l}\left(1+\frac{n}{l+1}\right)$. But because $l \geq 1$, $(1+\frac{n}{l+1}) \leq (1+n)$. The inequality now follows. □

Proposition 4.2.7. *Suppose that $\mathbb{X}$ is a set of points in generic position in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with $k > 1$ and $|\mathbb{X}| = s > 1$. Then* $\operatorname{depth} R/I_{\mathbb{X}} = 1$.

PROOF. By Corollary 4.2.2 we know $\operatorname{depth} R/I_{\mathbb{X}} \geq 1$. We show that equality holds. Without loss of generality, we assume that $n_1 \leq n_2 \leq \cdots \leq n_k$. Let l be the minimal integer such that $\binom{n_1+l}{l} > s$. Then

$$H_{\mathbb{X}}(l,0,\ldots,0) = \min\left\{\binom{n_1+l}{l}, s\right\} = s.$$

Claim. If $\underline{j} \in \mathbb{N}^k$ and $\underline{j} > (l-1,0,\ldots,0)$, then $H_{\mathbb{X}}(\underline{j}) = s$.

Proof of the Claim. There are two cases to consider: (1) $j_1 > l-1$, and (2) $j_1 = l-1$.

If $j_1 > l-1$, then $\binom{n_1+j_1}{j_1} \geq \binom{n_1+l}{l}$. Thus

$$\binom{n_1+j_1}{j_1}\cdots\binom{n_k+j_k}{j_k} \geq \binom{n_1+l}{l} \geq \min\left\{\binom{n_1+l}{l}, s\right\} = s,$$

and hence, $H_{\mathbb{X}}(\underline{j}) = s$.

So, suppose $j_1 = l-1$. Since $\underline{j} > (l-1,0,\ldots,0)$, there exists $m \in \{2,\ldots,k\}$ such that $j_m > 0$. Since $n_1 \leq n_m$, we have the following inequalities:

$$\binom{n_1+j_1}{j_1}\cdots\binom{n_k+j_k}{j_k} \geq \binom{n_1+j_1}{j_1}\binom{n_m+j_m}{j_m} \geq \binom{n_1+l-1}{l-1}\binom{n_1+1}{1}.$$

By Lemma 4.2.6, we also have $\binom{n_1+l-1}{l-1}(n_1+1) \geq \binom{n_1+l}{l}$. Hence,

$$\binom{n_1+j_1}{j_1}\cdots\binom{n_k+j_k}{j_k} \geq \binom{n_1+l}{l} \geq \min\left\{\binom{n_1+l}{l}, s\right\} = s.$$

Therefore, $H_{\mathbb{X}}(\underline{j}) = s$, as desired. □

By Lemma 2.2.12 there exists a non-zero divisor, say $\overline{L}$, of $R/I_{\mathbb{X}}$ such that $\deg L = e_1$. Let $J = (I_{\mathbb{X}}, L)$. From the short exact sequence

$$0 \longrightarrow (R/I_{\mathbb{X}})(-1,0,\ldots,0) \xrightarrow{\times\overline{L}} R/I_{\mathbb{X}} \longrightarrow R/(I_{\mathbb{X}},L) = R/J \longrightarrow 0$$

it follows that the Hilbert function of $H_{R/J}$ is

$$H_{R/J}(\underline{j}) = H_{\mathbb{X}}(\underline{j}) - H_{\mathbb{X}}(\underline{j} - e_1) \quad \text{for all } \underline{j} \in \mathbb{N}^k,$$

where $H_{\mathbb{X}}(\underline{j}) = 0$ if $\underline{j} \not\geq \underline{0}$.

From the claim, it follows that if $\underline{j} > (l, 0, \ldots, 0)$, then

$$H_{R/J}(\underline{j}) = H_{\mathbb{X}}(\underline{j}) - H_{\mathbb{X}}(\underline{j} - e_1) = s - s = 0.$$

On the other hand, if $\underline{j} = (l, 0, \ldots, 0)$, then

$$H_{R/J}(\underline{j}) = H_{\mathbb{X}}(le_1) - H_{\mathbb{X}}((l-1)e_1) = s - \binom{n_1 + l - 1}{l - 1} > 0.$$

Hence, there exists an element $F \in R_{le_1}$ such that $0 \neq \overline{F} \in R/J$.

We claim that all the non-zero elements of R/J are annihilated by $\overline{F}$, and hence, $\operatorname{depth} R/J = 0$. So, suppose that $G \in R$ is such that $0 \neq \overline{G} \in R/J$. Without loss of generality we can take G to be an $\mathbb{N}^k$-homogeneous element. with $\deg G = (j_1, \ldots, j_k) > \underline{0}$. We need to check that $FG \in J$. Now $\deg FG = (j_1 + l, j_2, \ldots, j_k) > (l, 0, \ldots, 0)$. Since $H_{R/J}(j_1 + l, j_2, \ldots, j_k) = 0$, if follows that $FG \in J$, i.e., $\overline{G}$ is annihilated by $\overline{F}$. Thus, $\operatorname{depth} R/I_{\mathbb{X}} = 1$. □

Remark 4.2.8. From the above proposition it follows that a set of s points in generic position in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with $s, k > 1$ is not ACM because $\operatorname{depth} R/I_{\mathbb{X}} = 1 < \text{K-}\dim R/I_{\mathbb{X}} = k$. By the Auslander-Buchsbaum formula, sets of points in generic position will also have the largest possible projective dimension, specifically, $\operatorname{proj.dim}_R R/I_{\mathbb{X}} = \sum_{i=1}^{k}(n_i + 1) - 1$. We need to omit the case that $|\mathbb{X}| = s = 1$ in the previous proposition because a point is a complete intersection, and hence, is ACM.

3. Hilbert Functions of ACM Sets of Points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$

Theorem 2.3.10 characterizes the Hilbert functions of sets of distinct points in $\mathbb{P}^n$. We recall that if $H : \mathbb{N} \to \mathbb{N}$ is a numerical function, then H is the Hilbert function of a set of distinct points in $\mathbb{P}^n$ if and only if the first difference function ΔH, where $\Delta H(i) := H(i) - H(i-1)$ for all $i \in \mathbb{N}$, is the Hilbert function of a graded artinian quotient of $\mathbf{k}[x_1, \ldots, x_n]$. This result was first demonstrated by Geramita, Maroscia, and

Roberts [19] (also see Corollary 2.5 of Geramita, Gregory, and Roberts [16]). The proof of the necessary condition relies on the fact that any set of distinct points $\mathbb{X} \subseteq \mathbb{P}^n$ is ACM, and hence, there exists a regular sequence of length K-$\dim R/I_{\mathbb{X}} = 1$ in $R/I_{\mathbb{X}}$. As we saw in the previous section, sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ need not be ACM, so we do not expect a similar result for arbitrary sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. However, we will show in this section that if we restrict to the ACM sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, an analogous result holds. We begin with a preparatory lemma.

Lemma 4.3.1. *Let $\mathbb{X}$ be a set of distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and let $R/I_{\mathbb{X}}$ be the $\mathbb{N}^k$-graded coordinate ring associated to $\mathbb{X}$. Suppose that $L_1, \ldots, L_t$ give rise to a regular sequence in $R/I_{\mathbb{X}}$ with $t \leq k$. Furthermore, suppose that $\deg L_i = e_i$, where e_i is the i^{th} standard basis vector for $\mathbb{N}^k$. Then there exists an $l \in \mathbb{N}$ such that*

$$(x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{t,0}, \ldots, x_{t,n_t})^l \subseteq (I_{\mathbb{X}}, L_1, L_2, \ldots, L_t).$$

Proof. Because $L_1, \ldots, L_t$ give rise to a regular sequence in $R/I_{\mathbb{X}}$, and because $\deg L_i = e_i$ for $1 \leq i \leq t$, we have the following short exact sequences with degree $(\underbrace{0, \ldots, 0}_{k})$ maps:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (R/I_{\mathbb{X}})(-1,0,\ldots,0) & \stackrel{\times \overline{L}_1}{\longrightarrow} & R/I_{\mathbb{X}} & \longrightarrow & R/J_1 & \longrightarrow & 0 \\
0 & \longrightarrow & (R/J_1)(0,-1,0,\ldots,0) & \stackrel{\times \overline{L}_2}{\longrightarrow} & R/J_1 & \longrightarrow & R/J_2 & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & (R/J_{t-1})(0,\ldots,-1,\ldots,0) & \stackrel{\times \overline{L}_t}{\longrightarrow} & R/J_{t-1} & \longrightarrow & R/J_t & \longrightarrow & 0
\end{array}$$

where $J_i := (I_{\mathbb{X}}, L_1, \ldots, L_i)$ for $i = 1, \ldots, t$.

We derive a formula for $\dim_{\mathbf{k}}(R/J_t)_{\underline{i}} = \dim_{\mathbf{k}}(R/(I_{\mathbb{X}}, L_1, \ldots, L_t))_{\underline{i}}$ for each tuple $\underline{i} = (i_1, \ldots, i_k) \in \mathbb{N}^k$ from the short exact sequences. Specifically, we have

$$\dim_{\mathbf{k}}(R/J_t)_{\underline{i}} = \sum_{\underline{0} \leq (j_1,\ldots,j_t) \leq (\underbrace{1,\ldots,1}_{t})} (-1)^{(j_1+\cdots+j_t)} \dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{i_1-j_1,\ldots,i_t-j_t,i_{t+1},\ldots,i_k}$$

where we take $\dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{\underline{i}} = 0$ if $\underline{i} \not\geq \underline{0}$.

For each integer $1 \leq j \leq t$, set $l_j = |\pi_j(\mathbb{X})|$. By Corollary 3.2.4, if $i_j \geq l_j$, then

$$\dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{0,\ldots,i_j,\ldots,0} = \dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{0,\ldots,i_j-1,\ldots,0} = l_j.$$

Hence, if $i_j = l_j$, then

$$\begin{aligned}\dim_{\mathbf{k}}(R/(I_{\mathbb{X}}, L_1, \ldots, L_t))_{0,\ldots,l_j,\ldots,0} &= \dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{0,\ldots,l_j,\ldots,0} - \dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{0,\ldots,l_j-1,\ldots,0}\\ &= l_j - l_j = 0.\end{aligned}$$

This fact implies that $R_{0,\ldots,l_j,\ldots,0} = (I_{\mathbb{X}}, L_1, \ldots, L_t)_{0,\ldots,l_j,\ldots,0}$, or equivalently, the ideal $(x_{j,0}, \ldots, x_{j,n_j})^{l_j} \subseteq (I_{\mathbb{X}}, L_1, \ldots, L_t)$. Since this holds true for each integer $1 \leq j \leq t$, there exists an integer $l \gg 0$ such that

$$(x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{t,0}, \ldots, x_{t,n_t})^l \subseteq (I_{\mathbb{X}}, L_1, \ldots, L_t).$$

This is the desired conclusion. □

Lemma 4.3.2. ([**51**] Lemma 3.55) *Let $\wp$ be a prime ideal of a commutative ring A, and let $I_1, \ldots, I_n$ be ideals of A. Then the following are equivalent:*

(*i*) $I_j \subseteq \wp$ *for some j with $1 \leq j \leq n$.*

(*ii*) $\bigcap_{i=1}^{n} I_i \subseteq \wp$.

Proposition 4.3.3. *Suppose that $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is an ACM set of distinct points. Then there exists elements $\overline{L}_1, \ldots, \overline{L}_k$ in $R/I_{\mathbb{X}}$ such that $L_1, \ldots, L_k$ give rise to a regular sequence in $R/I_{\mathbb{X}}$, and* $\deg L_i = e_i$, *where e_i is the i^{th} standard basis vector of $\mathbb{N}^k$.*

PROOF. By Lemma 2.2.12 there exists a form $L_1 \in R$ such that $\overline{L}_1$ is a non-zero divisor of the ring $R/I_{\mathbb{X}}$ and $\deg L_1 = e_1$.

So, suppose that t is an integer such that $2 \leq t \leq k$ and that we have shown that there exists forms $\overline{L}_1, \ldots \overline{L}_{t-1}$ in $R/I_{\mathbb{X}}$ such that $\deg L_i = e_i$ and such that $L_1, \ldots, L_{t-1}$ give rise to a regular sequence in $R/I_{\mathbb{X}}$. To complete the proof, it is sufficient to show that there exists an element $L_t \in R_{e_t}$ such that $\overline{L}_t$ is a non-zero divisor of the ring $R/(I_{\mathbb{X}}, L_1, \ldots, L_{t-1})$.

Let $(I_{\mathbb{X}}, L_1, \ldots, L_{t-1}) = Q_1 \cap \cdots \cap Q_r$ be the primary decomposition of $(I_{\mathbb{X}}, L_1, \ldots, L_{t-1})$. Set $\wp_i := \sqrt{Q_i}$ for $i = 1, \ldots, r$. Then the set of zero divisors of $R/(I_{\mathbb{X}}, L_1, \ldots, L_{t-1})$, denoted $\mathbf{Z}(R/(I_{\mathbb{X}}, L_1, \ldots, L_{t-1}))$, is precisely the elements of

$$\mathbf{Z}(R/(I_{\mathbb{X}}, L_1, \ldots, L_{t-1})) = \bigcup_{i=1}^{r} \overline{\wp}_i.$$

We want to show that $\mathbf{Z}(R/(I_{\mathbb{X}}, L_1, \ldots, L_{t-1}))_{e_t} \subsetneq (R/(I_{\mathbb{X}}, L_1, \ldots, L_{t-1}))_{e_t}$, or equivalently, $\bigcup_{i=1}^{r} (\wp_i)_{e_t} \subsetneq R_{e_t}$. If we can demonstrate that $(\wp_i)_{e_t} \subsetneq R_{e_t}$ for each i, then we can use Lemma 2.2.11 to show that $\bigcup_{i=1}^{r} (\wp_i)_{e_t} \subsetneq R_{e_t}$. It would then follow that there exists a form $L_t \in R_{e_t}$ such that $\overline{L}_t$ is non-zero divisor in $R/(I_{\mathbb{X}}, L_1, \ldots, L_{t-1})$.

So, suppose there exists an i in $\{1, \ldots, r\}$ such that $(\wp_i)_{e_t} = R_{e_t}$, and hence, the ideal $(x_{t,0}, \ldots, x_{t,n_t}) \subseteq \wp_i$. By Lemma 4.3.1 there also exists a positive integer l such that

$$(x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{t-1,0}, \ldots, x_{t-1,n_{t-1}})^l \subseteq (I_{\mathbb{X}}, L_1, \ldots, L_{t-1}) \subseteq Q_i.$$

Hence, the ideal $(x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{t,0}, \ldots, x_{t,n_t}) \subseteq \wp_i$.

The ideal $\wp_i$ also contains the ideal $I_{\mathbb{X}}$, and hence $I_{P_1} \cap \ldots \cap I_{P_s} \subseteq \wp_i$ where I_{P_j} is the prime ideal associated to the point of $P_j \in \mathbb{X}$. By Lemma 4.3.2, at least one of the prime ideals $I_{P_1}, \ldots, I_{P_s}$ is contained in $\wp_i$. We assume, after a possible relabelling, that $I_{P_1} \subseteq \wp_i$.

By Proposition 2.2.7 we have

$$I_{P_1} = (L_{1,1}, \ldots, L_{1,n_1}, \ldots, L_{t,1}, \ldots, L_{t,n_t}, \ldots, L_{k,1}, \ldots, L_{k,n_k})$$

where $\deg L_{m,n} = e_m$. But then, since $I_{P_1} \subseteq \wp_i$ and $(x_{1,0}, \ldots, x_{t,n_t}) \subseteq \wp_i$, the prime ideal

$$\wp := (x_{1,0}, \ldots, x_{t,n_t}, L_{t+1,1}, \ldots, L_{t+1,n_{t+1}}, \ldots, L_{k,1}, \ldots, L_{k,n_k})$$

is contained within $\wp_i$. The height of the prime ideal $\wp$ is $\mathrm{ht}_R(\wp) = \left(\sum_{i=1}^{k} n_i\right) + t$, and therefore, $\mathrm{ht}_R(\wp_i) \geq \left(\sum_{i=1}^{k} n_i\right) + t$.

On the other hand, because $\mathbb{X}$ is an ACM set of points, the ring $R/(I_{\mathbb{X}}, L_1, \ldots, L_{t-1})$ is Cohen-Macaulay by Lemma 4.1.18. Since the $\mathbb{N}^k$-homogeneous ideal $(I_{\mathbb{X}}, L_1, \ldots, L_{t-1})$ is also homogeneous with respect to the usual $\mathbb{N}$-grading, we can use Theorem 4.1.11 to compute the height of $(I_{\mathbb{X}}, L_1, \ldots, L_{t-1})$:

$$\begin{aligned} \operatorname{ht}_R((I_{\mathbb{X}}, L_1, \ldots, L_{t-1})) &= \text{K-}\dim R - \text{K-}\dim R/(I_{\mathbb{X}}, L_1, \ldots, L_{t-1}) \\ &= \left(\sum_{i=1}^{k}(n_i+1)\right) - (k-(t-1)) \\ &= \left(\sum_{i=1}^{k} n_i\right) + (t-1). \end{aligned}$$

Because $\wp_i \in \operatorname{Ass}_R(R/(I_{\mathbb{X}}, L_1, \ldots, L_{t-1}))$, from Theorem 4.1.11 it follows that

$$\left(\sum_{i=1}^{k} n_i\right) + (t-1) = \operatorname{ht}_R((I_{\mathbb{X}}, L_1, \ldots, L_{t-1})) = \operatorname{ht}_R(\wp_i) \geq \left(\sum_{i=1}^{k} n_i\right) + t.$$

But this is a contradiction. Therefore $(\wp_i)_{e_t} \subsetneq R_{e_t}$ for all $i = 1, \ldots, r$. □

We can generalize the notion of a graded artinian quotient to an $\mathbb{N}^k$-graded artinian quotient in the natural way.

Definition 4.3.4. An $\mathbb{N}^k$-homogeneous ideal $I \subseteq R = \mathbf{k}[x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{k,0}, \ldots, x_{k,n_k}]$ is an *artinian ideal* if any of the following equivalent statements hold:

- (*i*) $\text{K-}\dim R/I = 0$.
- (*ii*) $\sqrt{I} = (x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{k,0}, \ldots, x_{k,n_k})$.
- (*iii*) For each integer $1 \leq i \leq k$, there exists a positive integer t_i such that the ideal $(x_{i,0}, \ldots, x_{i,n_i})^{t_i} \subseteq I$.
- (*iv*) $H_{R/I}(i_1, 0, \ldots, 0) = 0$ for all $i_1 \gg 0$, $H_{R/I}(0, i_2, 0, \ldots, 0) = 0$ for all $i_2 \gg 0$, $\ldots$, and $H_{R/I}(0, \ldots, 0, i_k) = 0$ for all $i_k \gg 0$.

A ring $S = R/I$ is an $\mathbb{N}^k$*-graded artinian quotient* if the $\mathbb{N}^k$-homogeneous ideal I is an artinian ideal.

Remark 4.3.5. An $\mathbb{N}^k$-graded artinian quotient of R is always Cohen-Macaulay. Indeed, if R/I is such a ring, then $0 \leq \operatorname{depth} R/I \leq \text{K-}\dim R/I = 0$.

Corollary 4.3.6. *Suppose that $\mathbb{X}$ is an ACM set of distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with Hilbert function $H_{\mathbb{X}}$. Then*

$$\Delta H_{\mathbb{X}}(i_1, \ldots, i_k) := \sum_{\underline{0} \leq \underline{l} = (l_1, \ldots, l_k) \leq (1, \ldots, 1)} (-1)^{|\underline{l}|} H_{\mathbb{X}}(i_1 - l_1, \ldots, i_k - l_k),$$

where $H_{\mathbb{X}}(i_1, \ldots, i_k) = 0$ *if* $(i_1, \ldots, i_k) \not\geq \underline{0}$, *is the Hilbert function of some* $\mathbb{N}^k$*-graded artinian quotient of the ring* $\mathbf{k}[x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{k,1}, \ldots, x_{k,n_k}]$.

PROOF. By Proposition 4.3.3 there exist k forms $L_1, \ldots, L_k$ that give rise to a regular sequence in $R/I_{\mathbb{X}}$ and which have the property that $\deg L_i = e_i$. After making a linear change of variables in the $x_{1,i}$'s, a linear change of variables in the $x_{2,i}$'s, etc., we can assume that $\{L_1, \ldots, L_k\} = \{x_{1,0}, \ldots, x_{k,0}\}$. Set $J := (I_{\mathbb{X}}, x_{1,0}, \ldots, x_{k,0})/(x_{1,0}, \ldots, x_{k,0})$. Then J is an ideal of $S = \mathbf{k}[x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{k,1}, \ldots, x_{k,n_k}]$. Set $A := S/J$. Then

$$A \cong \frac{R/(x_{1,0}, \ldots, x_{k,0})}{(I_{\mathbb{X}}, x_{1,0}, \ldots, x_{k,0})/(x_{1,0}, \ldots, x_{k,0})} \cong \frac{R}{(I_{\mathbb{X}}, x_{1,0}, \ldots, x_{k,0})}.$$

Using the fact that $x_{1,0}, \ldots, x_{k,0}$ give rise to a regular sequence in $R/I_{\mathbb{X}}$ we have k short exact sequences of graded R-modules with degree $\underbrace{(0, \ldots, 0)}_{k}$ morphisms:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (R/I_{\mathbb{X}})(-1,0,\ldots,0) & \stackrel{\times \overline{x}_{1,0}}{\longrightarrow} & R/I_{\mathbb{X}} & \longrightarrow & R/J_1 & \longrightarrow & 0 \\
0 & \longrightarrow & (R/J_1)(0,-1,0,\ldots,0) & \stackrel{\times \overline{x}_{2,0}}{\longrightarrow} & R/J_1 & \longrightarrow & R/J_2 & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & (R/J_{k-1})(0,\ldots,0,-1) & \stackrel{\times \overline{x}_{k,0}}{\longrightarrow} & R/J_{k-1} & \longrightarrow & R/J_k & \longrightarrow & 0
\end{array}$$

where $J_i := (I_{\mathbb{X}}, x_{1,0}, \ldots, x_{i,0})$ for $i = 1, \ldots, k$. From the k short exact sequences it follows that

$$H_A(i_1, \ldots, i_k) = \Delta H_{\mathbb{X}}(i_1, \ldots, i_k) := \sum_{\underline{l}=(l_1,\ldots,l_k) \leq (1,\ldots,1)} (-1)^{|\underline{l}|} H_{\mathbb{X}}(i_1 - l_1, \ldots, i_k - l_k)$$

where $H_{\mathbb{X}}(\underline{i}) = 0$ if $\underline{i} \not\geq \underline{0}$, for all $(i_1, \ldots, i_k) \in \mathbb{N}^k$. That is, $\Delta H_{\mathbb{X}}$ is the Hilbert function of the $\mathbb{N}^k$-graded ring A.

By Lemma 4.3.1 there exists $l \gg 0$ such that $(x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{k,0}, \ldots, x_{k,n_k})^l \subseteq (I_{\mathbb{X}}, x_{1,0}, \ldots, x_{k,0})$. Therefore,

$$\sqrt{(I_{\mathbb{X}}, x_{1,0}, x_{2,0}, \ldots, x_{k,0})} = (x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{k,0}, \ldots, x_{k,n_k}),$$

and hence, $A \cong R/(I_{\mathbb{X}}, x_{1,0}, \ldots, x_{k,0})$ is an artinian quotient. □

In light of the previous corollary, it is natural to ask if the converse is true. We show that this is indeed the case. To demonstrate that the converse statement holds, we need to describe how to *lift* an ideal.

Definition 4.3.7. Let $R = \mathbf{k}[x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{k,0}, \ldots, x_{k,n_k}]$ and $S = \mathbf{k}[x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{k,1}, \ldots, x_{k,n_k}]$ be $\mathbb{N}^k$-graded rings. Let $I \subseteq R$ and $J \subseteq S$ be $\mathbb{N}^k$-homogeneous ideals. Then we say I is a *lifting of J to R* if

(*i*) I is radical in R

(*ii*) $\dfrac{(I, x_{1,0}, \ldots, x_{k,0})}{(x_{1,0}, \ldots, x_{k,0})} \cong J$

(*iii*) $x_{1,0}, \ldots, x_{k,0}$ give rise to a regular sequence in R/I.

If J is a monomial ideal in $S = \mathbf{k}[x_1, \ldots, x_n]$ (here S is considered as an $\mathbb{N}$-graded ring), then Hartshorne [**32**] was the first to show that J could be lifted to an ideal $I \subseteq R = \mathbf{k}[x_1, \ldots, x_n, u_1]$. This result was reproved by Geramita, Gregory, and Roberts [**16**] to show that if $J \subseteq S$ is an artinian monomial ideal, then the lifted ideal I is the ideal of a reduced set of points in $\mathbb{P}^n$. Recently, Migliore and Nagel [**40**] have generalized the construction used by Geramita, *et al.* [**16**] to show that after making some general choices, if J is a monomial ideal of S, then J can be lifted to an ideal $I \subseteq \mathbf{k}[x_1, \ldots, x_n, u_1, \ldots, u_t]$ for any t (cf. Theorem 3.4 of [**40**]). They also show, among other things, how some properties, for example the graded Betti numbers, are passed from J to the lifted ideal I.

By using the method of [**40**] we will construct from a monomial ideal J in $S = \mathbf{k}[x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{k,1}, \ldots, x_{k,n_k}]$ an $\mathbb{N}^k$-homogeneous ideal $I \subseteq R = \mathbf{k}[x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{k,0}, \ldots, x_{k,n_k}]$ that has properties (*ii*) and (*iii*) of Definition 4.3.7. The main idea is to use [**40**] to make a homogeneous ideal from J that is also $\mathbb{N}^k$-homogeneous. We begin by giving some notation and by describing the construction and results of [**40**] that we will require.

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$, and suppose that S and R are as above, but that they are $\mathbb{N}^1$-graded. For each indeterminate $x_{i,j}$ with $1 \leq i \leq k$ and $1 \leq j \leq n_i$, choose infinitely many linear forms $L_{i,j,l} \in \mathbf{k}[x_{i,j}, x_{1,0}, x_{2,0}, \ldots, x_{k,0}]$ with $l = 1, 2, \ldots$. We only assume that the coefficient of $x_{i,j}$ in $L_{i,j,l}$ is not zero. The infinite matrix A, where

$$A := \begin{bmatrix} L_{1,1,1} & L_{1,1,2} & L_{1,1,3} & \cdots \\ \vdots & \vdots & \vdots & \\ L_{1,n_1,1} & L_{1,n_1,2} & L_{1,n_1,3} & \cdots \\ \vdots & \vdots & \vdots & \\ L_{k,1,1} & L_{k,1,2} & L_{k,1,3} & \cdots \\ \vdots & \vdots & \vdots & \\ L_{k,n_k,1} & L_{k,n_k1,2} & L_{k,n_k,3} & \cdots \end{bmatrix}$$

is called a *lifting matrix*. By using the lifting matrix, we associate to each monomial $m = x_{1,1}^{a_{1,1}} \cdots x_{1,n_1}^{a_{1,n_1}} \cdots x_{k,1}^{a_{k,1}} \cdots x_{k,n_k}^{a_{k,n_k}}$ of the ring S the element

$$\overline{m} = \left[\prod_{i=1}^{n_1}\left(\prod_{j=1}^{a_{1,i}} L_{1,i,j}\right)\right] \cdots \left[\prod_{i=1}^{n_k}\left(\prod_{j=1}^{a_{k,i}} L_{k,i,j}\right)\right] \in R.$$

Note that depending on our choice of $L_{i,j,l}$'s, $\overline{m}$ may or may not be $\mathbb{N}^k$-homogeneous. However, $\overline{m}$ is homogeneous. If $J = (m_1, \ldots, m_r)$ is a monomial ideal of S, then we use I to denote the homogeneous ideal $(\overline{m}_1, \ldots, \overline{m}_r) \subseteq R$. Migliore and Nagel gave the following properties, among others, about S/J and R/I.

Proposition 4.3.8. ([**40**] Corollary 2.10) *Let $J = (m_1, \ldots, m_r) \subseteq S$ be a monomial ideal, and let $I = (\overline{m}_1, \ldots, \overline{m}_r)$ be the ideal constructed from J via any lifting matrix. Then*

(*i*) *S/J is CM if and only if R/I is CM.*
(*ii*) $\dfrac{(I, x_{1,0}, \ldots, x_{k,0})}{(x_{1,0}, \ldots, x_{k,0})} \cong J$
(*iii*) *$x_{1,0}, \ldots, x_{k,0}$ give rise to a regular sequence in R/I.*

Remark 4.3.9. Note that the construction of I from J that we have given above does not guarantee that I is a lifting of J since we do not know if I is reduced. Migliore and Nagel [**40**], however, also give some conditions on $L_{i,j,k}$ to ensure that I is also reduced.

We will now show how to lift a monomial ideal of S to an $\mathbb{N}^k$-homogeneous ideal of R. The main idea is to pick the $L_{i,j,l}$s with enough care so that $\overline{m}$ is also $\mathbb{N}^k$-homogeneous.

We begin by describing the needed notation. If

$$(\underline{\alpha}_1, \ldots, \underline{\alpha}_k) := ((a_{1,1}, \ldots, a_{1,n_1}), \ldots, (a_{k,1}, \ldots, a_{k,n_k})) \in \mathbb{N}^{n_1} \times \cdots \times \mathbb{N}^{n_k},$$

then we define $X_1^{\underline{\alpha}_1} \cdots X_k^{\underline{\alpha}_k} := x_{1,1}^{a_{1,1}} \cdots x_{1,n_1}^{a_{1,n_1}} \cdots x_{k,1}^{a_{k,1}} \cdots x_{k,n_k}^{a_{k,n_k}}$. Since $X_1^{\underline{\alpha}_1} \cdots X_k^{\underline{\alpha}_k}$ is $\mathbb{N}^k$-homogeneous, $\deg X_1^{\underline{\alpha}_1} \cdots X_k^{\underline{\alpha}_k} = (|\underline{\alpha}_1|, \ldots, |\underline{\alpha}_k|)$. Let P be the set of all monomials of S including the monomial 1. It follows that there exists a bijection between P and $\mathbb{N}^{n_1} \times \cdots \times \mathbb{N}^{n_k}$ given by the map $X_1^{\underline{\alpha}_1} \cdots X_k^{\underline{\alpha}_k} \leftrightarrow (\underline{\alpha}_1, \ldots, \underline{\alpha}_k)$. We also partially order the elements of $\mathbb{N}^{n_1} \times \cdots \times \mathbb{N}^{n_k}$ as follows: If $(\underline{\beta}_1, \ldots, \underline{\beta}_k) := ((b_{1,1}, \ldots, b_{1,n_1}), \ldots, (b_{k,1}, \ldots, b_{k,n_k}))$, then we say $(\underline{\alpha}_1, \ldots, \underline{\alpha}_k) \leq (\underline{\beta}_1, \ldots, \underline{\beta}_k)$ if $a_{i,j} \leq b_{i,j}$ for all i,j with $1 \leq i \leq k$ and $1 \leq j \leq n_i$. The statement $(\underline{\alpha}_1, \ldots, \underline{\alpha}_k) \leq (\underline{\beta}_1, \ldots, \underline{\beta}_k)$ is equivalent to the statement that $X_1^{\underline{\alpha}_1} \cdots X_k^{\underline{\alpha}_k}$ divides $X_1^{\underline{\beta}_1} \cdots X_k^{\underline{\beta}_k}$.

To each $m = X_1^{\underline{\alpha}_1} \cdots X_k^{\underline{\alpha}_k} \in P$ we associate the following $\mathbb{N}^k$-homogeneous form of R:

$$\overline{m} = \left[\prod_{i=1}^{n_1}\left(\prod_{j=1}^{a_{1,i}}(x_{1,i} - (j-1)x_{1,0})\right)\right] \cdots \left[\prod_{i=1}^{n_k}\left(\prod_{j=1}^{a_{k,i}}(x_{k,i} - (j-1)x_{k,0})\right)\right].$$

We observe that $\deg \overline{m} = \deg m = (|\underline{\alpha}_1|, \ldots, |\underline{\alpha}_k|)$. If $J = (m_1, \ldots, m_r)$ is a monomial ideal of S, then we use I to denote the $\mathbb{N}^k$-homogeneous ideal $(\overline{m}_1, \ldots, \overline{m}_r) \subseteq R$. Then, by Proposition 4.3.8, we have

Proposition 4.3.10. *Suppose $J = (m_1, \ldots, m_r)$ is a monomial ideal in the $\mathbb{N}^k$-graded ring S. Let $I = (\overline{m}_1, \ldots, \overline{m}_r)$ be the $\mathbb{N}^k$-homogeneous ideal of R constructed from J via the method described above. Then*

(i) *S/J is CM if and only if R/I is CM.*
(ii) $\dfrac{(I, x_{1,0}, \ldots, x_{k,0})}{(x_{1,0}, \ldots, x_{k,0})} \cong J$
(iii) *$x_{1,0}, \ldots, x_{k,0}$ give rise to a regular sequence in R/I.*

Proof. The crucial point is to realize that our construction of $\overline{m}$ from a monomial $m \in S$ is identical to the method described by Migliore and Nagel [**40**] using the lifting matrix

$$A = \begin{bmatrix} x_{1,1} & x_{1,1} - x_{1,0} & x_{1,1} - 2x_{1,0} & x_{1,1} - 3x_{1,0} & \cdots \\ x_{1,2} & x_{1,2} - x_{1,0} & x_{1,2} - 2x_{1,0} & x_{1,2} - 3x_{1,0} & \cdots \\ & \vdots & & & \\ x_{1,n_1} & x_{1,n_1} - x_{1,0} & x_{1,n_1} - 2x_{1,0} & x_{1,n_1} - 3x_{1,0} & \cdots \\ & \vdots & & & \\ x_{k,1} & x_{k,1} - x_{k,0} & x_{k,1} - 2x_{k,0} & x_{k,1} - 3x_{k,0} & \cdots \\ x_{k,2} & x_{k,2} - x_{k,0} & x_{k,2} - 2x_{k,0} & x_{k,2} - 3x_{k,0} & \cdots \\ & \vdots & & & \\ x_{k,n_k} & x_{k,n_k} - x_{k,0} & x_{k,n_k} - 2x_{k,0} & x_{k,n_k} - 3x_{k,0} & \cdots \end{bmatrix}.$$

The conclusions now follow from Proposition 4.3.8 because this proposition describes the properties of ideals constructed from a monomial ideal J via any lifting matrix. □

To each tuple $(\underline{\alpha}_1, \ldots, \underline{\alpha}_k) = ((a_{1,1}, \ldots, a_{1,n_1}), \ldots, (a_{k,1}, \ldots, a_{k,n_k})) \in \mathbb{N}^{n_1} \times \cdots \times \mathbb{N}^{n_k}$ we associate the point $\overline{(\underline{\alpha}_1, \ldots, \underline{\alpha}_k)} \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ where

$$\overline{(\underline{\alpha}_1, \ldots, \underline{\alpha}_k)} := [1 : a_{1,1} : a_{1,2} : \cdots : a_{1,n_1}] \times \cdots \times [1 : a_{k,1} : a_{k,2} : \cdots : a_{k,n_k}].$$

We define $\deg \overline{(\underline{\alpha}_1, \ldots, \underline{\alpha}_k)} := \deg X_1^{\underline{\alpha}_1} \cdots X_k^{\underline{\alpha}_k} = (|\underline{\alpha}_1|, \ldots, |\underline{\alpha}_k|)$. We note that if $m = X_1^{\underline{\alpha}_1} \cdots X_k^{\underline{\alpha}_k} \in P$ and if $\overline{m}$ is constructed from $X_1^{\underline{\alpha}_1} \cdots X_k^{\underline{\alpha}_k}$ as above, then $\overline{m}(\overline{(\underline{\alpha}_1, \ldots, \underline{\alpha}_k)}) \neq 0$. The following lemma is also a consequence of the definition of $\overline{(\underline{\alpha}_1, \ldots, \underline{\alpha}_k)}$ and $\overline{m}$.

Lemma 4.3.11. *Let $m = X_1^{\underline{\alpha}_1} \cdots X_k^{\underline{\alpha}_k} \in P$. Then*

(i) *$\overline{m}(\overline{(\underline{\beta}_1, \ldots, \underline{\beta}_k)}) = 0$ if and only if $(\underline{\alpha}_1, \ldots, \underline{\alpha}_k) \not\preceq (\underline{\beta}_1, \ldots, \underline{\beta}_k)$, that is, some coordinate of $(\underline{\beta}_1, \ldots, \underline{\beta}_k)$ is strictly less than some coordinate of $(\underline{\alpha}_1, \ldots, \underline{\alpha}_k)$.*

(ii) *$\overline{m}(\overline{(\underline{\beta}_1, \ldots, \underline{\beta}_k)}) = 0$ for all $(\underline{\beta}_1, \ldots, \underline{\beta}_k)$ with the property: there exists an integer i in $1 \leq i \leq k$ such that the tuple $\underline{\beta}_i \in (\underline{\beta}_1, \ldots, \underline{\beta}_k)$ satisfies $|\underline{\beta}_i| = \beta_{i,1} + \cdots + \beta_{i,n_i} \leq \alpha_{i,1} + \cdots + \alpha_{i,n_i} = |\underline{\alpha}_i|$ (except for the case that $\underline{\beta}_i = \underline{\alpha}_i$).*

Proof. Statement (i) follows immediately from the construction of $\overline{m}$.

(ii) Suppose that the point $\overline{(\underline{\beta}_1, \ldots, \underline{\beta}_k)} \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ has the property that the tuple $\underline{\beta}_i \in (\underline{\beta}_1, \ldots, \underline{\beta}_k)$ is such that $\underline{\beta}_i \neq \underline{\alpha}_i$ and $|\underline{\beta}_i| \leq |\underline{\alpha}_i|$. Then, because $\underline{\beta}_i, \underline{\alpha}_i \in \mathbb{N}^{n_i}$, there is $\beta_{i,j}$ with $1 \leq j \leq n_i$ in the tuple $\underline{\beta}_i := (\beta_{i,1}, \ldots, \beta_{i,n_i})$ such that $\beta_{i,j} < \alpha_{i,j}$

where $\alpha_{i,j}$ is the j^{th} coordinate of $\underline{\alpha}_i$. But then $(\underline{\alpha}_1, \ldots, \underline{\alpha}_k) \not\leq (\underline{\beta}_1, \ldots, \underline{\beta}_k)$, and so by (i), $\overline{m}(\overline{(\underline{\beta}_1, \ldots, \underline{\beta}_k)}) = 0$. □

If J is a monomial ideal of S, then let N be the set of monomials in J. Set $M := P \backslash N$. The elements of M are representatives for a $\mathbf{k}$-basis of the $\mathbb{N}^k$-graded ring S/J. Set

$$\overline{M} := \{ \ \overline{(\underline{\alpha}_1, \ldots, \underline{\alpha}_k)} \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \ \mid X_1^{\underline{\alpha}_1} \cdots X_k^{\underline{\alpha}_k} \in M \} \ .$$

We mimic the proof of Geramita, *et al.* [**16**] to show:

Lemma 4.3.12. *Let $J = (m_1, \ldots, m_r)$ be a a monomial ideal of S, and let $I = (\overline{m}_1, \ldots, \overline{m}_r) \subseteq R$ and $\overline{M}$ be constructed as above. Then*

$$I = \left\{ f \in R \ \middle| \ f(\overline{(\underline{\alpha}_1, \ldots, \underline{\alpha}_k)}) = 0, \overline{(\underline{\alpha}_1, \ldots, \underline{\alpha}_k)} \in \overline{M} \right\} .$$

In particular, I is a reduced ideal.

Because the proof of this lemma is very technical, we will postpone the proof until the last section of this chapter, Section 5. As a corollary, we have

Corollary 4.3.13. *Suppose $J = (m_1, \ldots, m_r)$ is a monomial ideal in the $\mathbb{N}^k$-graded ring S. Let $I = (\overline{m}_1, \ldots, \overline{m}_r)$ be the $\mathbb{N}^k$-homogeneous ideal of R constructed from J via the method described above. Then I is a lifting of J to R.*

We now state and prove the main result of the chapter.

Theorem 4.3.14. *Let $H : \mathbb{N}^k \to \mathbb{N}$ be a numerical function. Then H is the Hilbert function of an ACM set of distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ if and only if*

$$\Delta H(i_1, \ldots, i_k) = \sum_{\underline{l}=(l_1,\ldots,l_k) \leq (1,\ldots,1)} (-1)^{|\underline{l}|} H(i_1 - l_1, \ldots, i_k - l_k),$$

where $H(i_1, \ldots, i_k) = 0$ if $(i_1, \ldots, i_k) \not\geq \underline{0}$, is the Hilbert function of some $\mathbb{N}^k$-graded artinian quotient of $S = \mathbf{k}[x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{k,1}, \ldots, x_{k,n_k}]$.

Proof. Because of Corollary 4.3.6, we only need to show one direction. So, suppose ΔH is the Hilbert function of some $\mathbb{N}^k$-graded artinian quotient of S. There then exists an $\mathbb{N}^k$-homogeneous ideal $J \subseteq S$ with $\Delta H(\underline{i}) = H_{S/J}(\underline{i})$ for all $\underline{i} \in \mathbb{N}^k$. By replacing J with the leading term ideal of J (see Section 1 of Chapter 2), we can assume that $J = (m_1, \ldots, m_r)$ is a monomial ideal of S.

Let $I = (\overline{m}_1, \ldots, \overline{m}_r) \subseteq R$, where $\overline{m}_i$ is the $\mathbb{N}^k$-homogeneous form constructed from the monomial m_i via the method described after Proposition 4.3.8. By Proposition 4.3.10, $(I, x_{1,0}, \ldots, x_{k,0})/(x_{1,0}, \ldots, x_{k,0}) \cong J$ and $x_{1,0}, \ldots, x_{k,0}$ give rise to a regular sequence in R/I. Because $\deg x_{i,0} = e_i$, where e_i is the i^{th} standard basis vector of $\mathbb{N}^k$, we have the following k short exact sequences with degree $(\underbrace{0, \ldots, 0}_{k})$ maps:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (R/I)(-1, 0, \ldots, 0) & \stackrel{\times \overline{x}_{1,0}}{\longrightarrow} & R/I & \longrightarrow & R/J_1 & \longrightarrow & 0 \\
0 & \longrightarrow & (R/J_1)(0, -1, 0, \ldots, 0) & \stackrel{\times \overline{x}_{2,0}}{\longrightarrow} & R/J_1 & \longrightarrow & R/J_2 & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & (R/J_{k-1})(0, \ldots, 0, -1) & \stackrel{\times \overline{x}_{k,0}}{\longrightarrow} & R/J_{k-1} & \longrightarrow & R/J_k & \longrightarrow & 0
\end{array}$$

where $J_i := (I, x_{1,0}, \ldots, x_{i,0})$ for $i = 1, \ldots, k$. Furthermore,

$$S/J \cong \frac{R/(x_{1,0}, \ldots, x_{k,0})}{(I, x_{1,0}, \ldots, x_{k,0})/(x_{1,0}, \ldots, x_{k,0})} \cong R/(I, x_{1,0}, \ldots, x_{k,0}) = R/J_k.$$

We then use the k short exact sequences to compute the Hilbert function of R/I. This calculation will show that $H(i_1, \ldots, i_k) = H_{R/I}(i_1, \ldots, i_k)$ for all $(i_1, \ldots, i_k) \in \mathbb{N}^k$.

To complete the proof, we only need to show that I is the reduced ideal of a finite set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. It will then follow from Proposition 4.3.10 that this set of points will also be an ACM set of points because S/J is artinian, and hence, CM. If N is the set of monomials in J, then $M = P \backslash N$ is a finite set of monomials because the ring S/J is artinian. Hence

$$\overline{M} = \left\{ \overline{(\underline{\alpha}_1, \ldots, \underline{\alpha}_k)} \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \mid X_1^{\underline{\alpha}_1} \cdots X_k^{\underline{\alpha}_k} \in M \right\}$$

is a finite collection of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. By Lemma 4.3.12, the ideal I is the reduced ideal of the set of points $\overline{M} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. □

Remark 4.3.15. From Theorem 4.3.14, we see that characterizing the Hilbert functions of ACM sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is equivalent to characterizing the Hilbert functions of multi-graded artinian quotients of $\mathbf{k}[x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{k,1}, \ldots, x_{k,n_k}]$. Since we do not have a theorem like Macaulay's Theorem (Theorem 2.1.2) for $\mathbb{N}^k$-graded rings, the above theorem translates one open problem into another open problem. However, we will show in the next section that there is a Macaulay-type theorem for bigraded quotients of $\mathbf{k}[x_1, y_1, \ldots, y_m]$ and $\mathbb{N}^k$-graded quotients of $\mathbf{k}[x_{1,1}, x_{2,1}, \ldots, x_{k,1}]$. As a consequence, we can explicitly describe all the Hilbert functions of ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^m$ (cf. Corollary 4.4.15) and $\underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{k}$ (cf. Corollary 4.4.18) for any positive integer k.

In Proposition 4.2.7 we showed that if $\mathbb{X}$ is a set of points in generic position in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, then $\operatorname{depth} R/I_{\mathbb{X}} = 1$. Since $\operatorname{K-dim} R/I_{\mathbb{X}} = k$, if $k > 1$, then $\mathbb{X}$ cannot be an ACM set of points. We show that this result is also a corollary of the above theorem.

Corollary 4.3.16. *Let $s, k \in \mathbb{N}$ be such that $s, k > 1$. If $\mathbb{X}$ is a set of s distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ that is in generic position, then $\mathbb{X}$ is not an ACM set of points.*

PROOF. Without loss of generality, we assume that $n_1 \leq n_2$. Let l be the minimal integer such that $\binom{n_1+l}{l} < s$ but $\binom{n_1+l+1}{l+1} \geq s$. Since $n_2 \geq n_1$, from Lemma 4.2.6 it follows that

$$s \leq \binom{n_1+l+1}{l+1} \leq \binom{n_1+l}{l}(n_1+l) \leq \binom{n_1+l}{l}(n_2+1).$$

Because $\mathbb{X}$ is in generic position, the above inequalities imply:

$$\begin{aligned}
H_{\mathbb{X}}(l,0,\ldots,0) &= \min\left\{s, \binom{n_1+l}{l}\right\} = \binom{n_1+l}{l}, \\
H_{\mathbb{X}}(l+1,0,\ldots,0) &= \min\left\{s, \binom{n_1+l+1}{l+1}\right\} = s, \\
H_{\mathbb{X}}(l,1,\ldots,0) &= \min\left\{s, \binom{n_1+l}{l}\binom{n_2+1}{1}\right\} = s, \\
H_{\mathbb{X}}(l,0,\ldots,0) &= \min\left\{s, \binom{n_1+l+1}{l+1}\binom{n_2+1}{1}\right\} = s.
\end{aligned}$$

If $\mathbb{X}$ is an ACM set of points, then $\Delta H_{\mathbb{X}}(i_1,\ldots,i_k) \geq 0$ for all $(i_1,\ldots,i_k) \in \mathbb{N}^k$. But from the above values for $H_{\mathbb{X}}$, one finds that

$$\begin{aligned}
\Delta H_{\mathbb{X}}(l+1,1,0,\ldots,0) &= \sum_{\underline{l}=(l_1,\ldots,l_k)\leq(1,\ldots,1)} (-1)^{|\underline{l}|} H_{\mathbb{X}}(l+1-l_1, 1-l_2, 0-l_3, \ldots, 0-l_k) \\
&= H_{\mathbb{X}}(l+1,1,0,\ldots,0) - H_{\mathbb{X}}(l+1,0,\ldots,0) \\
&\quad - H_{\mathbb{X}}(l,1,0,\ldots,0) + H_{\mathbb{X}}(l,0,\ldots,0) \\
&= s - s - s + \binom{n_1+l}{l} = -s + \binom{n_1+l}{l} < 0.
\end{aligned}$$

Therefore $\mathbb{X}$ cannot be an ACM set of points. $\square$

Remark 4.3.17. We need to omit the case that $s = 1$ in the above corollary because a single point is a complete intersection, and hence, ACM.

4. The Hilbert Functions of Some $\mathbb{N}^k$-graded Artinian Quotients

The goal of this section is to characterize the Hilbert functions of $\mathbb{N}^k$-graded artinian quotients in two special cases. The first case is the case that $S = \mathbf{k}[x_1, y_1, \ldots, y_m]$ where $\deg x_1 = (1,0)$ and $\deg y_i = (0,1)$. The second case is the case that $S = \mathbf{k}[x_1, x_2, \ldots, x_k]$ with $\deg x_i = e_i$, where e_i is the i^{th} standard basis vector of $\mathbb{N}^k$, for any k. As a consequence, we can completely characterize the Hilbert functions of ACM sets of points in either $\mathbb{P}^1 \times \mathbb{P}^m$ for any positive integer m or in $\underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{k}$ for any positive integer k.

4.1. Artinian Quotients of $\mathbf{k}[x_1, y_1, \ldots, y_m]$ and their Hilbert Functions. If $S = \mathbf{k}[x_1, y_1, \ldots, y_m]$ with $\deg x_1 = (1,0)$ and $\deg y_i = (0,1)$, then in this section we characterize not only the Hilbert functions of the bigraded artinian quotients of S, but the Hilbert functions of *all* bigraded quotients of S. As a consequence, we can determine if a numerical function $H : \mathbb{N}^2 \to \mathbb{N}$ is the Hilbert function of an ACM set of points in $\mathbb{P}^1 \times \mathbb{P}^m$.

Remark 4.4.1. Suppose that $T = \mathbf{k}[x_1, y_1, \ldots, y_m]$ with $\deg y_i = (1,0)$ and $\deg x_1 = (0,1)$, that is, T is the coordinate ring associated to $\mathbb{P}^m \times \mathbb{P}^1$. The ring S and T are identical except that we have swapped the degrees of indeterminates. Hence, if I is any $\mathbb{N}^2$-homogeneous ideal of S, then I can also be considered as an $\mathbb{N}^2$-homogeneous ideal of T. Because we have switched the degrees of the indeterminates, it follows that

$$H_{S/I}(i,j) = H_{T/I}(j,i) \quad \text{for all } (i,j) \in \mathbb{N}^2.$$

Hence, to classify the Hilbert functions of quotients of T, it is enough to classify the Hilbert functions of quotients of S. Furthermore, any result that we give about the Hilbert functions of sets of points in $\mathbb{P}^1 \times \mathbb{P}^m$ is also a result about the Hilbert functions of sets of points in $\mathbb{P}^m \times \mathbb{P}^1$ by using the above identification.

To characterize the Hilbert functions of quotients of S, we need to recall some more general results about the Hilbert function of a bigraded ring. These results are due primarily to Aramova, Crona, and De Negri [**2**].

Suppose that $S = \mathbf{k}[x_1, \ldots, x_n, y_1, \ldots, y_m]$ with $\deg x_i = (1,0)$ and $\deg y_i = (0,1)$. If $x_1^{a_1} \cdots x_m^{a_m} y_1^{b_1} \cdots y_m^{b_m}$ is a monomial of S, then we write this monomial as $X^{\underline{a}}Y^{\underline{b}}$ where $\underline{a} := (a_1, \ldots, a_n) \in \mathbb{N}^n$ and $\underline{b} := (b_1, \ldots, b_m) \in \mathbb{N}^m$. Note that $\deg X^{\underline{a}}Y^{\underline{b}} = (|\underline{a}|, |\underline{b}|)$. We let $>_x$ denote the degree-lexicographical monomial ordering on S induced by $x_1 >_x x_2 >_x \cdots >_x x_n >_x y_1 >_x \cdots >_x y_m$. Similarly, we let $>_y$ denote the degree-lexicographical monomial ordering on S induced by $y_1 >_y y_2 >_y \cdots >_y y_m >_y x_1 >_y \cdots >_y x_n$. We let $M_{i,j}$ be the set of all monomials of S of degree (i,j).

Definition 4.4.2. A subset of monomials $L \subseteq M_{i,j}$ is called *bilex* if for every monomial $X^{\underline{a}}Y^{\underline{b}} \in L$, the following conditions are satisfied:

(i) if $X^{\underline{c}} \in M_{i,0}$ and $X^{\underline{c}} >_x X^{\underline{a}}$, then $X^{\underline{c}}Y^{\underline{b}} \in L$.
(ii) if $Y^{\underline{d}} \in M_{0,j}$ and $Y^{\underline{d}} >_x Y^{\underline{b}}$, then $X^{\underline{a}}Y^{\underline{d}} \in L$.

Definition 4.4.3. A monomial ideal $J \subseteq R$ is called a *bilex ideal* if $J_{i,j}$ is generated, as a $\mathbf{k}$-vector space, by a bilex set of monomials for every $(i,j) \in \mathbb{N}^2$.

For every integer $1 \leq l \leq |M_{i,j}|$, there exists a bilex subset $L \subseteq M_{i,j}$ with $|L| = l$. Indeed, order the elements of $M_{i,j}$ with respect to the monomial ordering $>_x$, and let L be the l largest elements of $M_{i,j}$. Suppose that $X^{\underline{a}}Y^{\underline{b}} \in L$, and suppose that $X^{\underline{c}} >_x X^{\underline{a}}$. Then, because $>_x$ is a monomial ordering, $X^{\underline{c}}Y^{\underline{b}} >_x X^{\underline{a}}Y^{\underline{b}}$. Since L consists of the $|L|$ largest elements of $M_{i,j}$ with respect to $>_x$, $X^{\underline{c}}Y^{\underline{b}} \in L$. A similar argument will verify the other condition of Definition 4.4.2 is satisfied, and thus, L is a bilex subset of size l. We give this special set a name.

Definition 4.4.4. If L is any bilex subset consisting of the $|L|$ largest monomials of $M_{i,j}$ with respect to the ordering $>_x$, then we call the bilex subset L the *lexsegment with respect to* $>_x$.

Remark 4.4.5. Suppose that the elements of $M_{i,j}$ are instead ordered with respect to $>_y$. For each integer $1 \leq l \leq |M_{i,j}|$, the subset L' of $M_{i,j}$ consisting of the l largest elements is a bilex subset with respect to the ordering $>_y$, that is, if $X^{\underline{a}}Y^{\underline{b}} \in L'$, then

(i) if $X^{\underline{c}} \in M_{i,0}$ and $X^{\underline{c}} >_y X^{\underline{a}}$, then $X^{\underline{c}}Y^{\underline{b}} \in L$.
(ii) if $Y^{\underline{d}} \in M_{0,j}$ and $Y^{\underline{d}} >_y Y^{\underline{b}}$, then $X^{\underline{a}}Y^{\underline{d}} \in L$.

But note that if $X^{\underline{a}}, X^{\underline{c}} \in M_{i,0}$, then $X^{\underline{c}} >_y X^{\underline{a}}$ if and only if $X^{\underline{c}} >_x X^{\underline{a}}$. Similarly, if $Y^{\underline{b}}, Y^{\underline{d}} \in M_{0,j}$, then $Y^{\underline{d}} >_y Y^{\underline{b}}$ if and only if $Y^{\underline{d}} >_x Y^{\underline{b}}$. So L' is also a bilex subset of $M_{i,j}$ with respect to the monomial ordering $>_x$, or simply, L' is a bilex set. For this reason, if L is a bilex subset of $M_{i,j}$ that consists of the $|L|$ largest elements of $M_{i,j}$ with respect to $>_y$, then L is a bilex set, and we say L is the *lexsegment with respect to* $>_y$.

Example 4.4.6. If l is an integer such that $1 \leq l \leq |M_{i,j}|$, then there may be more than one bilex subset of $M_{i,j}$ with cardinality equal to l. For example, suppose $S = \mathbf{k}[x_1, x_2, y_1, y_2]$. Then $M_{1,1} = \{x_1y_1, x_1y_2, x_2y_1, x_2y_2\}$. The subsets $L_1 = \{x_1y_1, x_1y_2\}$ and $L_2 = \{x_1y_1, x_2y_1\}$ are two different bilex subsets of $M_{i,j}$ that contain two elements. Note that L_1 is the lexsegment with respect to $>_x$ and L_2 is the lexsegment with respect to $>_y$.

Definition 4.4.7. If L is a bilex subset of $M_{i,j}$, then we denote by $\langle L \rangle$ the $\mathbf{k}$-vector subspace of $S_{i,j}$ spanned by the elements of L. We denote by $S_{0,1}\langle L \rangle$ the $\mathbf{k}$-vector subspace of $S_{i,j+1}$ spanned by the elements of the set $\{FG \mid F \in S_{0,1} \text{ and } G \in \langle L \rangle\}$. We define $S_{1,0}\langle L \rangle$ similarly.

Lemma 4.4.8.

(a) *Let L be a lexsegment with respect to $>_x$ in $M_{i,j}$. Define $YL = \{y_1, \ldots, y_m\}L := \{y_iX^{\underline{a}}Y^{\underline{b}} \mid 1 \leq i \leq m, X^{\underline{a}}Y^{\underline{b}} \in L\}$. Then*
   - (i) *YL is a lexsegment with respect to $>_x$ in $M_{i,j+1}$.*
   - (ii) *YL is a monomial basis for $S_{0,1}\langle L \rangle$ as a $\mathbf{k}$-vector subspace of $S_{i,j+1}$.*

(b) *Let L be a lexsegment with respect to $>_y$ in $M_{i,j}$. Define $XL = \{x_1, \ldots, x_n\}L := \{x_iX^{\underline{a}}Y^{\underline{b}} \mid 1 \leq i \leq n, X^{\underline{a}}Y^{\underline{b}} \in L\}$. Then*
   - (i) *XL is a lexsegment with respect to $>_y$ in $M_{i+1,j}$.*
   - (ii) *XL is a monomial basis for $S_{1,0}\langle L \rangle$ as a $\mathbf{k}$-vector subspace of $S_{i+1,j}$.*

PROOF. For (i) of (a), this is Lemma 4.6 of [**2**]. The second conclusion of (a) is immediate. The proof of (b) is the same. □

If i and a are two positive integers, then we recall that the i-binomial expansion of a is the unique expression

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$$

where $a_i > a_{i-1} > \cdots > a_j \geq j \geq 1$. The function ${}^{<i>} : \mathbb{N} \to \mathbb{N}$ is defined by

$$a \longmapsto a^{<i>} = \binom{a_i + 1}{i+1} + \binom{a_{i-1}+1}{i} + \cdots + \binom{a_j + 1}{j+1}$$

where $a_i, a_{i-1}, \ldots, a_j$ are as in the i-binomial expansion of a. With this notation we have:

Proposition 4.4.9. *Let $S = \mathbf{k}[x_1, \ldots, x_n, y_1, \ldots, y_m]$.*

(*a*) *Let L be a lexsegment with respect to $>_x$ in $M_{i,j}$. Let*

$$\dim_{\mathbf{k}} S_{i,j}/\langle L\rangle = \binom{m-1+j}{j} q + r$$

be the Euclidean division of $\dim_{\mathbf{k}} S_{i,j}/\langle L\rangle$ by $\binom{m-1+j}{j}$. Then

$$\dim_{\mathbf{k}} S_{i,j+1}/S_{0,1}\langle L\rangle = \binom{m+j}{j+1} q + r^{<j>}.$$

(*b*) *Let L be a lexsegment with respect to $>_y$ in $M_{i,j}$. Let*

$$\dim_{\mathbf{k}} S_{i,j}/\langle L\rangle = \binom{n-1+i}{i} q_1 + r_1$$

be the Euclidean division of $\dim_{\mathbf{k}} S_{i,j}/\langle L\rangle$ by $\binom{n-1+i}{i}$. Then

$$\dim_{\mathbf{k}} S_{i,j+1}/S_{1,0}\langle L\rangle = \binom{n+i}{i+1} q_1 + r_1^{<i>}.$$

PROOF. This is Proposition 4.16 of [**2**]. □

With these definitions and results, among others, Aramova, *et al.* were able to place bounds on the values of the Hilbert function of a bigraded ring S/I. This result is given below.

Theorem 4.4.10. ([**2**] Theorem 4.18) *Let I be a bihomogeneous ideal of the bigraded ring $S = \mathbf{k}[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Also, let $H_{S/I}(i,j) = \dim_{\mathbf{k}}(S/I)_{i,j}$ be the Hilbert function of S/I. Moreover, let*

$$H_{S/I}(i,j) = \binom{m-1+j}{j} q + r \quad \text{and} \quad H_{S/I}(i,j) = \binom{n-1+i}{i} q_1 + r_1$$

be the Euclidean division of $H_{S/I}(i,j)$ by $\binom{m-1+j}{j}$ and $\binom{n-1+i}{i}$, respectively. Then

(*i*) $H_{S/I}(i, j+1) \leq \binom{m+j}{j+1} q + r^{<j>}$.

(*ii*) $H_{S/I}(i+1, j) \leq \binom{n+i}{i+1} q_1 + r_1^{<i>}$.

We our now in position to prove the first major result of this section.

Theorem 4.4.11. *Let $H : \mathbb{N}^2 \to \mathbb{N}$ be a numerical function. Then there exists a bihomogeneous ideal $I \subsetneq S = \mathbf{k}[x_1, y_1, \ldots, y_m]$ such that the Hilbert function $H_{S/I} = H$ if and only if*

(*i*) $H(0,0) = 1$,
(*ii*) $H(0,1) \leq m$,
(*iii*) $H(i+1,j) \leq H(i,j)$ *for all* $(i,j) \in \mathbb{N}^2$, *and*
(*iv*) $H(i,j+1) \leq H(i,j)^{<j>}$ *for all* $(i,j) \in \mathbb{N}^2$ *with* $j \geq 1$.

Proof. Let I be the bihomogeneous ideal of S with Hilbert function $H_{S/I} = H$. Then assertions (*i*) and (*ii*) are immediate. For (*iii*), we observe that $\binom{n-1+i}{i} = \binom{1-1+i}{i} = 1$ for each positive integer i. Hence, the Euclidean division of $H_{S/I}(i,j)$ by $\binom{1-1+i}{i}$ is

$$H_{S/I}(i,j) = \binom{1-1+i}{i} H_{S/I}(i,j) + 0.$$

From Theorem 4.4.10, it follows that

$$H_{S/I}(i+1,j) \leq \binom{1+i}{i+1} H_{S/I}(i,j) + 0^{<i>} = H_{S/I}(i,j).$$

To prove that (*iv*) holds, we need to first recall that

$$H_{S/I}(i,j) = \dim_{\mathbf{k}} S_{i,j} - \dim_{\mathbf{k}} I_{i,j} = \binom{m-1+j}{j} - \dim_{\mathbf{k}} I_{i,j}.$$

If $\dim_{\mathbf{k}} I_{i,j} = 0$, then the Euclidean division of $H_{S/I}(i,j)$ by $\binom{m-1+j}{j}$ is

$$H_{S/I}(i,j) = \binom{m-1+j}{j} 1 + 0.$$

Using Theorem 4.4.10 to calculate an upper bound for $H_{S/I}(i,j+1)$, we get

$$H_{S/I}(i,j+1) \leq \binom{m+j}{j+1} 1 + 0^{<j>} = \binom{m-1+j}{j}^{<j>} = H_{S/I}(i,j)^{<j>}.$$

On the other hand, if $\dim_{\mathbf{k}} I_{i,j} > 0$, then the Euclidean division of $H_{S/I}(i,j)$ by $\binom{m-1+j}{j}$ yields

$$H_{S/I}(i,j) = \binom{m-1+j}{j} 0 + \left[\binom{m-1+j}{j} - \dim_{\mathbf{k}} I_{i,j}\right].$$

By applying Theorem 4.4.10 we therefore have

$$H_{S/I}(i,j+1) \leq \binom{m+j}{j+1}0 + \left[\binom{m-1+j}{j} - \dim_{\mathbf{k}} I_{i,j}\right]^{<j>} = H_{S/I}(i,j)^{<j>}.$$

This completes the proof that conditions (i)-(iv) are necessary.

To prove the converse, we require some lemmas that describe some of the properties of bilex subsets in $S = \mathbf{k}[x_1, y_1, \ldots, y_m]$.

Lemma 4.4.12. *Let $M_{i,j}$ be the set of monomials of degree (i,j) in $S = \mathbf{k}[x_1, y_1, \ldots, y_m]$. Then, for each integer $1 \leq l \leq |M_{i,j}|$, there is exactly one bilex subset $L \subseteq M_{i,j}$ with $|L| = l$.*

PROOF. Let L be the l largest elements of $M_{i,j}$ with respect to $>_x$. Then, as noted earlier, L is a bilex set with l elements. Now suppose that there exists a bilex set $L' \subseteq M_{i,j}$ with $L' \neq L$, but $|L'| = |L| = l$. Because $L' \neq L$, there exists a monomial $m \in L'$ such that $m \notin L$. Let $\widetilde{m}$ be any element of L. Then $\widetilde{m} >_x m$. Because $\widetilde{m}, m \in M_{i,j}$, we therefore have

$$\widetilde{m} = x_1^i y_1^{b_1} \cdots y_m^{b_m} >_x x_1^i y_1^{c_1} \cdots y_m^{c_m} = m \;\Leftrightarrow\; y_1^{b_1} \cdots y_m^{b_m} >_x y_1^{c_1} \cdots y_m^{c_m}.$$

Since L' is bilex, it follows that $\widetilde{m} \in L'$. Hence, $L \subsetneq L'$. But then $l = |L| < |L'| = l$. □

Lemma 4.4.13. *Let L_1, L_2 be two bilex subsets of $M_{i,j}$ in $S = \mathbf{k}[x_1, y_1, \ldots, y_m]$. If $|L_1| \leq |L_2|$, then $L_1 \subseteq L_2$.*

PROOF. The only bilex subset consisting of $|L_1|$ (respectively, $|L_2|$) elements is the bilex subset consisting of the $|L_1|$ (respectively, $|L_2|$) largest elements of $M_{i,j}$ with respect to $>_x$. The conclusion follows from this observation. □

We now return to the proof of the theorem. Assertions (ii) and (iv) imply that $H(0,j) \leq \binom{m-1+j}{j}$ for all j. It follows from (iii) that $H(i,j) \leq H(0,j) \leq \binom{m-1+j}{j}$ for all $(i,j) \in \mathbb{N}^2$.

Let $S = \mathbf{k}[x_1, y_1, \ldots, y_m]$ and let $M_{i,j}$ be the $\binom{m-1+j}{j}$ monomials of degree (i,j) in S. For each $(i,j) \in \mathbb{N}^2$, let $L_{i,j}$ be a bilex subset of $M_{i,j}$ consisting of $\left[\binom{m-1+j}{j} - H(i,j)\right] \geq 0$ elements. Because of Lemma 4.4.12, there is only choice for $L_{i,j}$.

Claim. For all (i,j), $S_{0,1}\langle L_{i,j}\rangle \subseteq \langle L_{i,j+1}\rangle$ and $S_{1,0}\langle L_{i,j}\rangle \subseteq \langle L_{i+1,j}\rangle$.

Proof of the Claim. A basis for $S_{0,1}\langle L_{i,j}\rangle$ is the set of monomials $YL_{i,j}$. If we can show that $|YL_{i,j}| \le |L_{i,j+1}|$, it would then follow from Lemma 4.4.13 that $YL_{i,j} \subseteq L_{i,j+1}$, or equivalently, $S_{0,1}\langle L_{i,j}\rangle \subseteq \langle L_{i,j+1}\rangle$.

Let $\dim_{\mathbf{k}} S_{i,j}/\langle L_{i,j}\rangle = \binom{m-1+j}{j}q + r$ be the Euclidean division of $\dim_{\mathbf{k}} S_{i,j}/\langle L_{i,j}\rangle$ by $\binom{m-1+j}{j}$. Because $L_{i,j}$ is also the lexsegment with respect to $>_x$, we calculate from Proposition 4.4.9 that

$$\dim_{\mathbf{k}} S_{i,j+1}/S_{0,1}\langle L_{i,j}\rangle = \binom{m-1+j}{j}q + r^{<j>} = H(i,j)^{<j>}.$$

Hence

$$\begin{aligned} |YL_{i,j}| = \dim_{\mathbf{k}} S_{0,1}\langle L_{i,j}\rangle &= \binom{m+j}{j+1} - H(i,j)^{<j>} \\ &\le \binom{m+j}{j+1} - H(i,j+1) \\ &= \dim_{\mathbf{k}}\langle L_{i,j+1}\rangle = |L_{i,j+1}|. \end{aligned}$$

Thus $|YL_{i,j}| \le |L_{i,j+1}|$ as desired.

Similarly, a basis for $S_{1,0}\langle L_{i,j}\rangle$ is the set of monomials $XL_{i,j}$. The set $XL_{i,j}$ is a bilex set because of Lemma 4.4.8. Moreover, since there is only one bilex set of size $|XL_{i,j}|$, the set $XL_{i,j}$ must also be the lexsegment with respect to $>_y$. By using Proposition 4.4.9, we calculate that

$$\begin{aligned} |XL_{i,j}| = \dim_{\mathbf{k}} S_{1,0}\langle L_{i,j}\rangle &= \dim_{\mathbf{k}} S_{i+1,j} - H(i,j) \\ &\le \binom{m-1+j}{j} - H(i+1,j) \\ &= \dim_{\mathbf{k}}\langle L_{i+1,j}\rangle = |L_{i+1,j}|. \end{aligned}$$

Because $|XL_{i,j}| \le |L_{i+1,j}|$, we conclude from Lemma 4.4.13 that $S_{1,0}\langle L_{i,j}\rangle \subseteq \langle L_{i+1,j}\rangle$. □

Let I be the ideal generated by all the monomials in all the bilex sets $L_{i,j}$ where $(i,j) \in \mathbb{N}^2$. Since I is generated by monomials, it is bihomogeneous. We claim that for every $(i,j) \in \mathbb{N}^2$, $I_{i,j} = \langle L_{i,j}\rangle$, that is, I is a bilex ideal. Indeed, let $F \in I$ be bihomogeneous of degree (i,j). Then, either $F \in \langle L_{i,j}\rangle$ and then clearly $F \in I_{i,j}$, or $F = HG$ where $G \in \langle L_{i',j'}\rangle$ with $(i',j') < (i,j)$ and $H \in S_{i-i',j-j'}$. But from the above claim, it follows that

$$S_{i-i',j-j'}\langle L_{i',j'}\rangle \subseteq \langle L_{i,j}\rangle.$$

Thus, $F \in I_{i,j}$. This then completes the proof because the Hilbert function of S/I is

$$\begin{aligned} H_{S/I}(i,j) &= \dim_{\mathbf{k}} R_{i,j} - \dim_{\mathbf{k}} I_{i,j} = \binom{m-1+j}{j} - \dim_{\mathbf{k}} \langle L_{i,j} \rangle \\ &= \binom{m-1+j}{j} - \left[\binom{m-1+j}{j} - H(i,j) \right] = H(i,j) \end{aligned}$$

for all $(i,j) \in \mathbb{N}^2$. □

Corollary 4.4.14. *Let $H : \mathbb{N}^2 \to \mathbb{N}$ be a numerical function. Then H is the Hilbert function of a bigraded artinian quotient of $\mathbf{k}[x_1, y_1, \ldots, y_m]$ if and only if*

(*i*) $H(0,0) = 1$,

(*ii*) $H(0,1) \leq m$,

(*iii*) $H(i+1,j) \leq H(i,j)$ *for all* $(i,j) \in \mathbb{N}^2$,

(*iv*) $H(i,j+1) \leq H(i,j)^{<j>}$ *for all* $(i,j) \in \mathbb{N}^2$ *with* $j \geq 1$,

(*v*) *there exists a positive integer t such that* $H(t,0) = 0$, *and*

(*vi*) *there exists a positive integer r such that* $H(0,r) = 0$.

Proof. Suppose that $I \subseteq S = \mathbf{k}[x_1, y_1, \ldots, y_m]$ is a bihomogeneous artinian ideal such that $H_{S/I} = H$. Then conditions (i)-(iv) are a consequence of Theorem 4.4.11. Assertions (v) and (vi) follow from the definition of an artinian quotient.

Conversely, conditions (i)-(iv) imply the existence of a bihomogeneous ideal I in the ring $S = \mathbf{k}[x_1, y_1, \ldots, y_m]$ such that the Hilbert function of S/I is equal to H. The final two conditions would then imply that I is an artinian ideal. □

By coupling the above corollary with Proposition 4.3.14, we get a complete description of the Hilbert functions of ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^m$. We express this formally as a corollary.

Corollary 4.4.15. *Let $H : \mathbb{N}^2 \to \mathbb{N}$ be a numerical function. Then H is the Hilbert function of an ACM set of points in $\mathbb{P}^1 \times \mathbb{P}^m$ if and only if the numerical function ΔH satisfies conditions (i)-(vi) of Corollary 4.4.14.*

4.2. Artinian Quotients of $\mathbf{k}[x_1, \ldots, x_k]$ and their Hilbert Functions. Suppose that $S = \mathbf{k}[x_1, \ldots, x_k]$ and $\deg x_i = e_i$, where e_i is the i^{th} standard basis vector of $\mathbb{N}^k$.

Just as in the previous section, we will show a stronger result by characterizing the Hilbert functions of all quotients of S, not only the artinian quotients.

Theorem 4.4.16. *Let $S = \mathbf{k}[x_1, \ldots, x_k]$ be an $\mathbb{N}^k$-graded ring with $\deg x_i = e_i$, the i^{th} standard basis vector of $\mathbb{N}^k$, and let $H : \mathbb{N}^k \to \mathbb{N}$ be a numerical function. Then there exists a proper ideal $I \subsetneq S$ such that the Hilbert function $H_{S/I} = H$ if and only if*

(*i*) $H(0, \ldots, 0) = 1$,
(*ii*) $H(\underline{i}) = 1$ *or* 0 *if* $\underline{i} > \underline{0}$, *and*
(*iii*) *if* $H(\underline{i}) = 0$, *then* $H(\underline{j}) = 0$ *for all* $\underline{j} \geq \underline{i}$.

Proof. Suppose that $I \subsetneq S$ and that $H_{S/I} = H$. Then condition (i) is a consequence of the fact that $I \subsetneq S$. For (ii), we recall the definition of $H_{S/I}(\underline{i})$:

$$0 \leq H_{S/I}(\underline{i}) \quad := \quad \dim_{\mathbf{k}}(S/I)_{\underline{i}} = \dim_{\mathbf{k}} S_{\underline{i}} - \dim_{\mathbf{k}} I_{\underline{i}} = 1 - \dim_{\mathbf{k}} I_{\underline{i}} \leq 1.$$

Hence, $H_{S/I}(\underline{i}) = 1$ or 0. Finally, if $H_{S/I}(\underline{i}) = 0$, this implies that $x_1^{i_1} \cdots x_k^{i_k} \in I$, or equivalently, $S_{\underline{i}} \subseteq I$ because $x_1^{i_1} \cdots x_k^{i_k}$ is a monomial basis for $S_{\underline{i}}$. But then, if $\underline{j} \geq \underline{i}$, then $S_{\underline{j}} \subseteq I$, that is, $H_{S/I}(\underline{j}) = 0$.

Conversely, suppose that H is a numerical function that satisfies conditions $(i) - (iii)$. If $H(\underline{i}) = 1$ for all $\underline{i} \in \mathbb{N}^k$, then the ideal $I = (0) \subseteq S = \mathbf{k}[x_1, \ldots, x_k]$ has the property that $H_{S/I} = H$.

So, suppose $H(\underline{i}) \neq 1$ for all $\underline{i}$. Set $\mathcal{I} := \{(i_1, \ldots, i_k) \mid H(\underline{i}) = 0\}$. Note that $\mathcal{I} \neq \mathbb{N}^k$ because $\underline{0} \notin \mathcal{I}$. In the ring $S = \mathbf{k}[x_1, \ldots, x_k]$, let I be the ideal $I := \langle \{x_1^{i_1} \cdots x_k^{i_k} \mid \underline{i} \in \mathcal{I}\} \rangle$. We claim that $H_{S/I}(\underline{i}) = H(\underline{i})$ for all $\underline{i} \in \mathbb{N}^k$. It is immediate that $H_{S/I}(\underline{0}) = H(\underline{0}) = 1$. Moreover, if $H(\underline{i}) = 0$, then $H_{S/I}(\underline{i}) = 0$ because $x_1^{i_1} \cdots x_k^{i_k} \in I_{\underline{i}} \subseteq I$, i.e., $S_{\underline{i}} \subseteq I$.

So, we only need to check: if $H(\underline{i}) = 1$, then $H_{S/I}(\underline{i}) = 1$. Suppose $H_{S/I}(\underline{i}) = 0$. This implies that $x_1^{i_1} \cdots x_k^{i_k} \in I$. But because $\underline{i} \notin \mathcal{I}$, there is a monomial $x_1^{j_1} \cdots x_k^{j_k} \in I$ with $\underline{j} \in \mathcal{I}$, such that $x_1^{j_1} \cdots x_k^{j_k}$ divides $x_1^{i_1} \cdots x_k^{i_k}$. But this is equivalent to the statement that $\underline{j} \leq \underline{i}$. But this contradicts hypothesis (iii). So $H_{S/I}(\underline{i}) = 1$. □

Corollary 4.4.17. *Let $S = \mathbf{k}[x_1, \ldots, x_k]$ be an $\mathbb{N}^k$-graded ring where $\deg x_i = e_i$, and let $H : \mathbb{N}^k \to \mathbb{N}$ be a numerical function. Then H is the Hilbert function of an $\mathbb{N}^k$-graded artinian quotient of S if and only if* (*i*) $H(0,\ldots,0) = 1$,

(*ii*) $H(\underline{i}) = 1$ *or* 0 *if* $\underline{i} > (0,\ldots,0)$,

(*iii*) *if* $H(\underline{i}) = 0$, *then* $H(\underline{j}) = 0$ *for all* $(\underline{j}) \geq (\underline{i})$, *and*

(*iv*) *for each integer* $1 \leq i \leq k$, *there exists an integer* t_i *such that* $H(t_1,0,\ldots,0) = H(0,t_2,0,\ldots,0) = \cdots = H(0,\ldots,0,t_k) = 0$.

Proof. This result follows from Theorem 4.4.16 and the definition of an $\mathbb{N}^k$-graded artinian quotient of S. □

Corollary 4.4.18. *Let* $H : \mathbb{N}^k \rightarrow \mathbb{N}$ *be a numerical function. Then* H *is the Hilbert function of an ACM set of distinct points in* $\underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{k}$ *if and only* ΔH *satisfies conditions* $(i) - (iv)$ *of Corollary 4.4.17.*

Remark 4.4.19. It follows from the previous corollaries that H is the Hilbert function of an ACM set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ if and only if

(*i*) $\Delta H(i,j) = 1$ or 0,

(*ii*) if $\Delta H(i,j) = 0$, then $\Delta H(i',j') = 0$ for all $(i',j') \in \mathbb{N}^2$ with $(i',j') > (i,j)$, and

(*iii*) there exists integers t and r such that $\Delta H(t,0) = 0$ and $\Delta H(0,r) = 0$.

Giuffrida, Maggioni, and Ragusa proved precisely this result in Theorem 4.1 and Theorem 4.2 of [**26**].

5. The Proof of Lemma 4.3.12

For this section we will use the standard notation $(i_1,\ldots,\hat{i}_i,\ldots,i_k)$ to denote the tuple $(i_1,\ldots,i_{i-1},i_{i+1},\ldots,i_k)$. In this section we prove Lemma 4.3.12. The proof of this lemma relies on the following lemma.

Lemma 4.5.1. *Suppose that* $R = \mathbf{k}[x_{1,0},\ldots,x_{1,n_1},\ldots,x_{k,0},\ldots,x_{k,n_k}]$ *and that* R *is* $\mathbb{N}^k$*-graded. Let* $f \in R$ *be a form of degree* $(d_1,\ldots,d_k)$. *If* $f(\overline{(\underline{\alpha}_1,\ldots,\underline{\alpha}_k)}) = 0$ *for all* $\overline{(\underline{\alpha}_1,\ldots,\underline{\alpha}_k)} \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ *of degree* $\leq (d_1,\ldots,d_k)$, *then* $f = 0$.

PROOF. If $k = 1$ and n_1 is any positive integer, then this lemma is Lemma 2.3 of Geramita, Gregory, and Roberts [**16**]. We will generalize this result to all positive $k \in \mathbb{N}$.

Set

$$\overline{P}_{d_1,\ldots,d_k} := \left\{ \overline{(\underline{\alpha}_1, \ldots, \underline{\alpha}_k)} \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \;\middle|\; \deg \overline{(\underline{\alpha}_1, \ldots, \underline{\alpha}_k)} \le (d_1, \ldots, d_k) \right\}.$$

The set $\overline{P}_{d_1,\ldots,d_k}$ consists of $\binom{d_1+n_1}{n_1} \cdots \binom{d_k+n_k}{n_k}$ points.

If k is any positive integer, $n_1, \ldots, n_k$ arbitrary positive integers, and $(d_1, \ldots, d_k) = (0, \ldots, 0)$, then $\overline{P}_{d_1,\ldots,d_k}$ consists of exactly 1 point. The only forms of degree $(0, \ldots, 0)$ in R are the constants; hence, if f vanishes at the single point of $\overline{P}_{d_1,\ldots,d_k}$, we must have $f = 0$.

Let $k \ge 1$ be a positive integer, and suppose that $n_1 = \ldots = n_k = 1$, and that $d_i > 0$ but $d_1 = \ldots = \hat{d_i} = \ldots = d_k = 0$. Then the set $\overline{P}_{0,\ldots,d_i,\ldots,0}$ consists of exactly $d_i + 1$ points. If $f \in R_{0,\ldots,d_i,\ldots,0}$, then f is a form of degree $(0, \ldots, d_i, \ldots, 0)$ in the indeterminates $x_{i,0}, x_{i,1}$. So, if f vanishes at the $d_i + 1$ distinct points of $\overline{P}_{0,\ldots,d_i,\ldots,0}$, then $f = 0$.

We now want to show that for any $k \in \mathbb{N}$, if $n_1 = \ldots = n_k = 1$ and $(d_1, \ldots, d_k) \in \mathbb{N}^k$ is arbitrary, then the lemma holds true. We proceed by induction on k and $(d_1, \ldots, d_k)$, that is, we assume that the lemma holds for all sets $\overline{P}_{c_1,\ldots,c_l} \subseteq \underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{l}$ if either $1 \le l < k$, or if $l = k$ and $(d_1, \ldots, d_k) >_{lex} (c_1, \ldots, c_k) \ge_{lex} \underline{0}$, where $>_{lex}$ denotes the lexicographical ordering. We will then show that the lemma is also true for $\overline{P}_{d_1,\ldots,d_k} \subseteq \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$.

So, suppose that $f \in R_{d_1,\ldots,d_k}$ and that f vanishes on $\overline{P}_{d_1,\ldots,d_k}$. Since $(d_1, \ldots, d_k) >_{lex} \underline{0}$, there is at least one coordinate of $(d_1, \ldots, d_k)$, say d_i, such that $d_i > 0$. If $d_j = 0$ for all j in $1 \le j \le k$ with $j \ne i$, then, as already noted, f must be zero if f vanishes on $\overline{P}_{d_1,\ldots,d_k}$. So, we can assume that at least two coordinates of $(d_1, \ldots, d_k)$, say d_i and d_j with $i \ne j$ are such that $d_i, d_j > 0$.

Now consider the subset of $\overline{P}_{d_1,\ldots,d_k}$ that vanishes on $x_{i,1} - d_i x_{i,0}$. We are assuming that $d_i > 0$. The elements of this subset are precisely the points:

$$\mathcal{P}_1 = \left\{ [1 : a_1] \times \cdots \times [1 : d_i] \times \cdots \times [1 : a_k] \;\middle|\; \begin{array}{l} ((a_1), \ldots, \widehat{(a_i)}, \ldots, (a_k)) \in \underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{k-1} \\ (a_1, \ldots, \hat{a}_i, \ldots, a_k) \le (d_1, \ldots, \hat{d}_i, \ldots, d_k) \end{array} \right\}.$$

We write f as $f = (x_{i,1} - d_i x_{i,0})g + r$ where $\deg g = (d_1, \ldots, d_i - 1, \ldots, d_k)$ and r is a degree $(d_1, \ldots, d_k)$ polynomial such that no term of r is divisible by $x_{i,1}$. Since $n_1 = \cdots = n_k = 1$, the term $r = r_0 x_{i,0}^{d_i}$ where r_0 is a polynomial in $R' = \mathbf{k}[x_{1,0}, x_{1,1}, \ldots, \hat{x}_{i,0}, \hat{x}_{i,1}, \ldots, x_{k,0}, x_{k,1}]$. But because r must vanish at the points in $\mathcal{P}_1$, this implies that r_0 vanishes at the $(d_1 + 1)\cdots\widehat{(d_i+1)}\cdots(d_k+1)$ points in the set

$$\left\{ [1:a_1] \times \cdots \times \widehat{[1:a_i]} \times \cdots \times [1:a_k] \;\middle|\; \begin{array}{l} ((a_1), \ldots, \widehat{(a_i)}, \ldots, (a_k)) \in \underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{k-1} \\ (a_1, \ldots, \hat{a}_i, \ldots, a_k) \leq (d_1, \ldots, \hat{d}_i, \ldots, d_k) \end{array} \right\}.$$

The above set is the set of points $\overline{P}_{d_1,\ldots,\hat{d}_i,\ldots,d_k} \subseteq \underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{k-1}$, and thus, by the induction hypothesis, $r_0 = 0$, and hence, $r = 0$.

Because of our assumption on $(d_1, \ldots, d_k)$, $\deg g = (d_1, \ldots, d_i - 1, \ldots, d_k) >_{lex} \underline{0}$. Moreover, g must vanish at the points of $\overline{P}_{d_1,\ldots,d_k}$ that do not vanish on $(x_{i,1} - d_i x_{i,0})$. These points are

$$\begin{aligned} \overline{P}_{d_1,\ldots,d_k} \backslash \mathcal{P}_1 &= \{[1:a_1] \times \cdots \times [1:a_i] \times \cdots \times [1:a_k] \in \overline{P}_{d_1,\ldots,d_k} \mid a_i \neq d_i\} \\ &= \left\{ [1:a_1] \times \cdots \times [1:a_i] \times \cdots \times [1:a_k] \;\middle|\; \begin{array}{l} \text{if } \underline{a} := (a_1, \ldots, a_k) \text{ then} \\ \underline{a} \leq (d_1, \ldots, d_i - 1, \ldots, d_k) \end{array} \right\} \\ &= \overline{P}_{d_1,\ldots,d_i-1,\ldots,d_k}. \end{aligned}$$

Thus, g is a form in $R_{d_1,\ldots,d_i-1,\ldots,d_k}$ that vanishes at all the points of $\overline{P}_{d_1,\ldots,d_i-1,\ldots,d_k}$. Since $(d_1, \ldots, d_k) >_{lex} (d_1, \ldots, d_i - 1, \ldots, d_k)$, we have $g = 0$ by the induction hypothesis. So $f = 0$, and thus the lemma holds for all positive integers k and all $(d_1, \ldots, d_k) \in \mathbb{N}^k$ in $\underbrace{\mathbb{P}^1 \times \ldots \times \mathbb{P}^1}_{k}$.

We will now show that for any $k > 1$, if $n_1, \ldots, n_k$ are arbitrary positive integers, and $(d_1, \ldots, d_k) \in \mathbb{N}^k$ is arbitrary, then the lemma holds for $\overline{P}_{d_1,\ldots,d_k} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. We proceed by induction on the tuples $(n_1, \ldots, n_k)$ and $(d_1, \ldots, d_k)$ and on k, that is, we assume that $\overline{P}_{c_1,\ldots,c_l} \subseteq \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_l}$ satisfies the lemma if either (i) $1 \leq l < k$, (ii) if $k = l$ and $(n_1, \ldots, n_k) >_{lex} (m_1, \ldots, m_k) \geq_{lex} \underline{1}$, or (iii) if $k = l$ and $(n_1, \ldots, n_k) = (m_1, \ldots, m_k)$, and $(d_1, \ldots, d_k) >_{lex} (c_1, \ldots, c_k) \geq_{lex} \underline{0}$.

So, suppose $\overline{P}_{d_1,\ldots,d_k} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, $f \in R_{d_1,\ldots,d_k}$ and f vanishes at all the points of $\overline{P}_{d_1,\ldots,d_k}$. If there is a d_i in $(d_1, \ldots, d_k)$ such that $d_i = 0$, then we can consider f as an $\mathbb{N}^{k-1}$-homogeneous element in the ring $R' = \mathbf{k}[x_{1,0}, \ldots, \hat{x}_{i,0}, \ldots \hat{x}_{i,n_i}, \ldots x_{k,n_k}]$, i.e.,

$f \in R'_{d_1,\ldots,\hat{d}_i,\ldots,d_k}$. But then f vanishes at all the points of

$$\overline{P}'_{d_1,\ldots,\hat{d}_i,\ldots,d_k} \subseteq \mathbb{P}^{n_1} \times \cdots \times \widehat{\mathbb{P}^{n_i}} \times \cdots \times \mathbb{P}^{n_k},$$

and hence, by the induction hypothesis, $f = 0$. Thus, we can assume that $(d_1, \ldots, d_k) \geq_{lex} \underline{1}$.

By assumption, the tuple $(n_1, \ldots, n_k) >_{lex} (1, \ldots, 1)$, so there is an n_i in $(n_1, \ldots, n_k)$ such that $n_i > 1$. Since f vanishes on the points of $\overline{P}_{d_1,\ldots,d_k}$, the form f must also vanish at the $\binom{d_1+n_1}{n_1} \cdots \binom{d_1+n_i-1}{n_i-1} \cdots \binom{d_k+n_k}{n_k}$ points of $\overline{P}_{d_1,\ldots,d_k}$ that vanish on the degree $e_i = (0, \ldots, 1, \ldots, 0)$ form $x_{i,1} = 0$. We write $f = x_{i,1}g + r$ where $\deg g = (d_1, \ldots, d_i - 1, \ldots, d_k)$ and r is a form of degree $(d_1, \ldots, d_k)$ in the indeterminates $x_{1,0}, \ldots, \hat{x}_{i,1}, \ldots, x_{k,n_k}$. Note that from our assumption about $(d_1, \ldots, d_k)$, $\deg g >_{lex} \underline{0}$.

Now the form r must vanish at the $\binom{d_1+n_1}{n_1} \cdots \binom{d_1+n_i-1}{n_i-1} \cdots \binom{d_k+n_k}{n_k}$ points of $\overline{P}_{d_1,\ldots,d_k}$ that vanish at $x_{i,1} = 0$. But we can consider this subset of points as the set of points $\overline{P}_{d_1,\ldots,d_k} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_i-1} \times \cdots \times \mathbb{P}^{n_k}$. Thus, by the induction hypothesis, $r = 0$.

It then follows that the form g must vanish on the points of $\overline{P}_{d_1,\ldots,d_k}$ not on the form $x_{i,1} = 0$, that is, if $\overline{(\underline{\alpha}_1, \ldots, \underline{\alpha}_k)} \in \overline{P}_{d_1,\ldots,d_k}$ and if the coordinate $a_{i,1} \in \underline{\alpha}_i$ is nonzero, then $g(\overline{(\underline{\alpha}_1, \ldots, \underline{\alpha}_k)}) = 0$. We set

$$\mathcal{P}_2 := \left\{ \overline{(\underline{\alpha}_1, \ldots, \underline{\alpha}_k)} \in \overline{P}_{d_1,\ldots,d_k} \;\middle|\; g(\overline{(\underline{\alpha}_1, \ldots, \underline{\alpha}_k)}) = 0 \right\}.$$

We define G to be the form of $R_{d_1,\ldots,d_i-1,\ldots,d_k}$ such that

$$G(x_{1,0}, \ldots, x_{k,n_k}) = g(x_{1,0}, \ldots, x_{i,0}, x_{i,0} + x_{i,1}, x_{i,2}, \ldots, x_{k,n_k}).$$

Claim. The form G vanishes at all the points of $\overline{P}_{d_1,\ldots,d_i-1,\ldots,d_k} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

Proof of the Claim. If $\overline{(\underline{\alpha}_1, \ldots, \underline{\alpha}_k)} \in \overline{P}_{d_1,\ldots,d_i-1,\ldots,d_k}$, then

$$G(\overline{(\underline{\alpha}_1, \ldots, \underline{\alpha}_k)}) = g(1, a_{1,1}, \ldots, a_{1,n_1}, \ldots, 1, a_{i,1} + 1, a_{i,2}, \ldots, a_{k,n_k}).$$

The point $[1 : a_{1,1} : \cdots : a_{1,n_1}] \times \cdots \times [1 : a_{i,1} + 1 : a_{i,2} : \cdots : a_{i,n_i}] \times \cdots \times [1 : a_{k,1} : \cdots : a_{k,n_k}] \in \mathcal{P}_2$ because $a_{i,1} + 1 \neq 0$ and the degree of this point $\leq (d_1, \ldots, d_i - 1 + 1, \ldots, d_k)$. Hence, $G(\overline{(\underline{\alpha}_1, \ldots, \underline{\alpha}_k)}) = 0$. □

By the induction hypothesis, $G = 0$ and thus, $g = 0$ because G is constructed from g by making a linear change of variables. Thus the form $f = 0$, as desired. □

PROOF. (of Lemma 4.3.12) Recall that $M := P\backslash N$ where P is the set of all monomials, including 1, in the polynomial ring S, and N is the set of all monomials contained in the monomial ideal J. We also define

$$\overline{M} := \{\ \overline{(\underline{\alpha}_1,\ldots,\underline{\alpha}_k)} \in \mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}\ \Big|\ X_1^{\underline{\alpha}_1}\cdots X_k^{\underline{\alpha}_k}\in M\}\ .$$

Suppose that $X_1^{\underline{\alpha}_1}\cdots X_k^{\underline{\alpha}_k} \in M$ and that $m_i := X_1^{\underline{\beta}_{i,1}}\cdots X_k^{\underline{\beta}_{i,k}}$ is one of the minimal generators of $J := (m_1,\ldots,m_r)$. Then, because $X_1^{\underline{\alpha}_1}\cdots X_k^{\underline{\alpha}_k} \in P\backslash N$, it follows that $m_i \nmid X_1^{\underline{\alpha}_1}\cdots X_k^{\underline{\alpha}_k}$, and hence $(\underline{\beta}_{i,1},\ldots,\underline{\beta}_{i,k}) \not\leq (\underline{\alpha}_1,\ldots,\underline{\alpha}_k)$. Thus, by Lemma 4.3.11, $\overline{m}_i(\overline{(\underline{\alpha}_1,\ldots,\underline{\alpha}_k)}) = 0$. This is true for all $\overline{m}_i \in \{\overline{m}_1,\ldots,\overline{m}_r\}$. Because this set of $\mathbb{N}^k$-homogeneous elements is the set of generators for the ideal I, we have

$$I\subseteq\left\{f\in R\ \Big|\ f(\overline{(\underline{\alpha}_1,\ldots,\underline{\alpha}_k)})=0, \overline{(\underline{\alpha}_1,\ldots,\underline{\alpha}_k)}\in\overline{M}\right\}\ .$$

Conversely, suppose that $f \in R$ is an $\mathbb{N}^k$-homogeneous element of degree $(d_1,\ldots,d_k)$ and that f vanishes at all the points of $\overline{M}$. Let

$$\{\overline{(\underline{\alpha}_{1,1},\ldots,\underline{\alpha}_{1,k})},\overline{(\underline{\alpha}_{2,1},\ldots,\underline{\alpha}_{2,k})},\overline{(\underline{\alpha}_{3,1},\ldots,\underline{\alpha}_{3,k})},\ldots\}$$

be the points of $\overline{P}_{d_1,\ldots,d_k}\backslash\overline{M}$, where $\overline{P}_{d_1,\ldots,d_k}$ is defined as in Lemma 4.5.1. Furthermore, order the elements of $\overline{P}_{d_1,\ldots,d_k}\backslash\overline{M}$ so that for each positive $i\in\mathbb{N}$,

$$\deg\overline{(\underline{\alpha}_{i,1},\ldots,\underline{\alpha}_{i,k})}\leq_{lex}\deg\overline{(\underline{\alpha}_{i+1,1},\ldots,\underline{\alpha}_{i+1,k})},$$

that is, $(|\underline{\alpha}_{i,1}|,\ldots,|\underline{\alpha}_{i,k}|) \leq_{lex} (|\underline{\alpha}_{i+1,1}|,\ldots,|\underline{\alpha}_{i+1,k}|)$. Those points that have the same degree may be put in any order.

Since $\overline{(\underline{\alpha}_{1,1},\ldots,\underline{\alpha}_{1,k})} \in \overline{P}_{d_1,\ldots,d_k}\backslash\overline{M}$, it follows that $h = X_1^{\underline{\alpha}_{1,1}}\cdots X_k^{\underline{\alpha}_{1,k}} \in J$, and hence $h = X_1^{\underline{\beta}_{1,1}}\cdots X_k^{\underline{\beta}_{1,k}}m_i$ for some minimal generator $m_i\in J$. So $\overline{h} = \overline{X_1^{\underline{\beta}_{1,1}}\cdots X_k^{\underline{\beta}_{1,k}}m_i}$ is a multiple of $\overline{m}_i$, and therefore $\overline{h}\in I$ and $\overline{h}$ vanishes at all the points of $\overline{M}$.

On the other hand, by Lemma 4.3.11, $\overline{h}(\overline{(\underline{\alpha}_{1,1},\ldots,\underline{\alpha}_{1,k})}) \neq 0$ because $h = X_1^{\underline{\alpha}_{1,1}}\cdots X_k^{\underline{\alpha}_{1,k}}$. Thus $\lambda_1 := f(\overline{(\underline{\alpha}_{1,1},\ldots,\underline{\alpha}_{1,k})})/h(\overline{(\underline{\alpha}_{1,1},\ldots,\underline{\alpha}_{1,k})}) \in \mathbf{k}$. Now consider the form $f_1 := f - \lambda_1\overline{h}x_{1,0}^{t_1}\cdots x_{k,0}^{t_k}$ where $(t_1,\ldots,t_k) = \deg f - \deg\overline{h}$. By construction, f_1 vanishes at all the points of $\overline{M}$ and at the point $\overline{(\underline{\alpha}_{1,1},\ldots,\underline{\alpha}_{1,k})}$.

We now repeat this process by replacing f with f_1 and using the element $\overline{(\underline{\alpha}_{2,1},\ldots,\underline{\alpha}_{2,k})} \in \overline{P}_{d_1,\ldots,d_k}\backslash\overline{M}$ to construct a form f_2, and so on, until we have used all the elements of

$\overline{P}_{d_1,\ldots,d_k}\backslash\overline{M}$. Our ordering of the elements ensures that when we change f_{j-1} to vanish at the new point $\overline{(\underline{\alpha}_{j,1},\ldots,\underline{\alpha}_{j,k})}$, the form f_j vanishes at all the previous points and on $\overline{M}$.

Indeed, suppose that f_{j-1} vanishes at $\overline{M}$ and at the points in the set

$$\left\{\overline{(\underline{\alpha}_{1,1},\ldots,\underline{\alpha}_{1,k})},\ldots,\overline{(\underline{\alpha}_{j-1,1},\ldots,\underline{\alpha}_{j-1,k})}\right\}.$$

Since $\overline{(\underline{\alpha}_{j,1},\ldots,\underline{\alpha}_{j,k})} \in \overline{P}_{d_1,\ldots,d_k}\backslash\overline{M}$, $h_j = X_1^{\underline{\alpha}_{j,1}}\cdots X_k^{\underline{\alpha}_{j,k}} \in J$. Thus $\overline{h}_j \in I$. But since $\overline{h}_j(\overline{(\underline{\alpha}_{j,1},\ldots,\underline{\alpha}_{j,k})}) \neq 0$, the number $\lambda_j = f(\overline{(\underline{\alpha}_{j,1},\ldots,\underline{\alpha}_{j,k})})/\overline{h}_j(\overline{(\underline{\alpha}_{j,1},\ldots,\underline{\alpha}_{j,k})}) \in \mathbf{k}$. Set $f_j := f_{j-1} - \lambda_j\overline{h}_j x_{1,0}^{t_1}\cdots x_{k,0}^{t_k}$ where $(t_1,\ldots,t_k) = \deg f_{j-1} - \deg\overline{h}_j$. Because $\overline{h}_j \in I$, f_j vanishes at all the points of $\overline{M}$. Because of the ordering of the elements in $\overline{P}_{d_1,\ldots,d_k}\backslash\overline{M}$, if

$$\overline{(\underline{\alpha}_{i,1},\ldots,\underline{\alpha}_{i,k})} \in \left\{\overline{(\underline{\alpha}_{1,1},\ldots,\underline{\alpha}_{1,k})},\ldots,\overline{(\underline{\alpha}_{j-1,1},\ldots,\underline{\alpha}_{j-1,k})}\right\},$$

then $(|\underline{\alpha}_{i,1}|,\ldots,|\underline{\alpha}_{i,k}|) \leq_{lex} (|\underline{\alpha}_{j,1}|,\ldots,|\underline{\alpha}_{j,k}|)$, i.e., there is an integer $l \in \{1,\ldots,k\}$ such that the tuple $\underline{\alpha}_{i,l}$ has the property that $|\underline{\alpha}_{i,l}| \leq |\underline{\alpha}_{j,l}|$ and $\underline{\alpha}_{i,l} \neq \underline{\alpha}_{j,l}$. But then by Lemma 4.3.11 (ii), we have $\overline{h}_j(\overline{(\underline{\alpha}_{i,1},\ldots,\underline{\alpha}_{i,k})}) = 0$ if $1 \leq i \leq j-1$.

When we have completed the above process, we end up with a form $f - G \in I$ such that $f - G$ vanishes at all the points of $\overline{P}_{d_1,\ldots,d_k}$. By Lemma 4.5.1 we must therefore have $f - G = 0$, and so $f = G \in I$, as desired. $\square$

CHAPTER 5

The Hilbert Function of Sets of Points in $\mathbb{P}^1 \times \mathbb{P}^1$

The Hilbert function of a set of points on the quadric surface $\mathcal{Q} \subseteq \mathbb{P}^3$ was first studied by Giuffrida, Maggioni, and Ragusa (see [**26**] but also [**24**], [**25**]). Because $\mathcal{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$, Giuffrida, *et al.* pioneered the study of Hilbert functions of sets of points in multi-projective space. Giuffrida, *et al.* [**26**] demonstrated a number of necessary conditions for the Hilbert function of a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. However, a complete characterization of these functions continues to be elusive.

The aim of this chapter is to extend the work of Giuffrida, *et al.* by using the results of the earlier chapters to study the Hilbert functions of sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$. This chapter is structured as follows. We begin this chapter by specializing some of our previous results to points in $\mathbb{P}^1 \times \mathbb{P}^1$ and by recalling some of the results of Giuffrida, *et al.* found in [**26**]. We also describe how one can "visualize" sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$.

In the second section we give a characterization of the tuples that can be the border of a Hilbert function of a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ (cf. Theorem 5.2.8 and Corollary 5.2.11). The proof of this result relies on a connection between points in $\mathbb{P}^1 \times \mathbb{P}^1$ and $(0,1)$-matrices. This result answers Question 3.1.10 for points in $\mathbb{P}^1 \times \mathbb{P}^1$. By answering this question, we have introduced a new necessary condition on the Hilbert function of a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$.

In the third section we demonstrate some applications of the border. Specifically, we can: (1) compute a lower bound for the number of distinct Hilbert functions for s points in $\mathbb{P}^1 \times \mathbb{P}^1$; (2) characterize the Hilbert functions of those sets of points $\mathbb{X}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ with either $|\pi_1(\mathbb{X})| = 2$ or $|\pi_2(\mathbb{X})| = 2$; and (3) compute the Hilbert function of certain subsets $\mathbb{Y} \subseteq \mathbb{X}$ from knowledge about $H_{\mathbb{X}}$.

In the last section we characterize the arithmetically Cohen-Macaulay sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ (cf. Theorem 5.4.4). Arithmetically Cohen-Macaulay sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ were first classified via their Hilbert function by Giuffrida, *et al.* [**26**]. We provide a new proof of this result. We also give a new characterization of ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ that only relies upon combinatoric information about $\mathbb{X}$. As a consequence, both the Hilbert function and the Betti numbers of the resolution of an ACM set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ depend only upon the configuration of the points of $\mathbb{X}$, that is, how the points are arranged in $\mathbb{P}^1 \times \mathbb{P}^1$, and not upon the coordinates of the points (cf. Theorem 5.4.9 and Theorem 5.4.11).

1. General Remarks on Points in $\mathbb{P}^1 \times \mathbb{P}^1$

Let $\mathbb{X}$ be a finite set of s distinct points in $\mathbb{P}^1 \times \mathbb{P}^1$, and let $H_{\mathbb{X}}$ be the Hilbert function of $\mathbb{X}$. In this section we study of the Hilbert function of $\mathbb{X}$ by applying some of the earlier results of this thesis, and by describing the results of Giuffrida, *et al.* [**26**].

Let $\pi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection morphism defined by $P_1 \times P_2 \mapsto P_1$. It follows that $\pi_1(\mathbb{X})$ is a finite set of points in $\mathbb{P}^1$. If we suppose that $|\pi_1(\mathbb{X})| = t \leq s$, then Proposition 2.2.10 and Proposition 2.3.8 can be combined to show

$$H_{\mathbb{X}}(i,0) = \begin{cases} i+1 & 0 \leq i \leq t-1 \\ t & i \geq t \end{cases}.$$

Similarly, if $\pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is the other projection morphism, and if $|\pi_2(\mathbb{X})| = r$, then

$$H_{\mathbb{X}}(0,j) = \begin{cases} j+1 & 0 \leq j \leq r-1 \\ r & j \geq r \end{cases}.$$

For this chapter we shall write $H_{\mathbb{X}}$ as an infinite matrix $(m_{i,j})$ where $m_{i,j} = H_{\mathbb{X}}(i,j)$ and $(i,j) \in \mathbb{N}^2$. By the above observations, and the fact that every Hilbert function of points has a border (see Corollary 3.1.7), we therefore have

$$(5.1.1) \qquad H_{\mathbb{X}} = \begin{bmatrix} 1 & 2 & \cdots & r-1 & \mathbf{r} & r & \cdots \\ 2 & & & & \mathbf{m_{1,r-1}} & m_{1,r-1} & \cdots \\ \vdots & & * & & \vdots & \vdots & \\ t-1 & & & & \mathbf{m_{2,r-1}} & m_{2,r-1} & \cdots \\ \mathbf{t} & \mathbf{m_{t-1,1}} & \cdots & \mathbf{m_{t-1,r-2}} & \mathbf{s} & s & \cdots \\ t & m_{t-1,1} & \cdots & m_{t-1,r-2} & s & s & \\ \vdots & & & \vdots & \vdots & & \ddots \end{bmatrix},$$

where the bold numbers are the border and the entries denoted by $*$ need to be calculated. If $\Delta H_{\mathbb{X}}$ is the first difference function of $H_{\mathbb{X}}$, i.e.,

$$\Delta H_{\mathbb{X}} = H_{\mathbb{X}}(i,j) - H_{\mathbb{X}}(i-1,j) - H_{\mathbb{X}}(i,j-1) + H_{\mathbb{X}}(i-1,j-1),$$

where $H_{\mathbb{X}}(i,j) = 0$ if $(i,j) \not\geq (0,0)$, then we also write $\Delta H_{\mathbb{X}}$ as an infinite matrix. Moreover, by (5.1.1), we have

$$\Delta H_{\mathbb{X}} = \begin{array}{c} 0 \\ \\ \\ \\ t-1 \\ \\ \\ \end{array} \overset{\begin{array}{cccccc} 0 & & & r-1 & & \end{array}}{\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 0 & \cdots \\ 1 & & & & & 0 & \cdots \\ \vdots & & * & & & \vdots & \\ 1 & & & & & 0 & \cdots \\ 1 & & & & & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \\ \vdots & & & \vdots & \vdots & & \ddots \end{bmatrix}}. \tag{5.1.2}$$

The properties of the matrix $M_{\mathbb{X}} = (m_{i,j})$ with $m_{i,j} = H_{\mathbb{X}}(i,j)$ were studied in [**26**]. In that paper the matrix $M_{\mathbb{X}}$ was called the *Hilbert matrix*; however, we will refrain from using this name to prevent any confusion with the *Hilbert-Burch matrix*. We recall the definition of an admissible matrix, as defined in [**26**], in order to state a necessary condition on the Hilbert function of a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$.

Definition 5.1.1. Let $M = (m_{i,j})$ be a matrix with $(i,j) \in \mathbb{N}^2$ and $m_{i,j} \in \mathbb{N}$. For every $(i,j) \in \mathbb{N}^2$, let $c_{i,j} = m_{i,j} - m_{i-1,j} - m_{i,j-1} + m_{i-1,j-1}$ where $m_{i,j} = 0$ if $(i,j) \not\geq (0,0)$. Set $\Delta M = (c_{i,j})$. The matrix M is an *admissible matrix* if $\Delta M = (c_{i,j})$ satisfies the following conditions:

(i) $c_{i,j} \leq 1$ and $c_{i,j} = 0$ for $i \gg 0$ and $j \gg 0$,
(ii) if $c_{i,j} \leq 0$, then $c_{r,s} \leq 0$ for all $(r,s) \geq (i,j)$, and
(iii) for every $(i,j) \in \mathbb{N}^2$,
- if $i > 1$, then $0 \leq \sum_{t=0}^{j} c_{i,t} \leq \sum_{t=0}^{j} c_{i-1,t}$, and
- if $j > 1$, then $0 \leq \sum_{t=0}^{i} c_{t,j} \leq \sum_{t=0}^{i} c_{t,j-1}$.

Theorem 5.1.2. ([**26**] Theorem 2.11) *Let $\mathbb{X}$ be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ with Hilbert function $H_{\mathbb{X}}$. Then $H_{\mathbb{X}}$, written as a matrix, is an admissible matrix.*

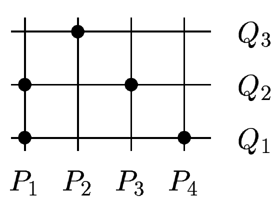
Remark 5.1.3. The conclusion of the previous theorem is only a necessary condition. Example 2.14 of [**26**] is an example of an admissible matrix that is not the Hilbert function of any set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Example 5.3.7 below shows that there exists an infinite family of such examples.

In this chapter we "draw" examples of sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$. We end this section by providing a justification for such "pictures". Because $\mathbb{P}^1 \times \mathbb{P}^1 \cong \mathcal{Q}$, where $\mathcal{Q}$ is the quadric surface of $\mathbb{P}^3$, from Exercise I.2.15 of Hartshorne [**31**] it follows that there exist two families of lines $\{L_P\}$ and $\{L'_P\}$, each parameterized by $P \in \mathbb{P}^1$, with the property that if $P \neq R \in \mathbb{P}^1$, then $L_P \cap L_R = \emptyset$ and $L'_P \cap L'_R = \emptyset$, and for all $P, R \in \mathbb{P}^1$, $L_P \cap L'_R = P \times R$, a point on $\mathcal{Q}$. In other words, $\mathcal{Q}$ is a ruled surface.

Hence, we can visualize a collection of points $\mathbb{X}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ as points on $\mathcal{Q}$. By first drawing $|\pi_1(\mathbb{X})| = t$ lines in one ruling and indexing the lines by the elements of $\pi_1(\mathbb{X})$, and then by drawing the $|\pi_2(\mathbb{X})| = r$ lines in the second ruling and indexing these line by $\pi_2(\mathbb{X})$, the set $\mathbb{X}$ is contained in the complete intersection (Definition 4.1.20) defined by these lines.

For example, if $\mathbb{X} = \{P_1 \times Q_1, P_1 \times Q_2, P_2 \times Q_3, P_3 \times Q_2, P_4 \times Q_1\} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$, then there are $|\pi_1(\mathbb{X})| = 4$ lines from one ruling which are indexed by $\{P_1, P_2, P_3, P_4\}$, and there are $|\pi_2(\mathbb{X})| = 3$ lines from the other ruling indexed by $\{Q_1, Q_2, Q_3\}$. We visualize this set as

$\mathbb{X} =$



where the dots represent the points in $\mathbb{X}$.

2. Classifying the Borders of Hilbert Functions of Points in $\mathbb{P}^1 \times \mathbb{P}^1$

In Chapter 3 we defined the border of a Hilbert function for points $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Question 3.1.10 asks what tuples can be the border of a Hilbert function of a set of points. For points $\mathbb{X} \subseteq \mathbb{P}^n \times \mathbb{P}^m$ this question reduces to describing all possible eventual column vectors B_C and eventual row vectors B_R. We wish to answer this question for points in $\mathbb{P}^1 \times \mathbb{P}^1$.

So, suppose that $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is a collection of s distinct points. We associate to $\mathbb{X}$ two tuples, $\alpha_{\mathbb{X}}$ and $\beta_{\mathbb{X}}$, as follows. For each $P_i \in \pi_1(\mathbb{X}) = \{P_1, \ldots, P_t\}$ we set $\alpha_i := |\pi_1^{-1}(P_i)|$. After relabelling the α_i's so that $\alpha_i \geq \alpha_{i+1}$ for $i = 1, \ldots, t-1$, we set $\alpha_{\mathbb{X}} := (\alpha_1, \ldots, \alpha_t)$. Analogously, for every $Q_i \in \pi_2(\mathbb{X}) = \{Q_1, \ldots, Q_r\}$ we set $\beta_i := |\pi_2^{-1}(Q_i)|$. After relabelling the β_i's so that $\beta_i \geq \beta_{i+1}$ for $i = 1, \ldots, r-1$, we let $\beta_{\mathbb{X}}$ be the r-tuple $\beta_{\mathbb{X}} := (\beta_1, \ldots, \beta_r)$. We note that $\alpha_{\mathbb{X}}$ and $\beta_{\mathbb{X}}$ are both partitions (see Definition 2.5.1) of the integer $s = |\mathbb{X}|$. Thus, we can write $\alpha_{\mathbb{X}} \vdash s$ and $\beta_{\mathbb{X}} \vdash s$. If we denote the length of $\alpha_{\mathbb{X}}$ (resp. $\beta_{\mathbb{X}}$) by $|\alpha_{\mathbb{X}}|$ (resp. $|\beta_{\mathbb{X}}|$), then we also observe that $|\pi_1(\mathbb{X})| = |\alpha_{\mathbb{X}}|$ and $|\pi_2(\mathbb{X})| = |\beta_{\mathbb{X}}|$.

As an application of Propositions 3.1.1 and 3.1.4, we demonstrate that for points $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ the eventual column vector B_C and the eventual row vector B_R can be computed directly from the tuples $\alpha_{\mathbb{X}}$ and $\beta_{\mathbb{X}}$.

Proposition 5.2.1. *Let $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a set of s distinct points and suppose that $\alpha_{\mathbb{X}} = (\alpha_1, \ldots, \alpha_t)$ and $\beta_{\mathbb{X}} = (\beta_1, \ldots, \beta_r)$. Let $B_C = (b_0, b_1, \ldots, b_{r-1})$ where $b_j = H_{\mathbb{X}}(t-1, j)$, be the eventual column vector of the Hilbert function $H_{\mathbb{X}}$. Then*

$$b_j = \#\{\alpha_i \in \alpha_{\mathbb{X}} \mid \alpha_i \geq 1\} + \#\{\alpha_i \in \alpha_{\mathbb{X}} \mid \alpha_i \geq 2\} + \cdots + \#\{\alpha_i \in \alpha_{\mathbb{X}} \mid \alpha_i \geq j+1\}.$$

Analogously, if $B_R = (b'_0, b'_1, \ldots, b'_{t-1})$, with $b'_j = H_{\mathbb{X}}(j, r-1)$, is the eventual row vector of $H_{\mathbb{X}}$, then

$$b'_j = \#\{\beta_i \in \beta_{\mathbb{X}} \mid \beta_i \geq 1\} + \#\{\beta_i \in \beta_{\mathbb{X}} \mid \beta_i \geq 2\} + \cdots + \#\{\beta_i \in \beta_{\mathbb{X}} \mid \beta_i \geq j+1\}.$$

Proof. After relabelling the elements of $\pi_1(\mathbb{X})$, we can assume that $|\pi_1^{-1}(P_i)| = \alpha_i$. By Proposition 3.1.1 and Remark 3.1.2 we have

$$\begin{aligned} b_j = H_{\mathbb{X}}(t-1, j) &= \#\left\{P_i \in \pi_1(\mathbb{X}) \;\middle|\; H_{Q_{P_i}}(j) \geq 1\right\} + \#\left\{P_i \in \pi_1(\mathbb{X}) \;\middle|\; H_{Q_{P_i}}(j) \geq 2\right\} + \cdots \\ &\quad + \#\left\{P_i \in \pi_1(\mathbb{X}) \;\middle|\; H_{Q_{P_i}}(j) \geq j+1\right\}. \end{aligned}$$

Now $Q_{P_i} = \pi_2(\pi_1^{-1}(P_i))$ is a subset of α_i points in $\mathbb{P}^1$. If $1 \leq k \leq j+1$, then $H_{Q_{P_i}}(j) \geq k$ if and only if $|\pi_1^{-1}(P_i)| \geq k$. This is a consequence of Proposition 2.3.8. This in turn implies that the sets $\left\{P_i \in \pi_1(\mathbb{X}) \;\middle|\; H_{Q_{P_i}}(j) \geq k\right\}$ and $\left\{P_i \in \pi_1(\mathbb{X}) \mid |\pi_1^{-1}(P_i)| \geq k\right\}$ are the same, and thus, the numbers $\#\{P_i \in \pi_1(\mathbb{X}) \mid H_{Q_{P_i}}(j) \geq k\}$ and $\#\{\alpha_i \in \alpha_{\mathbb{X}} \mid \alpha_i \geq k\}$ are equal.

The desired identity now follows from this result. The statement about the eventual row vector B_R is proved similarly. □

We can rewrite the above result more succinctly by invoking the language of combinatorics. Recall that the conjugate of a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is the tuple $\lambda^* = (\lambda_1^*, \ldots, \lambda_{\lambda_1}^*)$ where $\lambda_j^* := \#\{\lambda_i \in \lambda \mid \lambda_i \geq j\}$.

Definition 5.2.2. If $p = (p_1, p_2, \ldots, p_k)$, then $\Delta p := (p_1, p_2 - p_1, \ldots, p_k - p_{k-1})$.

Corollary 5.2.3. *Let $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be s distinct points with $\alpha_{\mathbb{X}}$ and $\beta_{\mathbb{X}}$. Then*

(i) $\Delta B_C = \alpha_{\mathbb{X}}^*$.
(ii) $\Delta B_R = \beta_{\mathbb{X}}^*$.

Proof. Using Proposition 5.2.1 to calculate ΔB_C we get

$$\Delta B_C = (\#\{\alpha_i \in \alpha_{\mathbb{X}} \mid \alpha_i \geq 1\}, \#\{\alpha_i \in \alpha_{\mathbb{X}} \mid \alpha_i \geq 2\}, \ldots, \#\{\alpha_i \in \alpha_{\mathbb{X}} \mid \alpha_i \geq r\}).$$

The conclusion follows by noting that $\#\{\alpha_i \in \alpha_{\mathbb{X}} \mid \alpha_i \geq j\}$ is by definition the j^{th} coordinate of $\alpha_{\mathbb{X}}^*$. The proof of (ii) is the same as (i). □

Remark 5.2.4. For each positive integer j we have the following identity:

$$\#\{\alpha_i \in \alpha_{\mathbb{X}} \mid \alpha_i \geq j\} - \#\{\alpha_i \in \alpha_{\mathbb{X}} \mid \alpha_i \geq j+1\} = \#\{\alpha_i \in \alpha_{\mathbb{X}} \mid \alpha_i = j\}.$$

Since Corollary 5.2.3 shows that

$$\#\{\alpha_i \in \alpha_{\mathbb{X}} \mid \alpha_i \geq j\} = H_{\mathbb{X}}(t-1, j-1) - H_{\mathbb{X}}(t-1, j-2)$$

it follows from the above identity that

$$\begin{aligned} \#\{\alpha_i \in \alpha_{\mathbb{X}} \mid \alpha_i = j\} &= [H_{\mathbb{X}}(t-1, j-1) - H_{\mathbb{X}}(t-1, j-2)] - \\ &\quad [H_{\mathbb{X}}(t-1, j) - H_{\mathbb{X}}(t-1, j-1)]. \end{aligned}$$

Thus, for each integer $1 \leq j \leq r$ there is precisely $[H_{\mathbb{X}}(t-1, j-1) - H_{\mathbb{X}}(t-1, j-2)] - [H_{\mathbb{X}}(t-1, j) - H_{\mathbb{X}}(t-1, j-1)]$ lines of degree $(1, 0)$ that pass through $\mathbb{X}$ that contain exactly j points of $\mathbb{X}$. This is the statement of Theorem 2.12 of Giuffrida, *et al.* [**26**]. Of course, a similar result holds for the lines of degree $(0, 1)$ that pass through $\mathbb{X}$.

Corollary 5.2.5. *Let $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be any collection of s distinct points with $\alpha_{\mathbb{X}}$ and $\beta_{\mathbb{X}}$, and suppose that $\alpha^*_{\mathbb{X}} = (\alpha^*_1, \ldots, \alpha^*_{\alpha_1})$ and $\beta^*_{\mathbb{X}} = (\beta^*_1, \ldots, \beta^*_{\beta_1})$. Let $\Delta H_{\mathbb{X}}$ be the first difference function of $H_{\mathbb{X}}$, and set $c_{i,j} := \Delta H_{\mathbb{X}}(i,j)$. Then*

(*i*) *for every $0 \leq j \leq r-1 = |\pi_2(\mathbb{X})| - 1$*

$$\alpha^*_{j+1} = \sum_{h \leq |\pi_1(\mathbb{X})|-1} c_{h,j}.$$

(*ii*) *for every $0 \leq i \leq t-1 = |\pi_1(\mathbb{X})| - 1$*

$$\beta^*_{i+1} = \sum_{h \leq |\pi_2(\mathbb{X})|-1} c_{i,h}.$$

Proof. We begin by noting that we have the following identity:

$$H_{\mathbb{X}}(i,j) = \sum_{(h,k) \leq (i,j)} c_{h,k}.$$

Fix an integer j such that $0 \leq j \leq |\pi_2(\mathbb{X})| - 1$ and set $t = |\pi_1(\mathbb{X})|$. Using Proposition 5.2.1 and the above identity to compute α^*_{j+1} we have

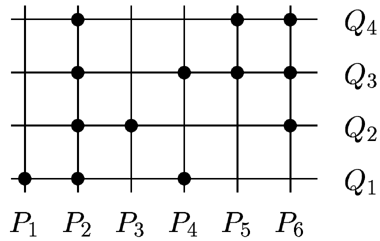
$$\begin{aligned} \alpha^*_{j+1} &= H_{\mathbb{X}}(t-1,j) - H_{\mathbb{X}}(t-1,j-1) \\ &= \sum_{(h,k)\leq(t-1,j)} c_{h,k} - \sum_{(h,k)\leq(t-1,j-1)} c_{h,k} = \sum_{h \leq t-1 = |\pi_1(\mathbb{X})|-1} c_{h,j}. \end{aligned}$$

The proof for the second statement is the same. □

Remark 5.2.6. Let $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a set of distinct points, and suppose that $\alpha_{\mathbb{X}} = (\alpha_1, \ldots, \alpha_t)$ and $\beta_{\mathbb{X}} = (\beta_1, \ldots, \beta_r)$. Suppose that j is an integer such that $\alpha_1 \leq j \leq r$. (We will see below that $\alpha_1 \leq r = |\beta_{\mathbb{X}}|$ always holds.) Then, by the definition of $\alpha^*_{\mathbb{X}}$, $\alpha^*_{j+1} = 0$. Hence, by the above corollary, the entries in the j^{th} row of $\Delta H_{\mathbb{X}}$, considered as a matrix where the top row is the 0^{th} row, must sum to zero.

Example 5.2.7. We illustrate how to use Corollary 5.2.3 to compute the Hilbert function for a set of points $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ for all but a finite number $(i,j) \in \mathbb{N}^2$. Suppose that

$\mathbb{X} =$



For this example $\alpha_{\mathbb{X}} = (4,3,2,2,1,1)$ because $|\pi_1^{-1}(P_1)| = 1$, $|\pi_1^{-1}(P_2)| = 4$, $|\pi_1^{-1}(P_3)| = 1$, $|\pi_1^{-1}(P_4)| = 2$, $|\pi_1^{-1}(P_5)| = 2$, and $|\pi_1^{-1}(P_6)| = 3$. The conjugate of $\alpha_{\mathbb{X}}$ is $\alpha_{\mathbb{X}}^* = (6,4,2,1)$, and hence, by Corollary 5.2.3 we know that $B_C = (6,10,12,13)$. Similarly, $\beta_{\mathbb{X}} = (4,3,3,3)$, and thus $\beta_{\mathbb{X}}^* = (4,4,4,1)$. Using Corollary 5.2.3 we have $B_R = (4,8,12,13,13,13)$. (Note that we need to add some 13's to the end of B_R to ensure that B_R has the correct length of $|B_R| = |\pi_1(\mathbb{X})| = 6$.) Visualizing the Hilbert function $H_{\mathbb{X}}$ as a matrix and using the tuples B_R and B_C, we have

$$H_{\mathbb{X}} = \begin{bmatrix} & & & \mathbf{4} & 4 & \cdots \\ & * & & \mathbf{8} & 8 & \cdots \\ & & & \mathbf{12} & 12 & \cdots \\ & & & \mathbf{13} & 13 & \cdots \\ & & & \mathbf{13} & 13 & \cdots \\ \mathbf{6} & \mathbf{10} & \mathbf{12} & \mathbf{13} & 13 & \cdots \\ 6 & 10 & 12 & 13 & 13 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

All that remains to be calculated are the entries in the upper left-hand corner of $H_{\mathbb{X}}$ denoted by $*$.

As is evident from Corollary 5.2.3 and Remark 5.2.4, the border of the Hilbert function for points $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is linked to combinatorial information describing some of the geometry of $\mathbb{X}$, e.g., the number of points whose first coordinate is P_1, the number of points whose first coordinate is P_2, etc. By utilizing the Gale-Ryser Theorem (Proposition 2.5.6) we show that the geometry of $\mathbb{X}$ forces a condition on $\alpha_{\mathbb{X}}$ and $\beta_{\mathbb{X}}$. As a corollary, we can answer Question 3.1.10 for points in $\mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 5.2.8. *Let $\alpha, \beta \vdash s$. Then there exists a set of points $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ such that $\alpha_{\mathbb{X}} = \alpha$ and $\beta_{\mathbb{X}} = \beta$ if and only if $\alpha^* \trianglerighteq \beta$.*

Proof. Suppose that there exists a set of points $\mathbb{X}$ such that $\alpha_{\mathbb{X}} = \alpha$ and $\beta_{\mathbb{X}} = \beta$. Suppose that $\pi_1(\mathbb{X}) = \{P_1, \ldots, P_t\}$ with $t = |\alpha|$. For $i = 1, \ldots, t$, let L_{P_i} be the line in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by a $(1,0)$-form such that $\pi_1^{-1}(P_i) \subseteq L_{P_i}$. Similarly, if $\pi_2(\mathbb{X}) = \{Q_1, \ldots, Q_r\}$, where $r = |\beta|$, let L_{Q_i} be the line defined by $(0,1)$-form such that $\pi_2^{-1}(Q_i) \subseteq L_{Q_i}$. For each pair (i,j) where $1 \leq i \leq t$ and $1 \leq j \leq r$, the lines L_{P_i} and L_{Q_j} intersect at a unique point $P_i \times Q_j$. We note that $\mathbb{X} \subseteq \{P_i \times Q_j \mid 1 \leq i \leq t, 1 \leq j \leq r\}$. We define an $r \times t$ $(0,1)$-matrix $A = (a_{i,j})$ where

$$a_{i,j} = \begin{cases} 1 & \text{if } L_{P_i} \cap L_{Q_j} = P_i \times Q_j \in \mathbb{X} \\ 0 & \text{if } L_{P_i} \cap L_{Q_j} = P_i \times Q_j \notin \mathbb{X} \end{cases} .$$

By construction this $(0,1)$-matrix has column sum vector $\alpha_A = \alpha_{\mathbb{X}}$ and row sum vector $\beta_A = \beta_{\mathbb{X}}$. Hence, $\mathcal{M}(\alpha, \beta) \neq \emptyset$ because $A \in \mathcal{M}(\alpha, \beta)$. The conclusion $\alpha^* \trianglerighteq \beta$ follows from the Gale-Ryser Theorem (Proposition 2.5.6).

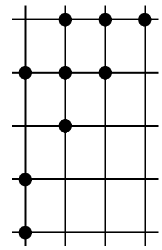
To prove the converse, it is sufficient to construct a set $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ with $\alpha_{\mathbb{X}} = \alpha$ and $\beta_{\mathbb{X}} = \beta$. Since $\alpha^* \trianglerighteq \beta$ there exists a $(0,1)$-matrix $A \in \mathcal{M}(\alpha, \beta)$. Fix such a matrix A. Let $L_{P_1}, \ldots, L_{P_t}$ be $t = |\alpha|$ distinct lines in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by forms of degree $(1,0)$, and let $L_{Q_1}, \ldots, L_{Q_r}$ be $r = |\beta|$ distinct lines in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by forms of degree $(0,1)$. For every pair (i,j), with $1 \leq i \leq t$ and $1 \leq j \leq r$, the lines L_{P_i} and L_{Q_j} intersect at the distinct point $P_i \times Q_j = L_{P_i} \cap L_{Q_j}$. We define a set of points $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ using the matrix $A = (a_{i,j})$ as follows:

$$\mathbb{X} := \{P_i \times Q_j \mid a_{i,j} = 1\}.$$

From our construction of $\mathbb{X}$ we have $\alpha_{\mathbb{X}} = \alpha$ and $\beta_{\mathbb{X}} = \beta$. □

Remark 5.2.9. Suppose that $\alpha, \beta \vdash s$ and $\alpha^* \trianglerighteq \beta$. Then by adopting the procedure described in Example 2.5.7, we can construct a set of points $\mathbb{X}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ with $\alpha_{\mathbb{X}} = \alpha$ and $\beta_{\mathbb{X}} = \beta$. For example, if $\alpha = (3,3,2,1)$ and $\beta = (3,3,1,1,1)$ are as in Example 2.5.7, then we saw how to construct a $(1,0)$-matrix from α and β. We then identify this matrix with a set of points as in the proof of Theorem 5.2.8. For that example

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \longleftrightarrow$$



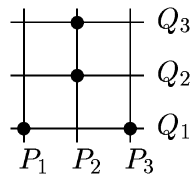
Remark 5.2.10. We will show that if $\alpha^*_{\mathbb{X}} = \beta_{\mathbb{X}}$, then the set $\mathbb{X}$ is also arithmetically Cohen-Macaulay (cf. Theorem 5.4.4).

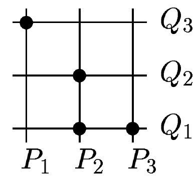
Corollary 5.2.11. *Suppose $B_C = (b_0, \ldots, b_{r-1})$ and $B_R = (b'_0, \ldots, b'_{t-1})$ are two tuples such that $b_0 = t$, $b'_0 = r$, and $\Delta B_C, \Delta B_R \vdash s$. Then B_C is the eventual column vector and B_R is the eventual row vector of a Hilbert function of a set of s points in $\mathbb{P}^1 \times \mathbb{P}^1$ if and only if $\Delta B_C \trianglerighteq (\Delta B_R)^*$.*

PROOF. For any partition λ, we have the identity $(\lambda^*)^* = \lambda$. If $B_{\mathbb{X}} = (B_C, B_R)$ is the border of a set of points, then $\Delta B_C = \alpha^*_{\mathbb{X}} \trianglerighteq \beta_{\mathbb{X}} = (\beta^*_{\mathbb{X}})^* = (\Delta B_R)^*$.

Conversely, suppose that $\Delta B_C \trianglerighteq (\Delta B_R)^*$. Let $\alpha = (\Delta B_C)^*$ and $\beta = (\Delta B_R)^*$. Since $\alpha^* \trianglerighteq \beta$, there exist a set of points $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ with $\alpha_{\mathbb{X}} = \alpha$ and $\beta_{\mathbb{X}} = \beta$. But then $\Delta B_C = \Delta B'_C$, where B'_C is the eventual column vector of the Hilbert function of $\mathbb{X}$. Since $|B'_C| = |\beta| = r$, and because first element of the tuple B'_C is t, we have $B_C = B'_C$. We show that the eventual row border B'_R of the Hilbert function of $\mathbb{X}$ is equal to B_R via the same argument. $\square$

Remark 5.2.12. It is possible for two sets of points to have the same border, but not the same Hilbert function. For example, let P_1, P_2, P_3 be three distinct points of $\mathbb{P}^1$, and let Q_1, Q_2, and Q_3 be another collection of three distinct points in $\mathbb{P}^1$. Let $\mathbb{X}_1 = \{P_1 \times Q_1, P_2 \times Q_2, P_2 \times Q_3, P_3 \times Q_1\}$, and let $\mathbb{X}_2 = \{P_1 \times Q_3, P_2 \times Q_1, P_2 \times Q_2, P_3 \times Q_1\}$. We can visualize these sets as

$\mathbb{X}_1 =$ 

$\mathbb{X}_2 =$ 

For this example, $\alpha_{\mathbb{X}_1} = \alpha_{\mathbb{X}_2} = (2,1,1)$ and $\beta_{\mathbb{X}_1} = \beta_{\mathbb{X}_2} = (2,1,1)$, and hence, both sets of points have the same border. However, using CoCoA to compute the Hilbert function of $\mathbb{X}_1$ and $\mathbb{X}_2$, we find that that the Hilbert functions are not equal. Specifically,

$$H_{\mathbb{X}_1} = \begin{bmatrix} 1 & 2 & 3 & \cdots \\ 2 & 3 & 4 & \cdots \\ 3 & 4 & 4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad H_{\mathbb{X}_2} = \begin{bmatrix} 1 & 2 & 3 & \cdots \\ 2 & 4 & 4 & \cdots \\ 3 & 4 & 4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

3. Applications of the Border

Corollary 5.2.11 characterizes the borders of the Hilbert functions of sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$, and thus, provides us with a new necessary condition on the Hilbert function of a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. In this section we examine some further consequences of Corollary 5.2.11. Specifically, we will show the following: (1) we give a lower bound on the number of distinct Hilbert functions for s points in $\mathbb{P}^1 \times \mathbb{P}^1$; (2) we characterize the Hilbert functions of points $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ with either $|\pi_1(\mathbb{X})| = 2$ or $|\pi_2(\mathbb{X})| = 2$; and (3) if $\mathbb{Y}$ is a subset of $\mathbb{X}$, we show that under some conditions the Hilbert function of $H_{\mathbb{Y}}$ can be determined from $H_{\mathbb{X}}$.

3.1. Counting Hilbert Functions. Let $\mathbb{X}$ be a set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and $H_{\mathbb{X}}$ its Hilbert function. Recall that for each $s \in \mathbb{N}$, we define

$$\mathcal{H}(s) := \{H_{\mathbb{X}} \mid \mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \text{ and } |\mathbb{X}| = s\}.$$

From Remark 3.3.5 we know that $\mathcal{H}(s)$ is a finite set, but we do not know how many elements are in the set.

By applying Corollary 5.2.11 we can calculate a lower bound for $\#\mathcal{H}(s)$ if $\mathcal{H}(s) = \{H_{\mathbb{X}} \mid \mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \text{ and } |\mathbb{X}| = s\}$. We first set some notation. For every positive integer

$s \in \mathbb{N}$ we let $\mathcal{P}_s$ denote the set of all partitions of s. For each $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathcal{P}_s$, we define

$$T_\lambda := \{\delta = (\delta_1, \ldots, \delta_r) \in \mathcal{P}_s \mid \lambda^* \trianglerighteq \delta\} .$$

Proposition 5.3.1. *Fix a positive integer s. Then*

$$\#\mathcal{H}(s) = \#\left\{H_{\mathbb{X}} \;\middle|\; \mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \text{ and } |\mathbb{X}| = s\right\} \geq \sum_{\lambda \in \mathcal{P}_s} \#T_\lambda.$$

Moreover, $\sum_{\lambda \in \mathcal{P}_s} \#T_\lambda$ *is equal to the number of distinct borders.*

Proof. Fix a partition $\lambda \in \mathcal{P}_s$. For each $\delta \in T_\lambda$, it follows from Theorem 5.2.8 that there exists a set of points $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ with $\alpha_{\mathbb{X}} = \lambda$, $\beta_{\mathbb{X}} = \delta$, and $|\mathbb{X}| = s$. Suppose that $\mathbb{X}_{\lambda,\delta}$ is such a set. By Corollary 5.2.11, it follows that $H_{\mathbb{X}_{\lambda,\delta}} \neq H_{\mathbb{X}_{\lambda,\delta'}}$ for any $\delta, \delta' \in T_\lambda$ with $\delta \neq \delta'$ because they cannot have the same borders. We can thus define a map $\varphi_\lambda : T_\lambda \to \mathcal{H}(s)$ by $\delta \mapsto H_{\mathbb{X}_{\lambda,\delta}}$. It follows that $\varphi_\lambda(\delta) = \varphi_\lambda(\delta')$ if and only if $\delta = \delta'$, and thus, φ_λ is an injective map.

If $\lambda \neq \lambda' \in \mathcal{P}_s$, then we claim that $\varphi_\lambda(T_\lambda) \cap \varphi_{\lambda'}(T_{\lambda'}) = \emptyset$. Indeed, if $H_{\mathbb{X}}$ is in the intersection, then this would mean that $\mathbb{X}$ has $\alpha_{\mathbb{X}} = \lambda$ and λ'. So, we have the following disjoint union $\bigcup_{\lambda \in \mathcal{P}_s} \varphi_\lambda(T_\lambda) \subseteq \mathcal{H}(s)$. Since $\#\varphi_\lambda(T_\lambda) = \#T_\lambda$, we get

$$\sum_{\lambda \in \mathcal{P}_s} \#T_\lambda \leq \#\mathcal{H}(s).$$

The last statement is immediate. □

Remark 5.3.2. Because $\lambda = (\underbrace{1, \ldots, 1}_{s}) \in \mathcal{P}_s$, we have $T_{(1,\ldots,1)} = \{\delta \in \mathcal{P}_s \mid (s) \trianglerighteq \delta\} = \mathcal{P}_s$. Thus, $\#\mathcal{H}(s) > \#\mathcal{P}_s$ where $\#\mathcal{P}_s$ is the number of partitions of s. The number $\#\mathcal{P}_s$ grows rapidly, so $\#\mathcal{H}(s)$ also grows rapidly.

3.2. Sets of Points with $|\pi_i(\mathbb{X})| = 2$. We consider all sets $\mathbb{X}$ of s points in $\mathbb{P}^1 \times \mathbb{P}^1$ with $|\pi_1(\mathbb{X})| = 2$, although everything we say will also hold if $|\pi_2(\mathbb{X})| = 2$. Hence, we consider sets of points which contain only two distinct first coordinates. Suppose that $|\pi_2(\mathbb{X})| = r$.

If follows from (5.1.1) that

$$H_{\mathbb{X}} = \begin{bmatrix} 1 & 2 & 3 & \cdots & r-1 & \mathbf{r} & r & \cdots \\ \mathbf{2} & \mathbf{m_{1,1}} & \mathbf{m_{1,2}} & \cdots & \mathbf{m_{1,r-2}} & \mathbf{s} & s & \cdots \\ 2 & m_{1,1} & m_{1,2} & \cdots & m_{1,r-2} & s & s & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where $m_{i,j} = H_{\mathbb{X}}(i,j)$. Hence, if we know $\alpha_{\mathbb{X}}$ and $\beta_{\mathbb{X}}$, we can compute the border of $H_{\mathbb{X}}$, and thus, completely determine $H_{\mathbb{X}}$. In fact, we have even a stronger result:

Theorem 5.3.3. *Let $H : \mathbb{N}^2 \to \mathbb{N}$ be a numerical function. Then H is the Hilbert function of a set of points $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ with $|\pi_1(\mathbb{X})| = 2$ if and only if the following conditions hold:*

(*i*)

$$H = \begin{bmatrix} 1 & 2 & 3 & \cdots & r-1 & \mathbf{r} & r & \cdots \\ \mathbf{2} & \mathbf{m_{1,1}} & \mathbf{m_{1,2}} & \cdots & \mathbf{m_{1,r-2}} & \mathbf{s} & s & \cdots \\ 2 & m_{1,1} & m_{1,2} & \cdots & m_{1,r-2} & s & s & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

(*ii*) $r \leq s$,

(*iii*) $2 \leq m_{1,1} \leq \cdots \leq m_{1,r-2} \leq s$ *and* $m_{1,j} \leq 2(j+1)$*, and*

(*iv*) *if* $B_1 = (2, m_{1,1}, \ldots, m_{1,r-2}, s)$ *and* $B_2 = (r, s)$*, then* $\Delta B_1, \Delta B_2$ *are partitions of* s*, and* $\Delta B_1 \trianglerighteq (\Delta B_2)^*$.

Proof. If H is the Hilbert function of a set of s points with $|\pi_1(\mathbb{X})| = 2$, then (*i*) follows from (5.1.1). Furthermore, $|\pi_2(\mathbb{X})| = r \leq s$. The first part of (*iii*) is a consequence of Lemma 2.2.13. The second part of (*iii*) holds because

$$m_{1,j} = H_{\mathbb{X}}(1,j) = \dim_{\mathbf{k}} R_{1,j} - \dim_{\mathbf{k}} (I_{\mathbb{X}})_{1,j} \leq \dim_{\mathbf{k}} R_{1,j} = 2(j+1)$$

for $1 \leq j \leq r-2$. Finally, (*iv*) is simply Corollary 5.2.11.

Conversely, suppose H is a numerical function that satisfies (*i*)-(*iv*). Because (*iv*) holds, by Corollary 5.2.11 there exists a set of points $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ with border equal to B_1 and B_2. But since the first coordinate of B_1 is 2 and the first coordinate of B_2 is r, we have $|\pi_1(\mathbb{X})| = 2$ and $|\pi_2(\mathbb{X})| = r$. It then follows from our construction of $\mathbb{X}$ that $H = H_{\mathbb{X}}$. □

3.3. Subsets of $\mathbb{X}$ and their Hilbert Function. Let $\mathbb{X}$ be a set of s points in $\mathbb{P}^1 \times \mathbb{P}^1$ and let $H_{\mathbb{X}}$ denote its Hilbert function. Suppose that $\mathbb{Y}$ is a subset of $\mathbb{X}$. We can then ask if the Hilbert function of $\mathbb{Y}$, that is $H_{\mathbb{Y}}$, is related to $H_{\mathbb{X}}$. We will consider the case that $\mathbb{Y}$ is a subset of $\mathbb{X}$ that lies on either on a $(1,0)$-line or on a $(0,1)$-line of $\mathbb{P}^1 \times \mathbb{P}^1$. We will investigate this problem using the results of the previous section.

So, suppose that $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is a set of s distinct points with $\alpha_{\mathbb{X}} = (\alpha_1, \ldots, \alpha_t)$ and $\beta_{\mathbb{X}} = (\beta_1, \ldots, \beta_r)$. Suppose that $\pi_1(\mathbb{X}) = \{P_1, \ldots, P_t\}$ and $\pi_2(\mathbb{X}) = \{Q_1, \ldots, Q_r\}$. After a possible relabelling, we can assume that $|\pi_1^{-1}(P_i)| = \alpha_i$ and $|\pi_2^{-1}(Q_i)| = \beta_i$.

If $P_i = [a_{i1} : a_{i2}] \in \pi_1(\mathbb{X})$, then let L_{P_i} be the $(1,0)$-line that contains the points of $\pi_1^{-1}(P_i)$. We sometimes abuse notation by letting L_{P_i} also denote the form of degree $(1,0)$ $L_{P_i} = a_{i2}x_0 - a_{i1}x_1 \in \mathbf{k}[x_0, x_1, y_0, y_1]$ that defines L_{P_i}. Similarly, if $Q_j \in \pi_2(\mathbb{X})$, then we let L_{Q_j} denote both the $(0,1)$-line that contains $\pi_2^{-1}(Q_j)$ and the degree $(0,1)$ form that defines the line. It follows that if $P_i \times Q_j \in \mathbb{X}$, then $I_{P_i \times Q_j} = (L_{P_i}, L_{Q_j})$.

For each $P_i \in \pi_1(\mathbb{X})$, we define

$$\mathbb{X}_{P_i} := \mathbb{X} \cap L_{P_i} = \pi_1^{-1}(P_i) = \{P_i \times Q_{i_1}, \ldots, P_i \times Q_{i_{\alpha_i}}\}.$$

The ideal associated to $\mathbb{X}_{P_i}$ is therefore

$$I_{\mathbb{X}_{P_i}} = \bigcap_{j=1}^{\alpha_i} \left(L_{P_i}, L_{Q_{i_j}}\right) = \left(L_{P_i}, L_{Q_{i_1}} L_{Q_{i_2}} \cdots L_{Q_{i_{\alpha_i}}}\right).$$

Analogously, we define $\mathbb{X}_{Q_i} := \mathbb{X} \cap L_{Q_i}$ for each $Q_i \in \pi_2(\mathbb{X})$. If $\mathbb{X}_{Q_i} := \{P_{i_1} \times Q_i, \ldots, P_{i_{\beta_i}} \times Q_i\}$, then it follows that $I_{\mathbb{X}_{Q_i}} := (L_{P_{i_1}} \cdots L_{P_{i_{\beta_i}}}, L_{Q_i})$.

Because $\alpha_{\mathbb{X}_{P_i}} = (\alpha_i)$ and $\beta_{\mathbb{X}_{P_i}} = (\underbrace{1, \ldots, 1}_{\alpha_i})$, the Hilbert function of $\mathbb{X}_{P_i}$ can be computed directly from Proposition 5.2.1. The same holds true for $H_{\mathbb{X}_{Q_i}}$. Indeed,

$$H_{\mathbb{X}_{P_i}} = \begin{bmatrix} 1 & 2 & 3 & \cdots & \alpha_{i-1} & \alpha_i & \alpha_i & \cdots \\ 1 & 2 & 3 & \cdots & \alpha_{i-1} & \alpha_i & \alpha_i & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and

$$H_{\mathbb{X}_{Q_i}} = \begin{bmatrix} 1 & 1 & \cdots \\ 2 & 2 & \cdots \\ \vdots & \vdots & \\ \beta_{i-1} & \beta_{i-1} & \cdots \\ \beta_i & \beta_i & \cdots \\ \beta_i & \beta_i & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

If we remove the points of $\mathbb{X}_{P_i}$ (respectively, $\mathbb{X}_{Q_i}$) from $\mathbb{X}$, then the next proposition shows that we can compute the Hilbert function of $\mathbb{X}\backslash\mathbb{X}_{P_i}$ (respectively, $\mathbb{X}\backslash\mathbb{X}_{Q_i}$) for *some* $(i,j) \in \mathbb{N}^2$ from $H_{\mathbb{X}}$ and $H_{\mathbb{X}_{P_i}}$ (respectively, $H_{\mathbb{X}_{Q_i}}$). This proposition is also the basis for some of our subsequent results.

Proposition 5.3.4. *Using the notation above, fix a $P \in \pi_1(\mathbb{X})$ and let $\mathbb{X}_P := \mathbb{X} \cap L_P$. Then for all $(i,j) \in \mathbb{N}^2$ with $j < |\mathbb{X}_P| = \alpha$,*

$$H_{\mathbb{X}\backslash\mathbb{X}_P}(i,j) = H_{\mathbb{X}}(i+1,j) - H_{\mathbb{X}_P}(i+1,j).$$

Similarly, fix a $Q \in \pi_2(\mathbb{X})$ and set $\mathbb{X}_Q = \mathbb{X} \cap L_Q$. Then, for all $(i,j) \in \mathbb{N}^2$ with $i < |\mathbb{X}_Q| = \beta$,

$$H_{\mathbb{X}\backslash\mathbb{X}_Q}(i,j) = H_{\mathbb{X}}(i,j+1) - H_{\mathbb{X}_Q}(i,j+1).$$

PROOF. Because the second statement is similar to the first, we show only the first conclusion. As we observed above, the defining ideal of $\mathbb{X}_P$ is $I_{\mathbb{X}_P} = (L_P, L_{Q_1} \cdots L_{Q_\alpha})$. We have a short exact sequence with degree $(0,0)$ maps:

$$0 \longrightarrow I_{\mathbb{X}_P}/I_{\mathbb{X}} \longrightarrow R/I_{\mathbb{X}} \longrightarrow \frac{(R/I_{\mathbb{X}})}{(I_{\mathbb{X}_P}/I_{\mathbb{X}})} \cong R/I_{\mathbb{X}_P} \longrightarrow 0$$

because $I_{\mathbb{X}} \subseteq I_{\mathbb{X}_P}$. This sequence induces a short exact sequence of vector spaces

$$0 \longrightarrow (I_{\mathbb{X}_P}/I_{\mathbb{X}})_{i,j} \longrightarrow (R/I_{\mathbb{X}})_{i,j} \longrightarrow (R/I_{\mathbb{X}_P})_{i,j} \longrightarrow 0$$

for all $(i,j) \in \mathbb{N}^2$.

Claim 1. For all $(i,j) \in \mathbb{N}^2$ with $j < \alpha$, $(I_{\mathbb{X}_P})_{i,j} \cong R_{i-1,j}$.

Proof of the Claim. Since $j < \alpha$, $(I_{\mathbb{X}_P})_{i,j} = (L_P, L_{Q_1} \cdots L_{Q_\alpha})_{i,j} = (L_P)_{i,j}$. The claim now follows because the vector space morphism $R_{i-1,j} \to (L_P)_{i,j}$ given by $F \mapsto F \cdot L_P$ is an isomorphism. □

Claim 2. For all $(i,j) \in \mathbb{N}^2$ with $j < \alpha$, $(I_{\mathbb{X}})_{i,j} \cong (I_{\mathbb{X}} : I_{\mathbb{X}_P})_{i-1,j}$.

Proof of the Claim. We define a map of vector spaces via multiplication by L_P, i.e.,

$$\varphi : (I_{\mathbb{X}} : I_{\mathbb{X}_P})_{i-1,j} \stackrel{\times L_P}{\longrightarrow} (I_{\mathbb{X}})_{i,j}.$$

This map is defined since $H \in (I_{\mathbb{X}} : I_{\mathbb{X}_P})_{i-1,j}$ implies $HI_{\mathbb{X}_P} \subseteq I_{\mathbb{X}}$, and in particular, $H \cdot L_P \in (I_{\mathbb{X}})_{i,j}$. The morphism φ is also injective because multiplication is defined in R.

To show φ is onto, let $H \in (I_{\mathbb{X}})_{i,j}$. But then $H \in (I_{\mathbb{X}_P})_{i,j}$ because $I_{\mathbb{X}} \subseteq I_{\mathbb{X}_P}$. Moreover, from the proof of Claim 1, $H \in (I_{\mathbb{X}_P})_{i,j} = (L_P)_{i,j}$, and thus, $H = L_P \cdot H'$ where $\deg H' = (i-1,j)$. Since $L_P \cdot H' \in I_{\mathbb{X}}$ and because L_P vanishes only at those points of $\mathbb{X}$ in $\mathbb{X}_P$, we have H' must vanish on $\mathbb{X} \backslash \mathbb{X}_P$. Hence, $H' \in I_{\mathbb{X} \backslash \mathbb{X}_P}$. The claim now follows because $I_{\mathbb{X} \backslash \mathbb{X}_P} = (I_{\mathbb{X}} : I_{\mathbb{X}_P})$. □

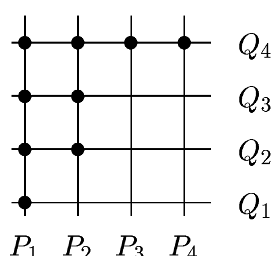
In light of Claim 1 and Claim 2, the exact sequence of vector spaces can be rewritten as

$$0 \longrightarrow (R/(I_{\mathbb{X}} : I_{\mathbb{X}_P}))_{i-1,j} \longrightarrow (R/I_{\mathbb{X}})_{i,j} \longrightarrow (R/I_{\mathbb{X}_P})_{i,j} \longrightarrow 0$$

for all $(i,j) \in \mathbb{N}^2$ with $j < \alpha$. If we now consider the dimension of each vector space, then the conclusion follows. □

Example 5.3.5. Set $P_i := [1:i] \in \mathbb{P}^1$ and $Q_i := [1:i] \in \mathbb{P}^1$. Let $\mathbb{X}$ be the following set of points:

$\mathbb{X} =$



Using **`CoCoA`** to compute the Hilbert function of $\mathbb{X}$ we find:

$$H_{\mathbb{X}} = \begin{bmatrix} 1 & 2 & 3 & 4 & 4 & \cdots \\ 2 & 4 & 6 & 7 & 7 & \cdots \\ 3 & 5 & 7 & 8 & 8 & \cdots \\ 4 & 6 & 8 & 9 & 9 & \cdots \\ 4 & 6 & 8 & 9 & 9 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Suppose that we remove $\mathbb{X}_{P_2}$ from $\mathbb{X}$. Since $|\mathbb{X}_{P_2}| = 3$, for all $(i,j) \in \mathbb{N}^2$ with $j < 3$, we have

$$H_{\mathbb{X}\backslash\mathbb{X}_P} = \begin{bmatrix} 2 & 4 & 6 & 7 & 7 & \cdots \\ 3 & 5 & 7 & 8 & 8 & \cdots \\ 4 & 6 & 8 & 9 & 9 & \cdots \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 & 3 & 3 & \cdots \\ 1 & 2 & 3 & 3 & 3 & \cdots \\ 1 & 2 & 3 & 3 & 3 & \cdots \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 2 & 3 & 4 & 4 & \cdots \\ 2 & 3 & 4 & 5 & 5 & \cdots \\ 3 & 4 & 5 & 6 & 6 & \cdots \end{bmatrix}.$$

We focus on the case that we remove the $(1,0)$-line (respectively, the $(0,1)$-line) with the largest number of points. That is, we remove the α_1 (respectively, β_1) points of $\mathbb{X}$ that lie on L_{P_1} (respectively, L_{Q_1}). By Proposition 5.2.1, we can compute the Hilbert function of $\mathbb{X}\backslash\mathbb{X}_{P_1}$ (respectively, $X\backslash\mathbb{X}_{Q_1}$) for all but a finite number of $(i,j) \in \mathbb{N}^2$ if we know $\alpha_{\mathbb{X}\backslash\mathbb{X}_{P_1}}$ and $\beta_{\mathbb{X}\backslash\mathbb{X}_{P_1}}$ (respectively, $\alpha_{\mathbb{X}\backslash\mathbb{X}_{Q_1}}$ and $\beta_{\mathbb{X}\backslash\mathbb{X}_{Q_1}}$). Therefore, a natural starting point is to ask if these two tuples can be computed from $\alpha_{\mathbb{X}}$ and $\beta_{\mathbb{X}}$. We consider only $\mathbb{X}\backslash\mathbb{X}_{P_1}$, although analogous results hold for $\mathbb{X}\backslash\mathbb{X}_{Q_1}$.

So, let $\mathbb{Y} = \mathbb{X}\backslash\mathbb{X}_{P_1}$, where $|\mathbb{X}_{P_1}| = \alpha_1$. It follows immediately that $\alpha_{\mathbb{Y}} = (\alpha_2, \ldots, \alpha_r)$. What cannot be easily determined is $\beta_{\mathbb{Y}}$. When we remove the α_1 points of $\mathbb{X}_{P_1}$, we are removing α_1 points from $\mathbb{X}$ with α_1 distinct second coordinates. Thus, $\beta_{\mathbb{Y}}$ is constructed from $\beta_{\mathbb{X}}$ by subtracting one from α_1 coordinates of $\beta_{\mathbb{X}}$. This is always possible, because $(\beta_{\mathbb{X}})^* \trianglerighteq \alpha_{\mathbb{X}}$, and hence $(\beta_{\mathbb{X}})^*_1 = |\beta_{\mathbb{X}}| = r \geq \alpha_1$, However, if $r > \alpha_1$, then it is not always evident from which entries of $\beta_{\mathbb{X}}$ we can subtract one. We therefore would like to know what $\beta_{\mathbb{Y}}$'s are possible.

Determining the possible $\beta_{\mathbb{Y}}$'s can be translated into a combinatorial question about $(0,1)$-matrices. Indeed, let A be an $r \times t$ $(0,1)$-matrix with column sum vector $\alpha_A = (\alpha_1, \ldots, \alpha_t)$ and row sum vector $\beta_A = (\beta_1, \ldots, \beta_r)$. We construct a new $(0,1)$-matrix, say A', by removing the column with α_1 ones. Then $\alpha_{A'} = (\alpha_2, \ldots, \alpha_r)$. The question of describing all the $\beta_{\mathbb{Y}}$'s is equivalent to giving a complete list of possible row sum vectors for A'. This problem appears to be unexplored.

If we consider the extremal case that $\alpha_1 = r = |\beta_1|$, it follows from the above discussion that there is only one possibility for $\beta_{\mathbb{Y}}$, namely, $\beta_{\mathbb{Y}} = (\beta_1 - 1, \beta_2 - 1, \ldots, \beta_r - 1)$. In this case, we have

Proposition 5.3.6. *Let $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ with $\alpha_{\mathbb{X}} = (\alpha_1, \ldots, \alpha_t)$ and $\beta_{\mathbb{X}} = (\beta_1, \ldots, \beta_r)$. Let $\mathbb{X}_{P_1} = \mathbb{X} \cap L_{P_1}$ where $|\pi_1^{-1}(P_1)| = \alpha_1$. Suppose that $\alpha_1 = r = |\beta_{\mathbb{X}}|$. Then*

$$H_{\mathbb{X} \backslash \mathbb{X}_P}(i,j) = H_{\mathbb{X}}(i+1,j) - H_{\mathbb{X}_P}(i+1,j)$$

for all $(i,j) \in \mathbb{N}^2$.

Proof. Let $\mathbb{Y} = \mathbb{X} \backslash \mathbb{X}_{P_1}$. By Proposition 5.3.4 we have $H_{\mathbb{Y}}(i,j) = H_{\mathbb{X}}(i+1,j) - H_{\mathbb{X}_{P_1}}(i+1,j)$ for all $j < \alpha_1$. So, suppose $(i,j) \in \mathbb{N}^2$ with $j \geq \alpha_1$. Now, because $\alpha_1 = r$, we have $\beta_{\mathbb{Y}} = (\beta_1 - 1, \ldots, \beta_r - 1)$. Hence, $|\pi_2(\mathbb{Y})| = |\beta_{\mathbb{Y}}| \leq r$. By Corollary 3.1.7, because $j \geq r \geq |\pi_2(\mathbb{Y})| - 1$,

$$H_{\mathbb{Y}}(i,j) = H_{\mathbb{Y}}(i,r-1) = H_{\mathbb{X}}(i+1,r-1) - H_{\mathbb{X}_P}(i+1,r-1).$$

But because $|\pi_2(\mathbb{X})| = |\pi_2(\mathbb{X}_{P_1})| = r$, then Corollary 3.1.7 also implies that the right hand side of the above equation is equal to $H_{\mathbb{X}}(i+1,j) - H_{\mathbb{X}_P}(i+1,j)$ for any $j \geq r-1$. The conclusion now follows. ☐

Example 5.3.7. We will use Propositions 5.3.3 and 5.3.4 to show that there exists an infinite family of admissible matrices (see Definition 5.1.1) such that no matrix in the family is equal to the Hilbert function of a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$.

Fix an integer $s \geq 4$ and let $M_s = (m_{i,j})$ with $(i,j) \in \mathbb{N}^2$ be the following infinite matrix:

$$M_s = \begin{bmatrix} 1 & 2 & 3 & \cdots & s-3 & s-2 & s-1 & s & s & \cdots \\ 2 & 3 & 4 & \cdots & s-2 & s-1 & s & s & s & \cdots \\ 3 & 4 & 5 & \cdots & s-1 & s & s & s & s & \cdots \\ 3 & 4 & 5 & \cdots & s-1 & s & s & s & s & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We set $\Delta M_s = (c_{i,j})$ where $c_{i,j} = m_{i,j} - m_{i-1,j} - m_{i,j-1} + m_{i-1,j-1}$ where $m_{i,j} = 0$ if $(i,j) \not\geq (0,0)$. Hence

$$\Delta M_s = \begin{bmatrix} \overset{0}{1} & 1 & \cdots & 1 & 1 & \overset{s-1}{1} & 0 & \cdots \\ 1 & 0 & \cdots & 0 & 0 & -1 & 0 & \cdots \\ 1 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The reader can verify that M_s is an admissible matrix.

Claim. There is no set of points $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ with $H_{\mathbb{X}} = M_s$.

Suppose, for a contradiction, that $\mathbb{X}$ is a set of points such that $H_{\mathbb{X}} = M_s$. Then from M_s we calculate that $\alpha^*_{\mathbb{X}} = (3, \underbrace{1,\ldots,1}_{s-3})$ and $\beta^*_{\mathbb{X}} = (s)$, and hence, $\alpha_{\mathbb{X}} = (s-2,1,1)$ and $\beta_{\mathbb{X}} = (\underbrace{1,\ldots,1}_{s})$.

From $\alpha_{\mathbb{X}}$, we deduce that there is $(1,0)$-line, say L, such that L contains the $s-2$ points of $\mathbb{X}$ that have the same first coordinate. Set $\mathbb{Y} = \mathbb{X}\backslash L$. It then follows from $\alpha_{\mathbb{X}}$ and $\beta_{\mathbb{X}}$ that $\alpha_{\mathbb{Y}} = (1,1)$ and $\beta_{\mathbb{Y}} = (1,1)$. From Theorem 5.3.3, the Hilbert function of $\mathbb{Y}$ is

$$H_{\mathbb{Y}} = \begin{bmatrix} 1 & 2 & 2 & \cdots \\ 2 & 2 & 2 & \cdots \\ 2 & 2 & 2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

On the other hand, when we use Proposition 5.3.4 to calculate $H_{\mathbb{Y}}(i,j)$ for all $(i,j) \in \mathbb{N}^2$ with $j < 2 \leq s-2$, we find

$$\begin{aligned} H_{\mathbb{Y}}(i,j) &= H_{\mathbb{X}}(i+1,j) - H_L(i+1,j) = m_{i+1,j} - H_L(i+1,j) \quad \text{ if } 0 \leq j < 2 \\ &= \begin{bmatrix} 2 & 3 & 4 & \cdots & s-2 & s-1 & s & s & \cdots \\ 3 & 4 & 5 & \cdots & s-1 & s & s & s & \cdots \end{bmatrix} - \begin{bmatrix} 1 & 2 & \cdots & s-3 & s-2 & s-2 & \cdots \\ 1 & 2 & \cdots & s-3 & s-2 & s-2 & \cdots \end{bmatrix} \\ &= \begin{bmatrix} \overbrace{1 & 1 & \cdots & 1 & 1}^{s-2} & 2 & 2 & \cdots \\ 2 & 2 & \cdots & 2 & 2 & 2 & 2 & \cdots \end{bmatrix}. \end{aligned}$$

Since $s-2 \geq 2$ we have $H_{\mathbb{Y}} \neq H_{\mathbb{Y}}$. Thus, M_s is not the Hilbert function of any set of points in $\mathbb{P}^1 \times \mathbb{P}^1$.

4. Characterizing ACM Sets of Points in $\mathbb{P}^1 \times \mathbb{P}^1$

Arithmetically Cohen-Macaulay sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ were first characterized via their Hilbert function by Giuffrida, Maggioni, and Ragusa [**26**]. In this section, we will give a new proof of this characterization. We will also demonstrate a new characterization for ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ via the tuples $\alpha_{\mathbb{X}}$ and $\beta_{\mathbb{X}}$ as defined in Section 2 of this chapter. As a consequence, the Hilbert function and the Betti numbers in the resolution of an ACM collection of points in $\mathbb{P}^1 \times \mathbb{P}^1$ is completely determined by the combinatorial information about $\mathbb{X}$ contained within $\alpha_{\mathbb{X}}$ and $\beta_{\mathbb{X}}$.

Before proceeding, we will require the following lemmas.

Lemma 5.4.1. *Let $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_m)$, and suppose that $\alpha, \beta \vdash s$. If $\alpha^* = \beta$, then*

(i) $\alpha_1 = |\beta|$.
(ii) $\beta_1 = |\alpha|$.
(iii) *if $\alpha' = (\alpha_2, \ldots, \alpha_n)$ and $\beta' = (\beta_1 - 1, \ldots, \beta_{\alpha_2} - 1)$, then $(\alpha')^* = \beta'$.*

Proof. The proof of (i) and (ii) are the same. We do (ii). By definition, $\alpha_1^* = \#\{\alpha_i \in \alpha \mid \alpha_i \geq 1\}$. Because $\alpha \vdash s$, $\alpha_i \geq 1$ for every i, and so $\alpha_1^* = n = |\alpha|$. But $\alpha^* = \beta$ implies $\alpha_1^* = \beta_1$, thus completing the proof.

For (iii), because $\alpha_j^* = \beta_j$, for every $1 \leq j \leq \alpha_1$ we have $\#\{\alpha_i \in \alpha \mid \alpha_i \geq j\} = \beta_j$. Now $\alpha_1 \geq \alpha_i$ for every coordinate α_i of $\alpha' = (\alpha_2, \ldots, \alpha_n)$. Thus, we can rewrite α_j^* as

$$\alpha_j^* = \begin{cases} \#\{\alpha_i \in \alpha' \mid \alpha_i \geq j\} + 1 = (\alpha')_j^* + 1 & \text{if } 1 \leq j \leq \alpha_2 \\ 1 & \text{if } \alpha_2 < j \leq \alpha_1 \end{cases}.$$

Hence

$$\beta_j - 1 = \alpha_j^* - 1 = \begin{cases} (\alpha')_j^* + 0 & \text{if } 1 \leq j \leq \alpha_2 \\ 0 & \text{if } \alpha_2 < j \leq \alpha_1 \end{cases}.$$

The conclusion now follows. □

Lemma 5.4.2. *Let $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ and suppose that $\alpha^*_{\mathbb{X}} = \beta_{\mathbb{X}}$. Let P be a point of $\pi_1(\mathbb{X})$ such that $|\pi_1^{-1}(P)| = \alpha_1$. Set $\mathbb{X}_P := \pi_1^{-1}(P)$. Then $\pi_2(\mathbb{X}_P) = \pi_2(\mathbb{X})$.*

PROOF. Since $\mathbb{X}_P \subseteq \mathbb{X}$, it is clear that $\pi_2(\mathbb{X}_P) \subseteq \pi_2(\mathbb{X})$. Now, by our choice of P, $|\pi_2(\mathbb{X}_P)| = \alpha_1$. But since $|\pi_2(\mathbb{X})| = |\beta_{\mathbb{X}}|$ and $\alpha^*_{\mathbb{X}} = \beta$, it follows from Lemma 5.4.1 that $|\pi_2(\mathbb{X})| = |\beta_{\mathbb{X}}| = \alpha_1 = |\pi_2(\mathbb{X}_P)|$, and hence $\pi_2(\mathbb{X}_P) = \pi_2(\mathbb{X})$. $\square$

Lemma 5.4.3. *Let $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a set of s distinct points. If $\alpha_{\mathbb{X}} = (s)$ and $\beta_{\mathbb{X}} = (\underbrace{1, 1, \ldots, 1}_{s})$, then $\mathbb{X}$ is ACM.*

PROOF. Because $|\alpha_{\mathbb{X}}| = 1$, there is only one distinct first coordinate, say P. We deduce from $\alpha_{\mathbb{X}}$ and $\beta_{\mathbb{X}}$ that $\mathbb{X} = \{P \times Q_1, \ldots, P \times Q_s\} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ where the Q_i are distinct points in $\mathbb{P}^1$. The ideal corresponding to the point $P \times Q_i \in \mathbb{X}$ is the bihomogeneous prime ideal $I_{P \times Q_i} = (L_P, L_{Q_i})$, where L_P is the form of degree $(1, 0)$ that vanishes at P and the L_{Q_i} is the form of degree $(0, 1)$ that vanishes at Q_i. But then

$$I_{\mathbb{X}} = \bigcap_{i=1}^{s} I_{P \times Q_i} = \bigcap_{i=1}^{s} (L_P, L_{Q_i}) = (L_P, L_{Q_1} L_{Q_2} \cdots L_{Q_s}).$$

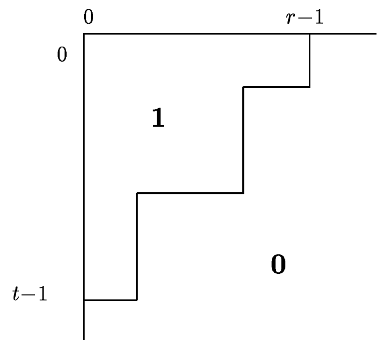
We observe the generators of $I_{\mathbb{X}}$ give rise to a regular sequence in R. Therefore, $\mathbb{X}$ is a complete intersection (see Definition 4.1.20), and thus, $\mathbb{X}$ is ACM. $\square$

Theorem 5.4.4. *Let $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a set of s distinct points, let $\alpha_{\mathbb{X}}$ and $\beta_{\mathbb{X}}$ be constructed from $\mathbb{X}$ as above, and let $H_{\mathbb{X}}$ be the Hilbert function of $\mathbb{X}$. Then the following are equivalent:*

(*i*) *$\mathbb{X}$ is ACM.*
(*ii*) *$\Delta H_{\mathbb{X}}$ is the Hilbert function of a bigraded artinian quotient of $\mathbf{k}[x_1, y_1]$.*
(*iii*) *$\alpha^*_{\mathbb{X}} = \beta_{\mathbb{X}}$.*

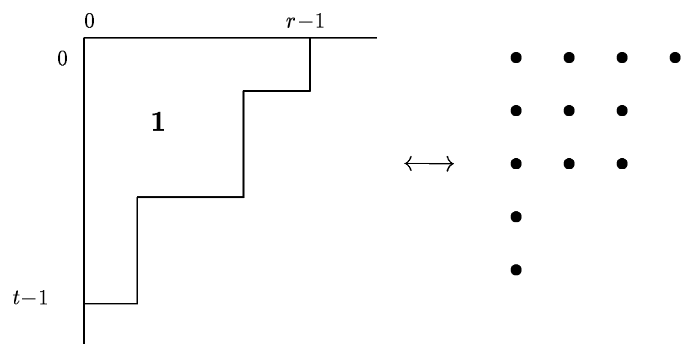
PROOF. The implication $(i) \Rightarrow (ii)$ is Corollary 4.3.6. So, let us now suppose that (ii) holds. Because $\Delta H_{\mathbb{X}}$ is the Hilbert function of a bigraded artinian quotient of $\mathbf{k}[x_1, y_1]$, Corollary 4.4.14, Remark 4.4.19, and (5.1.2) give

$\Delta H_{\mathbb{X}} =$



where $t = |\pi_1(\mathbb{X})|$ and $r = |\pi_2(\mathbb{X})|$. We have written $\Delta H_{\mathbb{X}}$ as an infinite matrix whose indexing starts from zero rather than one.

From Corollary 5.2.5, the number of 1's in the $(i-1)^{th}$ row of $\Delta H_{\mathbb{X}}$ for each integer $1 \leq i \leq t$ is simply the i^{th} coordinate of $\beta_{\mathbb{X}}^*$. Similarly, the number of ones in the $(j-1)^{th}$ column of $\Delta H_{\mathbb{X}}$ for each integer $1 \leq j \leq r$ is the j^{th} coordinate of $\alpha_{\mathbb{X}}^*$. Now $\Delta H_{\mathbb{X}}$ can be identified with the Ferrers diagram (cf. Definition 2.5.2) of $\beta_{\mathbb{X}}^*$ by associating to each 1 in $\Delta H_{\mathbb{X}}$ a dot in the Ferrers diagram in the natural way, that is,



By using the Ferrers diagram and Corollary 5.2.5, it is now straightforward to calculate that the conjugate of $\beta_{\mathbb{X}}^*$ is $(\beta_{\mathbb{X}}^*)^* = \beta_{\mathbb{X}} = \alpha_{\mathbb{X}}^*$, and so (iii) holds.

To demonstrate that (iii) implies (i), we will do a proof by induction on the tuple $(|\pi_1(\mathbb{X})|, |\mathbb{X}|)$. For any positive integer s, if $(|\pi_1(\mathbb{X})|, |\mathbb{X}|) = (1, s)$, then $\alpha_{\mathbb{X}} = (s)$ and $\beta_{\mathbb{X}} = (\underbrace{1, \ldots, 1}_{s})$. But then $\alpha_{\mathbb{X}}^* = \beta_{\mathbb{X}}$, and by Lemma 5.4.3, $\mathbb{X}$ is also ACM.

So, suppose that $(|\pi(\mathbb{X})), |\mathbb{X}|) = (t,s)$ and that result holds true for all $\mathbb{Y} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ with $\alpha^*_{\mathbb{Y}} = \beta_{\mathbb{Y}}$ and $(t,s) >_{lex} (|\pi_1(\mathbb{Y})|, |\mathbb{Y}|)$ where $>_{lex}$ is the lexicographical ordering on $\mathbb{N}^2$, i.e., $(a,b) >_{lex} (c,d)$ if $a > c$, or if $a = c$, then $b > d$.

Suppose that P_1 (after a possible relabelling) is the element of $\pi_1(\mathbb{X})$ such that $|\pi_1^{-1}(P_1)| = \alpha_1$. Let L_{P_1} be the form of degree $(1,0)$ that vanishes at P_1. By abusing notation, we also let L_{P_1} denote the $(1,0)$-line in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by L_{P_1}.

Set $\mathbb{X}_{P_1} := \mathbb{X} \cap L_{P_1} = \pi_1^{-1}(P_1)$ and $\mathbb{Z} := \mathbb{X} \backslash \mathbb{X}_{P_1}$. It follows that $\alpha_{\mathbb{Z}} = (\alpha_2, \ldots, \alpha_t)$ and $\beta_{\mathbb{Z}} = (\beta_1 - 1, \ldots, \beta_{\alpha_2} - 1)$. Now $(t,s) >_{lex} (|\pi_1(\mathbb{Z})|, |\mathbb{Z}|)$. Moreover, $\alpha^*_{\mathbb{Z}} = \beta_{\mathbb{Z}}$ by Lemma 5.4.1. Thus, by the induction hypothesis, $\mathbb{Z}$ is ACM.

Suppose that $\pi_2(\mathbb{X}) = \{Q_1, \ldots, Q_r\}$. Let L_{Q_i} be the degree $(0,1)$ form that vanishes at $Q_i \in \pi_2(\mathbb{X})$ and set $F := L_{Q_1} L_{Q_2} \cdots L_{Q_r}$. Because $\alpha^*_{\mathbb{X}} = \beta_{\mathbb{X}}$, from Lemma 5.4.2 we have $\pi_2(\mathbb{X}_{P_1}) = \pi_2(\mathbb{X})$. So, $\mathbb{X}_{P_1} = \{P_1 \times Q_1, \ldots, P_1 \times Q_r\}$, and hence

$$I_{\mathbb{X}_{P_1}} = \bigcap_{i=1}^{s} (L_{P_1}, L_{Q_i}) = (L_{P_1}, F).$$

Furthermore, if $P \times Q \in \mathbb{Z}$, then $Q \in \pi_2(\mathbb{Z}) \subseteq \pi_2(\mathbb{X})$, and thus $F(P \times Q) = 0$. Therefore $F \in I_{\mathbb{Z}}$. Because F is in $I_{\mathbb{Z}}$ and is also a generator of $I_{\mathbb{X}_{P_1}}$, we will be able to show that the following claim holds.

Claim. Let $I = L_{P_1} \cdot I_{\mathbb{Z}} + (F)$. Then $I = I_{\mathbb{X}}$.

Proof of the Claim. Since $I_{\mathbb{X}} = I_{\mathbb{Z} \cup \mathbb{X}_{P_1}} = I_{\mathbb{Z}} \cap I_{\mathbb{X}_{P_1}}$, we will demonstrate that $I_{\mathbb{Z}} \cap I_{\mathbb{X}_{P_1}} = L_{P_1} \cdot I_{\mathbb{Z}} + (F)$.

So, suppose that $G = L_{P_1} H_1 + H_2 F \in L_{P_1} \cdot I_{\mathbb{Z}} + (F)$ with $H_1 \in I_{\mathbb{Z}}$ and $H_2 \in R$. Because L_{P_1} and F are in $I_{\mathbb{X}_{P_1}}$, we have $G \in I_{\mathbb{X}_{P_1}}$. On the other hand, because $H_1, F \in I_{\mathbb{Z}}$, G is also in $I_{\mathbb{Z}}$. Hence $L_{P_1} \cdot I_{\mathbb{Z}} + (F) \subseteq I_{\mathbb{X}}$.

Conversely, let $G \in I_{\mathbb{Z}} \cap I_{\mathbb{X}_{P_1}}$. Since $G \in I_{\mathbb{X}_{P_1}}$, $G = L_{P_1} H_1 + F H_2$. If we can show that $H_1 \in I_{\mathbb{Z}}$, then we will have completed the proof. Now because $G, F \in I_{\mathbb{Z}}$, we also have $L_{P_1} H_1 \in I_{\mathbb{Z}}$. But for every $P \times Q \in \mathbb{Z}$, $P \neq P_1$, and thus $L_{P_1}(P \times Q) \neq 0$. Hence $L_{P_1} H_1 \in I_{\mathbb{Z}}$ if and only if $H_1(P \times Q) = 0$ for every $P \times Q \in \mathbb{Z}$. □

By Remark 4.1.17, $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is ACM if and only if the variety $\tilde{\mathbb{X}} \subseteq \mathbb{P}^3$ defined by $I_{\mathbb{X}}$, considered as a homogeneous ideal of $\mathbf{k}[x_0, x_1, y_0, y_1]$, is ACM, So, $\mathbb{X}$ is ACM if and only if the $\mathbb{N}^1$-graded ring $R/I_{\mathbb{X}}$ is Cohen-Macaulay. Because we wish to show that $R/I_{\mathbb{X}}$ is

Cohen-Macaulay, by Theorem 4.1.22 this is equivalent to showing that $\operatorname{proj.dim}_R R/I_{\mathbb{X}} = 4 - \operatorname{K-dim} R/I_{\mathbb{X}} = 2$. On the other hand, since $\operatorname{proj.dim}_R I_{\mathbb{X}} = \operatorname{proj.dim}_R R/I_{\mathbb{X}} - 1$ (see Theorem 2.4.6 (i)), it is enough to show that $\operatorname{proj.dim}_R I_{\mathbb{X}} = 1$.

Now because $I_{\mathbb{X}_{P_1}} = (L_{P_1}, F)$ where $\deg L_{P_1} = (1,0)$ and $\deg F = (0,r)$ (recall $r = |\pi_2(\mathbb{X})|$), when we consider $I_{\mathbb{X}_{P_1}}$ as a homogeneous ideal we have $\deg L_{P_1} = 1$ and $\deg F = r$. The graded resolution of $I_{\mathbb{X}_{P_1}}$ is therefore

$$0 \longrightarrow R(-(1+r)) \xrightarrow{\phi_2} R(-1) \oplus R(-r) \xrightarrow{\phi_1} (L_{P_1}, F) \longrightarrow 0$$

where $\phi_1 = [L_{P_1} \ F]$ and $\phi_2 = \begin{bmatrix} F \\ -L_{P_1} \end{bmatrix}$. We note that for every $G \in R(-(1+r))$, we have $\phi_2(G) = (FG, -L_{P_1}G)$. But because $F \in I_{\mathbb{Z}}$, we in fact have $\operatorname{im} \phi_2 \subseteq I_{\mathbb{Z}}(-1) \oplus R(-r)$. This fact, coupled with the claim, gives us the following short exact sequence of graded R-modules:

$$0 \longrightarrow R(-(1+r)) \xrightarrow{\phi_2} I_{\mathbb{Z}}(-1) \oplus R(-r) \xrightarrow{\phi_1} I_{\mathbb{X}} = L_{P_1} \cdot I_{\mathbb{Z}} + (F) \longrightarrow 0$$

where ϕ_1 and ϕ_2 are the same as the maps above.

The projective dimensions of $R(-(1+r))$ and $R(-r)$ are zero. By the induction hypothesis, $\operatorname{proj.dim}_R I_{\mathbb{Z}}(-1) = 1$. We therefore have $\operatorname{proj.dim}_R R(-(1+r)) < \operatorname{proj.dim}_R (I_{\mathbb{Z}}(-1) \oplus R(-r))$. From the above short exact sequence and Theorem 2.4.6 (ii), it follows that

$$\begin{aligned} \operatorname{proj.dim}_R I_{\mathbb{X}} &= \operatorname{proj.dim}_R (I_{\mathbb{Z}}(-1) \oplus R(-r)) \\ &= \max\{\operatorname{proj.dim}_R I_{\mathbb{Z}}(-1), \operatorname{proj.dim}_R R(-r)\} = 1. \end{aligned}$$

Therefore $\mathbb{X}$ is ACM, and so $(iii) \Rightarrow (i)$, as desired. □

Remark 5.4.5. Arithmetically Cohen-Macaulay sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ were first classified by Giuffrida, *et al.* (see Theorem 4.1 of [**26**]). They showed that $\mathbb{X}$ is an ACM set of points if and only if $H_{\mathbb{X}}$, considered as an infinite matrix, is an admissible matrix such that the entries of $\Delta H_{\mathbb{X}}$ are either 1 or 0. By Remark 4.4.19, this condition on $\Delta H_{\mathbb{X}}$ is equivalent to the statement that $\Delta H_{\mathbb{X}}$ is the Hilbert function of a bigraded artinian quotient of $\mathbf{k}[x_1, y_1]$. Our contribution is to show that the ACM sets of points are also characterized by the tuples $\alpha_{\mathbb{X}}$ and $\beta_{\mathbb{X}}$.

Remark 5.4.6. In light of the previous result, it is natural to ask if Theorem 4.3.14 classifies ACM sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. We phrase this more precisely in the following question:

Question 5.4.7. *Suppose that $\mathbb{X}$ is set of distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with Hilbert function $H_{\mathbb{X}}$. If $\Delta H_{\mathbb{X}}$ is the Hilbert function of a $\mathbb{N}^k$-graded artinian quotient of $\mathbf{k}[x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{k,1}, \ldots, x_{k,n_k}]$, then is $\mathbb{X}$ an ACM set of points?*

As we have just seen, this question has a positive answer if $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$. Although we do not have an answer to this question in the general case, we suspect that the answer is yes.

Corollary 5.4.8. *Let $\mathbb{X}$ be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ with $\alpha_{\mathbb{X}} = (\alpha_1, \ldots, \alpha_t)$, and $\pi_1(\mathbb{X}) = \{P_1, \ldots, P_t\}$. Suppose (after a possible relabelling) that $|\pi_1^{-1}(P_i)| = \alpha_i$. For each integer $0 \leq i \leq t-1$ define*

$$\mathbb{X}_i = \mathbb{X} \backslash \left\{ \pi_1^{-1}(P_1) \cup \cdots \cup \pi_1^{-1}(P_i) \right\},$$

where $\mathbb{X}_0 = \mathbb{X}$. If $\mathbb{X}$ is ACM, then $\mathbb{X}_i$ is ACM for each integer $0 \leq i \leq t-1$. Moreover, $\alpha_{\mathbb{X}_i} = (\alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_t)$.

PROOF. It is sufficient to show that for each integer $0 \leq i \leq t-2$, if $\mathbb{X}_i$ is ACM, then $\mathbb{X}_{i+1}$ is ACM. So, suppose that $\mathbb{X}_i$ is ACM. Then, by construction $\alpha_{\mathbb{X}_i} = (\alpha_{i+1}, \ldots, \alpha_t)$. Suppose that $\beta_{\mathbb{X}_i} = (\beta_1, \ldots, \beta_r)$. Because $\mathbb{X}_i$ is ACM, $\alpha^*_{\mathbb{X}_i} = \beta_{\mathbb{X}_i}$.

Since $\mathbb{X}_{i+1} = \mathbb{X}_i \backslash \{\pi_1^{-1}(P_{i+1})\}$, $\mathbb{X}_{i+1}$ is constructed from $\mathbb{X}_i$ by removing the α_{i+1} points of $\mathbb{X}_i$ which have P_{i+1} as its first coordinate. The tuple $\beta_{\mathbb{X}_{i+1}}$ is constructed from $\beta_{\mathbb{X}_i}$ by subtracting 1 from α_{i+1} coordinates in $\beta_{\mathbb{X}_i}$. But because $\alpha^*_{\mathbb{X}_i} = \beta_{\mathbb{X}_i}$, we have $r = \alpha_{i+1}$, and thus $\beta_{\mathbb{X}_{i+1}} = (\beta_1 - 1, \ldots, \beta_{\alpha_i} - 1) = (\beta_1 - 1, \ldots, \beta_{\alpha_{i+2}} - 1)$. But by Lemma 5.4.1, $\alpha^*_{\mathbb{X}_{i+1}} = \beta_{\mathbb{X}_{i+1}}$, and hence, $\mathbb{X}_{i+1}$ is ACM by Theorem 5.4.4. □

It is well known that if $\mathbb{X}$ is a set of points of $\mathbb{P}^1$, then the Hilbert function and the graded Betti numbers in the resolution depend only upon the number $s = |\mathbb{X}|$ and not upon the coordinates of the points themselves. As we will show below, the Hilbert function and the graded Betti numbers in the resolution of an ACM set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ share the property that they depend only upon the combinatorics of $\mathbb{X}$ and not upon the coordinates of the points.

Theorem 5.4.9. *Let $\mathbb{X}$ be an ACM set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ with $\alpha_{\mathbb{X}} = (\alpha_1, \ldots \alpha_t)$. Then*

$$H_{\mathbb{X}} = \begin{bmatrix} 1 & 2 & \cdots & \alpha_1 - 1 & \alpha_1 & \alpha_1 & \cdots \\ 1 & 2 & \cdots & \alpha_1 - 1 & \alpha_1 & \alpha_1 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 1 & 2 & \cdots & \alpha_2 - 1 & \alpha_2 & \alpha_2 & \cdots \\ 1 & 2 & \cdots & \alpha_2 - 1 & \alpha_2 & \alpha_2 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 1 & 2 & \cdots & \alpha_3 - 1 & \alpha_3 & \alpha_3 & \cdots \\ 1 & 2 & \cdots & \alpha_3 - 1 & \alpha_3 & \alpha_3 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{bmatrix} + \cdots + \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 1 & 2 & \cdots & \alpha_t - 1 & \alpha_t & \alpha_t & \cdots \\ 1 & 2 & \cdots & \alpha_t - 1 & \alpha_t & \alpha_t & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

PROOF. Our proof will be by induction on the tuple $(|\pi_1(\mathbb{X})|, |\mathbb{X}|)$. For any $s \in \mathbb{N}$, if $(|\pi_1(\mathbb{X}), |\mathbb{X}|) = (1, s)$, then $\alpha_{\mathbb{X}} = (s)$ and $\beta_{\mathbb{X}} = (\underbrace{1, \ldots, 1}_{s})$. The Hilbert function of $\mathbb{X}$, which can be computed directly from Proposition 5.2.1, is

$$H_{\mathbb{X}} = \begin{bmatrix} 1 & 2 & 3 & \cdots & s-1 & s & s & \cdots \\ 1 & 2 & 3 & \cdots & s-1 & s & s & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \ddots \end{bmatrix},$$

which is the desired outcome.

Now suppose that $(|\pi_1(\mathbb{X})|, |\mathbb{X}|) = (t, s)$ and that the theorem holds for all ACM sets of points $\mathbb{Y} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ with $(t, s) >_{lex} (|\pi_1(\mathbb{Y})|, |\mathbb{Y}|)$.

After a possible relabelling, we can assume that P_1 is an element of $\pi_1(\mathbb{X})$ with $|\pi_1^{-1}(P_1)| = \alpha_1$. Set $\mathbb{X}_{P_1} := \pi_1^{-1}(P_1)$ and $\mathbb{Z} := \mathbb{X} \backslash \mathbb{X}_{P_1}$. Because $\mathbb{X}$ is ACM, $\alpha_{\mathbb{X}}^* = \beta_{\mathbb{X}}$, and hence, $\alpha_1 = |\beta_{\mathbb{X}}|$. Therefore, from Proposition 5.3.6, the Hilbert function of $\mathbb{X}$ is

$$H_{\mathbb{X}}(i,j) = H_{\mathbb{X}_{P_1}}(i,j) + H_{\mathbb{Z}}(i-1,j) \quad \text{for all } (i,j) \in \mathbb{N}^2,$$

where we adopt the convention that $H_{\mathbb{Z}}(i,j) = 0$ if $i < 0$.

It follows from the construction of $\mathbb{Z}$ and Corollary 5.4.8 that $\mathbb{Z}$ is ACM. Since $(t,s) >_{lex} (|\pi_1(\mathbb{Z})|, |\mathbb{Z}|) = (t-1, |\mathbb{Z}|)$, by the induction hypothesis we have

$$H_{\mathbb{X}} = H_{\mathbb{X}_{P_1}} + \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 1 & 2 & \cdots & \alpha_2 - 1 & \alpha_2 & \alpha_2 & \cdots \\ 1 & 2 & \cdots & \alpha_2 - 1 & \alpha_2 & \alpha_2 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{bmatrix} + \cdots + \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 1 & 2 & \cdots & \alpha_t - 1 & \alpha_t & \alpha_t & \cdots \\ 1 & 2 & \cdots & \alpha_t - 1 & \alpha_t & \alpha_t & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The conclusion now follows because $\mathbb{X}_{P_1}$ is an ACM set of points with $\alpha_{\mathbb{X}_{P_1}} = (\alpha_1)$. □

We will require the following result to describe the resolution of an ACM set of points in $\mathbb{P}^1 \times \mathbb{P}^1$.

Proposition 5.4.10. *Let $\mathbb{X}$ be a set of $s = tr$ points in $\mathbb{P}^1 \times \mathbb{P}^1$ such that $\alpha_{\mathbb{X}} = (\underbrace{r, \ldots, r}_{t})$ and $\beta_{\mathbb{X}} = (\underbrace{t, \ldots, t}_{r})$. Then $\mathbb{X}$ is ACM. In fact, $\mathbb{X}$ is a complete intersection. Furthermore, the minimal free resolution of $I_{\mathbb{X}}$ is*

$$0 \longrightarrow R(-t,-r) \longrightarrow R(-t,0) \oplus R(0,-r) \longrightarrow I_{\mathbb{X}} \longrightarrow 0$$

where the morphisms have degree $(0,0)$.

Proof. The set $\mathbb{X}$ is ACM because $\alpha^*_{\mathbb{X}} = (\underbrace{t, \ldots, t}_{r}) = \beta_{\mathbb{X}}$. Because $|\alpha_{\mathbb{X}}| = t$ and $|\beta_{\mathbb{X}}| = r$, it follows that $\pi_1(\mathbb{X}) = \{P_1, \ldots, P_t\}$ and $\pi_2(\mathbb{X}) = \{Q_1, \ldots, Q_r\}$ where $P_i, Q_j \in \mathbb{P}^1$. Since $|\mathbb{X}| = tr$, the set $\mathbb{X}$ must be the set of points $\{P_i \times Q_j \mid 1 \le i \le t, 1 \le j \le r\}$. Hence, if $I_{P_i \times Q_j} = (L_{P_i}, L_{Q_j})$ is the bihomogeneous prime ideal associated to the point $P_i \times Q_j$, then the defining ideal of $\mathbb{X}$ is

$$I_{\mathbb{X}} = \bigcap_{\substack{1 \le i \le t \\ 1 \le j \le r}} (L_{P_i}, L_{Q_j}) = \bigcap_{1 \le i \le t} (L_{P_i}, L_{Q_1} L_{Q_2} \cdots L_{Q_r}) = (L_{P_1} L_{P_2} \cdots L_{P_t}, L_{Q_1} L_{Q_2} \cdots L_{Q_r}).$$

Since $\deg L_{P_1} L_{P_2} \cdots L_{P_t} = (t,0)$ and $\deg L_{Q_1} L_{Q_2} \cdots L_{Q_r} = (0,r)$, the two generators of $I_{\mathbb{X}}$ form a regular sequence on R, and hence, $\mathbb{X}$ is a complete intersection.

Because $I_{\mathbb{X}}$ is generated by a regular sequence, the minimal free resolution of $I_{\mathbb{X}}$ is given by a *Koszul resolution* (see page 35 of Migliore [**39**] or Corollary 4.5.5 of Weibel [**56**]).

Taking into consideration that $I_{\mathbb{X}}$ is bigraded, we get

$$0 \longrightarrow R(-t,-r) \xrightarrow{\phi_2} R(-t,0) \oplus R(0,-r) \xrightarrow{\phi_1} I_{\mathbb{X}} \longrightarrow 0$$

where $\phi_1 = [L_{P_1}L_{P_2}\cdots L_{P_t}\ L_{Q_1}L_{Q_2}\cdots L_{Q_r}]$ and $\phi_2 = \begin{bmatrix} L_{Q_1}L_{Q_2}\cdots L_{Q_r} \\ -L_{P_1}L_{P_2}\cdots L_{P_t}\end{bmatrix}$. □

To state our result about the resolution, we require the following notation. Suppose that $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is a set of points with $\alpha_{\mathbb{X}} = (\alpha_1, \ldots, \alpha_t)$. Define

$$C_{\mathbb{X}} := \{(t,0),(0,\alpha_1)\} \cup \{(i-1,\alpha_i) \mid \alpha_i - \alpha_{i-1} < 0\},$$

and

$$V_{\mathbb{X}} := \{(t,\alpha_t)\} \cup \{(i-1,\alpha_{i-1}) \mid \alpha_i - \alpha_{i-1} < 0\}.$$

We take $\alpha_{-1} = 0$. With this notation, we have

Theorem 5.4.11. *Suppose that $\mathbb{X}$ is an ACM set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ with $\alpha_{\mathbb{X}} = (\alpha_1, \ldots, \alpha_t)$. Let $C_{\mathbb{X}}$ and $V_{\mathbb{X}}$ be constructed from $\alpha_{\mathbb{X}}$ as above. Then the graded minimal free resolution of $I_{\mathbb{X}}$ is*

$$0 \longrightarrow \bigoplus_{(v_1,v_2)\in V_{\mathbb{X}}} R(-v_1,-v_2) \longrightarrow \bigoplus_{(c_1,c_2)\in C_{\mathbb{X}}} R(-c_1,-c_2) \longrightarrow I_{\mathbb{X}} \longrightarrow 0.$$

where the morphisms have degree $(0,0)$.

Proof. We will do a proof by induction on the tuple $(|\pi_1(\mathbb{X})|, |\mathbb{X}|)$. If s is any integer, and $(|\pi_1(\mathbb{X}), |\mathbb{X}|) = (1,s)$, then $\alpha_{\mathbb{X}} = (s)$ and $\beta_{\mathbb{X}} = (\underbrace{1,\ldots,1}_{s})$. The conclusion now follows from Theorem 5.4.10 because $C_{\mathbb{X}} = \{(1,0),(0,s)\}$ and $V_{\mathbb{X}} = \{(1,s)\}$.

So, suppose $(|\pi_1(\mathbb{X}), |\mathbb{X}|) = (t,s)$ and the theorem holds for all $\mathbb{Y} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ with $(t,s) >_{lex} (|\pi_1(\mathbb{Y})|, |\mathbb{Y}|)$. Suppose that $\alpha_{\mathbb{X}} = (\underbrace{\alpha_1,\ldots,\alpha_1}_{l}, \alpha_{l+1}, \ldots, \alpha_t)$, i.e., $\alpha_{l+1} < \alpha_1$, but $\alpha_l = \alpha_1$.

If $l = t$, then $\mathbb{X}$ is a complete intersection and the resolution is given by Theorem 5.4.10. The conclusion now follows because $C_{\mathbb{X}} = \{(l,0),(0,\alpha_1)\}$ and $V_{\mathbb{X}} = \{(l,\alpha_1)\}$.

So, suppose that $l < t$. Let $P_1, \ldots, P_l$ be the l points of $\pi_1(\mathbb{X})$ that have $|\pi_1^{-1}(P_i)| = \alpha_1$. Set $\mathbb{Y} = \pi_1^{-1}(P_1) \cup \cdots \cup \pi_1^{-1}(P_l)$. Because $\mathbb{X}$ is ACM, $\alpha_1 = |\beta_{\mathbb{X}}|$. Hence,

$$\mathbb{Y} := \{P_i \times Q_j \mid 1 \leq i \leq l,\ Q_j \in \pi_2(\mathbb{X})\}$$

So, $\alpha_{\mathbb{Y}} = (\underbrace{\alpha_1, \ldots, \alpha_1}_{l})$ and $\beta_{\mathbb{Y}} = (\underbrace{l, \ldots, l}_{\alpha_1})$. Also, $I_{\mathbb{Y}} = (L_{P_1} \cdots L_{P_l}, L_{Q_1} \cdots L_{Q_r})$ where L_{P_i} is the form of degree $(1,0)$ that vanishes at all the points of $\mathbb{P}^1 \times \mathbb{P}^1$ which have P_i as their first coordinate, and L_{Q_i} is the form of degree $(0,1)$ that vanishes at all points $P \times Q \in \mathbb{P}^1 \times \mathbb{P}^1$ such that $Q = Q_i$.

Let $F := L_{P_1} \cdots L_{P_l}$ and $G := L_{Q_1} \cdots L_{Q_r}$. From the proof of Theorem 5.4.10 we have

$$0 \longrightarrow R(-l,-r) \xrightarrow{\phi_2} R(-l,0) \oplus R(0,-r) \xrightarrow{\phi_1} I_{\mathbb{Y}} \longrightarrow 0$$

where $\phi_1 = [F\ G]$ and $\phi_2 = \begin{bmatrix} G \\ -F \end{bmatrix}$. Let $\mathbb{Z} := \mathbb{X} \backslash \mathbb{Y}$. Since $\pi_2(\mathbb{Z}) \subseteq \pi_2(\mathbb{X})$, it follows that $G = L_{Q_1} \cdots L_{Q_r} \in I_{\mathbb{Z}}$. Hence, $\operatorname{im} \phi_2 \subseteq I_{\mathbb{Z}}(-l,0) \oplus R(0,-\alpha_1)$ since $r = \alpha_1$. We also require the following claim.

Claim. $I_{\mathbb{X}} = F \cdot I_{\mathbb{Z}} + (G)$

Proof of the Claim. By construction, $\mathbb{X} = \mathbb{Z} \cup \mathbb{Y}$, and thus $I_{\mathbb{X}} = I_{\mathbb{Z}} \cap I_{\mathbb{Y}}$. Hence, we want to show that $I_{\mathbb{Z}} \cap I_{\mathbb{Y}} = F \cdot I_{\mathbb{Z}} + (G)$.

So, if $K \in F \cdot I_{\mathbb{Z}} + (G)$, then there exists $H_1 \in I_{\mathbb{Z}}$ and $H_2 \in R$ such that $K = FH_1 + GH_2$. Since $F, G \in I_{\mathbb{Y}}$, $K \in I_{\mathbb{Y}}$. But $H_1, G \in I_{\mathbb{Z}}$, so we have $K \in I_{\mathbb{Z}} \cap I_{\mathbb{Y}}$.

To show the reverse inclusion, let $K \in I_{\mathbb{Z}} \cap I_{\mathbb{Y}}$. Since $K \in I_{\mathbb{Y}}$, there exists $H_1, H_2 \in R$ such that $K = FH_1 + GH_2$. If we can show that $H_1 \in I_{\mathbb{Z}}$, we will be finished. Since $K, G \in I_{\mathbb{Z}}$, we have $FH_1 \in I_{\mathbb{Z}}$. So, FH_1 must vanish at all $P \times Q \in \mathbb{Z}$. By construction, no point in $\mathbb{Z}$ can have P_i, where $1 \leq i \leq l$, as its first coordinate. So, if $P \times Q \in \mathbb{Z}$, then $F(P \times Q) = (L_{P_1} \cdots L_{P_l})(P \times Q) \neq 0$. Hence $H_1(P \times Q) = 0$, and thus $H_1 \in I_{\mathbb{Z}}$. □

From the above resolution for $I_{\mathbb{Y}}$, the claim, and the fact that $\operatorname{im} \phi_2 \subseteq I_{\mathbb{Z}}(-l,0) \oplus R(0,-\alpha_1)$, we have the following short exact sequence of R-modules

$$0 \longrightarrow R(-l,-r) \xrightarrow{\phi_2} I_{\mathbb{Z}}(-l,0) \oplus R(0,-\alpha_1) \xrightarrow{\phi_1} I_{\mathbb{X}} = F \cdot I_{\mathbb{Z}} + (G) \longrightarrow 0$$

where ϕ_1 and ϕ_2 are as above.

By Corollary 5.4.8 the set of points $\mathbb{Z}$ is ACM with $\alpha_{\mathbb{Z}} = (\alpha_{l+1}, \ldots, \alpha_t)$. Therefore, the induction hypothesis holds for $\mathbb{Z}$. With the above short exact sequence, we can use the *mapping cone construction* (see Section 1.5 of Weibel [**56**], and Section 4 of Chapter 1) to construct a resolution for $I_{\mathbb{X}}$. In particular, we get

$$0 \longrightarrow \left[\bigoplus_{(v_1,v_2)\in V_{\mathbb{Z}}} R(-(v_1+l),-v_2)\right] \oplus R(-l,-\alpha_1) \longrightarrow$$

$$\left[\bigoplus_{(c_1,c_2)\in C_{\mathbb{Z}}} R(-(c_1+l),-c_2)\right] \oplus R(0,\alpha_1) \longrightarrow I_{\mathbb{X}} \longrightarrow 0.$$

Since the resolution has length 2, and because $\mathbb{X}$ is ACM, the resolution of $I_{\mathbb{X}}$ cannot be made shorter by the Auslander-Buchsbaum formula (cf. Theorem 4.1.22)

To show that this resolution is minimal, it is enough to show that no tuple in the set $\{(c_1+l,c_2) \mid (c_1,c_2) \in C_{\mathbb{Z}}\} \cup \{(0,\alpha_1)\}$ is in the set $\{(v_1+l,v_2) \mid (v_1,v_2) \in V_{\mathbb{Z}}\} \cup \{(l,\alpha_1)\}$. By the induction hypothesis, we can assume that no $(c_1,c_2) \in C_{\mathbb{Z}}$ is in $V_{\mathbb{Z}}$, and hence, if $(c_1+l,c_2) \in \{(c_1+l,c_2) \mid (c_1,c_2) \in C_{\mathbb{Z}}\}$, then (c_1+l,c_2) is not in $\{(v_1+l,v_2) \mid (v_1,v_2) \in V_{\mathbb{Z}}\}$ If $(c_1+l,c_2) = (l,\alpha_1)$ for some $(c_1,c_2) \in C_{\mathbb{Z}}$, then this implies that $(0,\alpha_1)$. But this contradictions the induction hypothesis. Similarly, if $(0,\alpha_1) \in \{(v_1+l,v_2) \mid (v_1,v_2) \in V_{\mathbb{Z}}\}$, this implies $(-l,\alpha_1) \in V_{\mathbb{Z}}$, which is again a contradiction of the induction hypothesis. So the resolution given above is also minimal.

To complete the proof we only need to verify that

(i) $C_{\mathbb{X}} = \{(c_1+l,c_2) \mid (c_1,c_2) \in C_{\mathbb{Z}}\} \cup \{(0,\alpha_1)\}$.
(ii) $V_{\mathbb{X}} = \{(v_1+l,v_2) \mid (v_1,v_2) \in V_{\mathbb{Z}}\} \cup \{(l,\alpha_1)\}$.

Let $C' = \{(c_1+l,c_2) \mid (c_1,c_2) \in C_{\mathbb{Z}}\} \cup \{(0,\alpha_1)\}$. By definition $C_{\mathbb{X}} := \{(t,0),(0,\alpha_1)\} \cup \{(i-1,\alpha_i) \mid \alpha_i - \alpha_{i-1} < 0\}$ and $C_{\mathbb{Z}} = \{(t-l,0),(0,\alpha_{l+1})\} \cup \{(k-1,\alpha_{l+k}) \mid \alpha_{l+k} - \alpha_{l+k-1} < 0\}$ since $\alpha_{\mathbb{X}} = (\underbrace{\alpha_1,\ldots,\alpha_1}_{l},\alpha_{l+1},\ldots,\alpha_t)$ and $\alpha_{\mathbb{Z}} = (\alpha_{l+1},\ldots,\alpha_t)$

We check that $C_{\mathbb{X}} \subseteq C'$. It is immediate that the elements $(0,\alpha_1)$ and $(t,0)$ are in C'. So, suppose $(d_1,d_2) \in \{(i-1,\alpha_i) \mid \alpha_i - \alpha_{i-1} < 0\}$. For $i \leq l$, $\alpha_i = \alpha_1$. So, if $(d_1,d_2) \in \{(i-1,\alpha_i) \mid \alpha_i - \alpha_{i-1} < 0\}$, then $(d_1,d_2) = (l,\alpha_{l+1})$, or there exists some positive k such that $(d_1,d_2) = (l+k-1,\alpha_{l+k})$ and $\alpha_{l+k} - \alpha_{l+k-1} < 0$. But in either case, $(d_1-l,d_2) \in C_{\mathbb{Z}}$, and hence, $(d_1,d_2) \in C'$.

Conversely, we only need to check that $(c_1+l,c_2) \in C_{\mathbb{X}}$ for every $(c_1,c_2) \in C_{\mathbb{Z}}$. It is straightforward to check that $((t-l)+l,0) \in C_{\mathbb{X}}$. Also, as noted, $(l,\alpha_{l+1}) \in C_{\mathbb{X}}$. If

$(c_1, c_2) \in \{(k-1, \alpha_{l+k}) \mid \alpha_{l+k} - \alpha_{l+k-1} < 0\}$, then $(c_1 + l, c_2) = (k + l - 1, \alpha_{l+k}) \in C_{\mathbb{Z}}$ because $\alpha_{l+k} - \alpha_{l+k-1}$. Thus $C_{\mathbb{X}} = C'$.

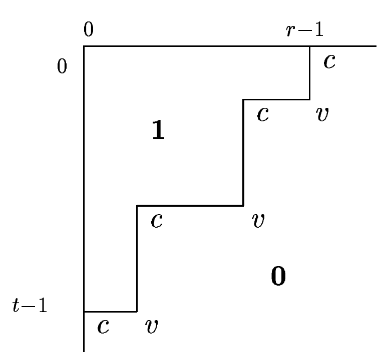
The proof of (ii) is similar in nature. However, for completeness, we will verify the details. Let V' denote the set $\{(v_1 + l, v_2) \mid (v_1, v_2) \in V_{\mathbb{Z}}\} \cup \{(l, \alpha_1)\}$. By definition, $V_{\mathbb{X}} = \{(t, \alpha_t)\} \cup \{(i-1, \alpha_{i-1}) \mid \alpha_i - \alpha_{i-1} < 0\}$ since $\alpha_{\mathbb{X}} = (\underbrace{\alpha_1, \ldots, \alpha_1}_{l}, \alpha_{l+1}, \ldots, \alpha_t)$, and $V_{\mathbb{Z}} = \{(t-l, \alpha_t)\} \cup \{(i-1, \alpha_{i+l-1}) \mid \alpha_{i+l} - \alpha_{i+l-1} < 0\}$ because $\alpha_{\mathbb{Z}} = (\alpha_{l+1}, \ldots, \alpha_t)$.

We will check that $V_{\mathbb{X}} \subseteq V'$. The element $(t, \alpha_t) \in V'$ because $(t-l, \alpha_t) \in V_{\mathbb{Z}}$. So, suppose $(d_1, d_2) \in \{(i-1, \alpha_{i-1}) \mid \alpha_i - \alpha_{i-1} < 0\}$. Because $\alpha_i = \alpha_1$ for $1 \leq i \leq l$, it is either the case that $(d_1, d_2) = (l, \alpha_l) = (l, \alpha_1)$, or $(d_1, d_2) = (l + i - 1, \alpha_{l+i-1})$ with $i > 1$. But in the first case, it is immediate that $(d_1, d_2) \in V'$. In the second case, because $(d_1, d_2) \in V_{\mathbb{X}}$, $\alpha_{l+i} - \alpha_{l+i-1} < 0$. But then $(i-1, \alpha_{l+i-1}) \in V_{\mathbb{Z}}$, and hence, $(d_1, d_2) \in V'$.

Conversely, because $(l, \alpha_1) \in V_{\mathbb{X}}$, we only need to check that $(v_1 + l, v_2) \in V_{\mathbb{X}}$ for all $(v_1, v_2) \in V_{\mathbb{Z}}$. It is immediate that $(t - l + l, \alpha_t) \in V_{\mathbb{X}}$. So, suppose that $(v_1, v_2) \in \{(i - 1, \alpha_{i+l-1}) \mid \alpha_{i+l} - \alpha_{i+l-1} < 0\}$. But then $(v_1 + l, v_2) \in \{(i+l-1, \alpha_{i+l-1}) \mid \alpha_{i+l} - \alpha_{i+l-1} < 0\}$. Because $\alpha_{\mathbb{X}} = (\underbrace{\alpha_1, \ldots, \alpha_1}_{l}, \alpha_{l+1}, \ldots, \alpha_t)$, we must have $\{(i + l - 1, \alpha_{i+l-1}) \mid \alpha_{i+l} - \alpha_{i+l-1} < 0\} = V_{\mathbb{X}} \backslash \{(t, \alpha_t)\}$, which completes the proof. $\square$

Remark 5.4.12. The resolution of an ACM set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ was first computed by Giuffrida, *et al.* (Theorem 4.1 [**26**]). Giuffrida, *et al.* showed that the graded Betti numbers for an ACM set of points $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ could be determined from the first difference function $\Delta H_{\mathbb{X}}$, i.e.,

$\Delta H_{\mathbb{X}} =$



An element of $C_{\mathbb{X}}$, which they called a *corner* of $\Delta H_{\mathbb{X}}$, corresponds to a tuple (i,j) that is either $(t,0),(0,\alpha_1) = (0,r)$, or has the property that $\Delta H_{\mathbb{X}}(i,j) = 0$, but $\Delta H_{\mathbb{X}}(i-1,j) = \Delta H_{\mathbb{X}}(i,j-1) = 1$. We have labelled the corners of $\Delta H_{\mathbb{X}}$ with a c in the above diagram. An element of $V_{\mathbb{X}}$ is a *vertex*. A tuple (i,j) is called a vertex if $\Delta H_{\mathbb{X}}(i,j) = \Delta H_{\mathbb{X}}(i-1,j) = \Delta H_{\mathbb{X}}(i,j-1) = 0$, but $\Delta H_{\mathbb{X}}(i-1,j-1) = 1$. We have labelled the vertices of $\Delta H_{\mathbb{X}}$ with a v in the above diagram. Our contribution, besides giving a new proof for the resolution of an ACM set of points in $\mathbb{P}^1 \times \mathbb{P}^1$, is to show that the graded Betti numbers can be computed directly from the tuple $\alpha_{\mathbb{X}}$.

Example 5.4.13. The set of points $\mathbb{X}$ in Example 5.2.7 is not ACM. Indeed, for that example, we saw that $\alpha_{\mathbb{X}} = (4,3,2,2,1,1)$ and $\beta_{\mathbb{X}} = (4,3,3,3)$. Since $\alpha^*_{\mathbb{X}} \neq \beta_{\mathbb{X}}$, $\mathbb{X}$ cannot be an ACM set of points.

Example 5.4.14. Let $\mathbb{X}$ be the set of points in Example 5.3.5. Then $\alpha_{\mathbb{X}} = (4,3,1,1)$ and $\beta_{\mathbb{X}} = (4,2,2,1)$. It is an easy exercise to verify that $\alpha^*_{\mathbb{X}} = (4,2,2,1) = \beta_{\mathbb{X}}$. Thus, $\mathbb{X}$ is ACM. Because $\mathbb{X}$ is ACM, the Hilbert function of $\mathbb{X}$ can be computed using Theorem 5.4.9. We have

$$H_{\mathbb{X}} = \begin{bmatrix} 1 & 2 & 3 & 4 & 4 & \cdots \\ 1 & 2 & 3 & 4 & 4 & \cdots \\ 1 & 2 & 3 & 4 & 4 & \cdots \\ 1 & 2 & 3 & 4 & 4 & \cdots \\ 1 & 2 & 3 & 4 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 3 & 3 & \cdots \\ 1 & 2 & 3 & 3 & \cdots \\ 1 & 2 & 3 & 3 & \cdots \\ 1 & 2 & 3 & 3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ 1 & 1 & \cdots \\ 1 & 1 & \cdots \\ 1 & 1 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ 1 & 1 & \cdots \\ 1 & 1 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 & 4 & \cdots \\ 2 & 4 & 6 & 7 & 7 & \cdots \\ 3 & 5 & 7 & 8 & 8 & \cdots \\ 4 & 6 & 8 & 9 & 9 & \cdots \\ 4 & 6 & 8 & 9 & 9 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We can also compute the resolution of $I_{\mathbb{X}}$ by using Theorem 5.4.11. We compute the sets $C_{\mathbb{X}}$ and $V_{\mathbb{X}}$ from $\alpha_{\mathbb{X}} = (4,3,1,1)$ to get:

(*i*) $C_{\mathbb{X}} = \{(4,0),(0,4),(1,3),(2,1)\}$.

(*ii*) $V_{\mathbb{X}} = \{(4,1),(1,4),(2,3)\}$.

The resolution of $I_{\mathbb{X}}$ is then

$$0 \longrightarrow \begin{array}{c} R(-1,-4) \oplus \\ R(-4,-1) \oplus R(-2,-3) \end{array} \longrightarrow \begin{array}{c} R(0,-4) \oplus R(-4,0) \oplus \\ R(-1,-3) \oplus R(-2,-1) \end{array} \longrightarrow I_{\mathbb{X}} \longrightarrow 0.$$

APPENDIX A

Using CoCoA to Compute the Hilbert Function of a Multi-graded Rings

The goal of this appendix is to describe how one can compute the Hilbert function of a multi-graded ring using **CoCoA**. Although the procedure that we describe is straightforward, using **CoCoA** to implement this method requires some care. We will begin by giving the mathematics behind the algorithm. We will then provide a step-by-step account of how to implement this procedure into **CoCoA**. In the last section of the appendix, we will provide some output of our algorithm. We will emphasize examples dealing with points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

We have written our code using **CoCoA** 4.0 for Linux. **CoCoA** can be obtained for free via anonymous FTP at **cocoa.dima.unige.it** or via the **CoCoA** home page:

http : //cocoa.dima.unige.it

There is also a comprehensive manual and a series of tutorials at this web address.

I want to thank John Abbott, Anna Bigatti, and Massimo Caboara for helping me with all my **CoCoA** related questions and problems. I would especially like to thank Anna Bigatti, who read an earlier version of this appendix, for making some very helpful suggestions to increase the readability of the appendix, and for bringing to my attention some features of **CoCoA** that simplified the following discussion.

1. The Mathematics of the Algorithm

Suppose that $\mathbf{k}$ is a field and let $R = \mathbf{k}[x_{1,0},\ldots,x_{1,n_1},\ldots,x_{k,0},\ldots,x_{k,n_k}]$. We suppose that $\deg x_{i,j} = e_i$ where e_i is the i^{th} standard basis vector of $\mathbb{N}^k$. The ring R is then $\mathbb{N}^k$-graded. Suppose that $I \subsetneq R$ is an $\mathbb{N}^k$-homogeneous ideal. Then the quotient ring $S = R/I$

is an $\mathbb{N}^k$-graded ring, i.e., $S = \bigoplus_{\underline{i}\in\mathbb{N}^k} S_{\underline{i}}$ where $S_{\underline{i}} = R_{\underline{i}}/I_{\underline{i}}$ for all $\underline{i} := (i_1,\ldots,i_k) \in \mathbb{N}^k$. Furthermore, $S_{\underline{i}}$ is a finite dimensional vector space over $\mathbf{k}$ for all $\underline{i} \in \mathbb{N}^k$.

The numerical function $H_S : \mathbb{N}^k \to \mathbb{N}$ defined by

$$H_S(\underline{i}) = \dim_{\mathbf{k}} S_{\underline{i}} = \dim_{\mathbf{k}} R_{\underline{i}} - \dim_{\mathbf{k}} I_{\underline{i}}$$

is the *Hilbert function* of $S = R/I$. To compute the Hilbert function of a multi-graded ring using CoCoA, we will use the Hilbert-Poincaré series. Recall that the *Hilbert-Poincaré series* of $S = R/I$ is the infinite series

$$HP_S(t_1,\ldots,t_k) = \sum_{\underline{i}\in\mathbb{N}^k} H_S(\underline{i}) t^{\underline{i}} \quad \text{where } t^{\underline{i}} := t_1^{i_1}\ldots t_k^{i_k}.$$

Using the Hilbert-Serre theorem (see [**7**]) we have

$$HP_S(t_1,\ldots,t_k) = \frac{Q(t_1,\ldots,t_k)}{(1-t_1)^{n_1+1}\cdots(1-t_k)^{n_k+1}} \tag{A.1.3}$$

where $Q(t_1,\ldots,t_k) \in \mathbb{Z}[t_1,\ldots,t_k]$.

The CoCoA function `Poincare` is able to compute the Hilbert-Poincaré series of a multi-graded ring. A description of the algorithm used by CoCoA is found in Bigatti [**3**]. The routine `Poincare` returns the rational function given in equation (A.1.3). We, therefore, need to extract the Hilbert function of $S = R/I$ from the rational function.

Because

$$\sum_{\underline{i}\in\mathbb{N}^k} H_S(\underline{i}) t^{\underline{i}} = \frac{Q(t_1,\ldots,t_k)}{(1-t_1)^{n_1+1}\cdots(1-t_k)^{n_k+1}},$$

to compute $H_S(\underline{i})$ for any $\underline{i} \in \mathbb{N}^k$, we need to compute the coefficient of $t^{\underline{i}}$ in the expression on the right. From the identity

$$\frac{1}{(1-t)^n} = 1 + \binom{n+1-1}{1} t^1 + \binom{n+2-1}{2} t^2 + \cdots + \binom{n+d-1}{d} t^d + \cdots$$

it follows that

$$\begin{aligned}\frac{Q(t_1,\ldots,t_k)}{(1-t_1)^{n_1+1}\cdots(1-t_k)^{n_k+1}} &= Q(t_1,\ldots,t_k)\left(1 + \binom{n_1+1}{1} t_1^1 + \cdots + \binom{n_1+d}{d} t_1^d + \cdots\right) \times \\ &\left(1 + \binom{n_2+1}{1} t_2^1 + \cdots + \binom{n_2+d}{d} t_2^d + \cdots\right) \times \cdots \times \\ &\left(1 + \binom{n_k+1}{1} t_k^1 + \cdots + \binom{n_k+d}{d} t_k^d + \cdots\right).\end{aligned}$$

By expanding out the right hand side, we can compute $H_S(\underline{i})$ for all $\underline{i} \in \mathbb{N}^k$. However, this approach is not feasible because it requires an infinite number of operations.

To get around this difficulty, we decide *a priori* on a finite number of $\underline{i} \in \mathbb{N}^k$ for which we wish to compute $H_S(\underline{i})$. We shall usually fix a $\underline{j} = (j_1, \ldots, j_k) \in \mathbb{N}^k$ and compute $H_S(\underline{i})$ for all $\underline{i} = (i_1, \ldots, i_k) \leq \underline{j} = (j_1, \ldots, j_k)$. Recall that we say $\underline{i} \leq \underline{j}$ if and only if $i_l \leq j_l$ for all l. Thus, we need to compute the coefficients of $t^{\underline{i}}$ of $HP_S(t_1, \ldots, t_k)$ for only those $\underline{i} \leq \underline{j}$. Hence, for each integer $1 \leq l \leq k$, we need to write out only the first j_l terms of $\frac{1}{(1+t_i)^{n_l+1}}$ because the larger terms do not contribute to any coefficient of $t^{\underline{i}}$ with $\underline{i} \leq \underline{j}$. We, therefore, only expand out

$$Q(t_1, \ldots, t_k)\left(1 + \binom{n_1+1}{1}t_1^1 + \cdots + \binom{n_1+j_1}{j_1}t_1^{j_1}\right) \times$$
$$\left(1 + \binom{n_2+1}{1}t_2^1 + \cdots + \binom{n_2+j_2}{j_2}t_2^{j_2}\right) \times \cdots \times$$
$$\left(1 + \binom{n_k+1}{1}t_k^1 + \cdots + \binom{n_k+j_k}{j_k}t_k^{j_k}\right)$$

to calculate the value of $H_S(\underline{i})$ for all $\underline{i} \leq \underline{j}$. Moreover, there are only a finite number of calculations required.

The following algorithm is a summary of the above discussion. The algorithm is also the basis for the actual implementation we give in the next section.

Algorithm A.1.1.

Input: *An $\mathbb{N}^k$-graded ring $S = R/I$ and $\underline{j} = (j_1, \ldots, j_k) \in \mathbb{N}^k$.*

Output: $H_S(\underline{i})$, the Hilbert function of $S = R/I$, for all $\underline{i} \leq \underline{j}$.

1. *Compute* $HP_S(t_1, \ldots, t_k) = \dfrac{Q(t_1, \ldots, t_k)}{(1-t_1)^{n_1+1} \cdots (1-t_k)^{n_k+1}}$.
2. *For each integer $1 \leq l \leq k$, set*

$$L_l := \left(1 + \binom{n_l+1}{1}t_l^1 + \cdots + \binom{n_l+j_l}{j_l}t_l^{j_l}\right).$$

3. $K(t_1, \ldots, t_k) := Q(t_1, \ldots, t_k)L_1L_2 \cdots L_k$.
4. *For each $\underline{i} \leq \underline{j}$, return the coefficient of $t^{\underline{i}} = t_1^{i_1} \cdots t_k^{i_k}$ in $K(t_1, \ldots, t_k)$.*

2. The Implementation of the Algorithm

To simplify our notation, we will only describe how to implement Algorithm A.1.1 for bigraded quotients of the $\mathbb{N}^2$-graded ring $R = \mathbf{k}[x_1, \ldots, x_n, y_1, \ldots, y_m]$ with $\deg x_i = (1,0)$ and $\deg y_i = (0,1)$. Moreover, we will also assume that $\mathbf{k} = \mathbb{Q}$. We cannot work in an algebraically closed field of characteristic zero, for example, $\mathbb{C}$, because all computers have to store numbers as finite pieces of information. Fortunately, if all the coordinates of the points that we consider are in $\mathbb{Q}$, then the computations over any extension, and in particular, over $\mathbb{C}$, are the same and will give the same result.

Before implementing Algorithm A.1.1 into `CoCoA`, we need to describe how to overcome the following two problems: (1) `CoCoA` does not allow one to give an indeterminate a degree of $(0,1)$; and (2) the output of the `Poincare` function is not returned as a rational function.

We will start by showing how to give our polynomial ring the appropriate grading. Suppose that $T = \mathbf{k}[x_1, \ldots, x_n]$. It is then possible in `CoCoA` to assign each indeterminate x_i a non-standard degree, that is, $\deg x_i := (a_{i,1}, a_{i,2}, \ldots, a_{i,r})$ where $r \leq n$ and $a_{i,j} \in \mathbb{Z}$ via the `Weights` function. For example, if $T = \mathbb{Q}[x_1, \ldots, x_3]$ and $\deg x_1 = (1,2)$, $\deg x_2 = (2,3)$, and $\deg x_3 = (3,4)$, then the commands to define this multi-graded ring are:

```
W:= Mat([[1,2,3],
         [2,3,4]]);
Use T::= Q[x[1..3]],Weights(W);
```

However, if we were to use this example as a guide to give $R = \mathbb{Q}[x_1, x_2, y_1, y_2]$ a bigrading with $\deg x_i = (1,0)$ and $\deg y_i = (0,1)$, the corresponding commands

```
W:= Mat([[1,1,0,0],
         [0,0,1,1]]);
Use R::= Q[x[1..2],y[1..2]],Weights(W);
```

will result in an error. This is because `CoCoA` requires the first row of the matrix to contain only positive integers. To circumvent this problem, we give R a tri-grading, that is, an $\mathbb{N}^3$-grading, by defining $\deg x_i = (1,1,0)$ and $\deg y_i = (1,0,1)$. Therefore, the command to define the ring above would be

```
W:= Mat([[1,1,1,1],
```

```
        [1,1,0,0],
        [0,0,1,1]]);
Use R::= Q[x[1..2],y[1..2]],Weights(W);
```

Although we no longer have a standard bigraded ring, we can still use this ring to make our calculations. Indeed, $F \in R$ is homogeneous with respect to this $\mathbb{N}^3$-grading if and only if F is homogeneous with respect to the $\mathbb{N}^2$-grading. Hence, if I is any $\mathbb{N}^2$ homogeneous ideal of R, it will also be $\mathbb{N}^3$-homogeneous with respect to this grading.

To use Algorithm A.1.1, we need to compute the Hilbert-Poincaré series of the bigraded ring $S = R/I$. Even though `CoCoA` will not allow us to define such a bigraded ring, the following proposition enables us to use the above non-standard grading on $R = \mathbf{k}[x_1, \ldots, x_n, y_1, \ldots, y_m]$ to compute the desired Hilbert-Poincaré series.

Proposition A.2.1. *Let* $R = \mathbf{k}[x_1, \ldots, x_n, y_1, \ldots, y_m]$ *with* $\deg x_i = (1,1,0)$ *and* $\deg y_i = (1,0,1)$. *Suppose that* I *is a homogeneous ideal of* R *with respect to this grading and that*

$$HP_S(t_0, t_1, t_2) = \frac{Q(t_0, t_1, t_2)}{(1 - t_0t_1)^n(1 - t_0t_2)^m}$$

where $Q(t_0, t_1, t_2) \in \mathbb{Z}[t_0, t_1, t_2]$, *is the Hilbert-Poincaré series of* $S = R/I$. *Then* I *is also a homogeneous ideal of* R *with respect to the grading on* R *induced by setting* $\deg x_i = (1,0)$ *and* $\deg y_i = (0,1)$. *Furthermore, let* $Q'(t_1, t_2) = Q(1, t_1, t_2)$. *Then the Hilbert-Poincaré series of* $S = R/I$ *with respect to this new grading is*

$$HP_S(t_1, t_2) = \frac{Q'(t_1, t_2)}{(1 - t_1)^n(1 - t_2)^m}.$$

PROOF. Since F is homogeneous with respect to the non-standard $\mathbb{N}^3$-grading if and only if F is homogeneous with respect to the $\mathbb{N}^2$ grading, then one needs to only verify that the coefficient of $t_0^i t_1^j t_2^k$ of $HP_S(t_0, t_1, t_2)$ is equal to the coefficient of $t_1^j t_2^k$ if $i = j + k$ and zero otherwise. □

From the above proposition, we see that we need to manipulate the numerator of the Hilbert-Poincaré series of the non-standard graded ring $S = R/I$ in order to get the desired numerator. However, when we wish to implement this step, we encounter our second difficulty. The `CoCoA` function `Poincare` which computes the Hilbert-Poincaré function returns an object that is not a rational function.

For example, suppose we use the ring $R = \mathbb{Q}[x_1, x_2, y_1, y_2]$ where $\deg x_i = (1,1,0)$ and $\deg y_i = (1,0,1)$ and we wish to compute the Hilbert-Poincaré series of the ring R/I where $I = (x_1 + x_2, y_1)$. Then the needed commands are:

```
W:= Mat([[1,1,1,1],
         [1,1,0,0],
         [0,0,1,1]]);
Use R::=Q[x[1..2],y[1..2]],Weights(W);
I:=Ideal(x[1]+x[2],y[1]);
P:=Poincare(R/I); P;
```

The output is:

```
---  Non-simplified HilbertPoincare' Series  ---
(x[1]^2x[2]y[1] - x[1]x[2] - x[1]y[1] + 1) /
( (1-x[1]x[2]) (1-x[1]x[2]) (1-x[1]y[1]) (1-x[1]y[1]) )
```

Although CoCoA outputs the result as rational function, it is not stored as such. However, we use the function HP.ToRatFun to turn the output of Poincare into a rational function. Continuing with our above example, we have:

```
HP.ToRatFun(P);
1/(x[1]^2x[2]y[1] - x[1]x[2] - x[1]y[1] + 1)
-------------------------------
```

Note that the output that is returned is simplified. We can now multiply the above output by the denominator to isolate the numerator.

Below is our code for the function BiHilbert that is based upon Algorithm A.1.1. We assume that the appropriate multi-graded ring has been defined. The function BiHilbert has been written to output the Hilbert function as a matrix $(m_{i,j})$ where $m_{i,j} := H_{R/I}(i,j)$. Note that we adopt the convention that the indexing of the matrix starts at $(0,0)$.

```
----------------------------------------------------------------------
-- BiHilbert(I,K1,K2)
--
-- (Assume that the appropriate ring R has been defined)
-- BiHilbert computes the Hilbert function of the bigraded ring S = R/I
-- for all (I,J) <= (K1,K2)
----------------------------------------------------------------------


Define BiHilbert(I,K1,K2)
 --  N = #indeterminates of degree (1,1,0) in R
 --  M = #indeterminates of degree (1,0,1) in R
  N:= Len([X In Indets() | MDeg(X) =[1,1,0]]);
  M:= NumIndets() - N;
 -- Compute Hilbert-Poincare Series
  P:=Poincare(CurrentRing()/I);
  BiHiRing::=Q[t[1..3]];
  Using BiHiRing Do
 -- Determine the numerator (Num) of the HP-series
    RationalP:=$cocoa/hp.ToRatFun(P);
    Num := RationalP*(1-t[1]t[2])^N*(1-t[1]t[3])^M;
 -- Derive the correct numerator by substituting 1 into Num
    Num:=Subst(Num,t[1],1);
 -- Write out the appropriate number of terms of the denominator
 -- and multiply by the numerator.  We use the
 -- routine Expansion
    Expanded:=Num*Expansion(N,M,K1,K2);
 -- Read off the coefficient of the term t[2]^I1*t[3]^I2
 -- for all (I1,I2) <= (K1,K2).  Store result in a matrix
    HilbertMatrix:=NewMat(K1+1,K2+1,0);
    Foreach M In Monomials(Expanded) Do
      If Deg(M,t[2]) <= K1 And Deg(M,t[3]) <= K2 Then
HilbertMatrix[Deg(M,t[2])+1,Deg(M,t[3])+1] := LC(M);
      End;
    End;
 -- Return desired values as a matrix
    Return(HilbertMatrix);
  End;
End;
```

```
------------------------------------------------------------
-- Expansion(N,M,K1,K2)
--
-- Expansion computes the first K1 terms of 1/(1-t[2])^N and
-- the first K2 terms of 1/(1-t[3]^M.  It then returns
-- the product of these two polynomials.
------------------------------------------------------------

Define Expansion(N,M,K1,K2)
  L1:=[Bin(D+N-1,N-1)*t[2]^D | D In 0..K1];
  L2:=[Bin(D+M-1,M-1)*t[3]^D | D In 0..K2];
  P1:=Sum(L1);
  P2:=Sum(L2);
  Return(P1*P2);
End; -- Expansion
```

3. Examples of the Algorithm

In this section, we will demonstrate how to use the function `BiHilbert` to compute the Hilbert function of a bigraded quotient of the ring $R = \mathbf{k}[x_1, \ldots, x_n, y_1, \ldots, y_m]$.

Example A.3.1. Let $R = \mathbb{Q}[x_1, x_2, x_3, x_4, y_1, y_2]$ where $\deg x_i = (1,0)$ and $\deg y_i = (0,1)$. As observed in Chapter 2,

$$H_R(i,j) = \dim_{\mathbb{Q}} R_{i,j} = \binom{4-1+i}{i}\binom{2-1+j}{j} \qquad \text{for all } (i,j) \in \mathbb{N}^2.$$

We use `BiHilbert` to compute the Hilbert function of R for all $(i,j) \leq (10,10)$ to verify that our algorithm has been properly coded.

```
W:=Mat([[1,1,1,1,1,1],
        [1,1,1,1,0,0],
        [0,0,0,0,1,1]]);
Use R::=Q[x[1..4],y[1..2]],Weights(W);
I:=Ideal(0);
BiHilbert(I,10,10);
Mat[
  [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11],
  [4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44],
```

```
  [10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110],
  [20, 40, 60, 80, 100, 120, 140, 160, 180, 200, 220],
  [35, 70, 105, 140, 175, 210, 245, 280, 315, 350, 385],
  [56, 112, 168, 224, 280, 336, 392, 448, 504, 560, 616],
  [84, 168, 252, 336, 420, 504, 588, 672, 756, 840, 924],
  [120, 240, 360, 480, 600, 720, 840, 960, 1080, 1200, 1320],
  [165, 330, 495, 660, 825, 990, 1155, 1320, 1485, 1650, 1815],
  [220, 440, 660, 880, 1100, 1320, 1540, 1760, 1980, 2200, 2420],
  [286, 572, 858, 1144, 1430, 1716, 2002, 2288, 2574, 2860, 3146]
]
-------------------------------
```

Example A.3.2. Let $R = \mathbb{Q}[x_1, x_2, y_1, y_2]$ with $\deg x_i = (1,0)$ and $\deg y_i = (0,1)$. Then R is the bigraded coordinate ring of $\mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$. Let $P_i = [1:i] \in \mathbb{P}^1_{\mathbb{Q}}$ and $Q_i = [1:i] \in \mathbb{P}^1_{\mathbb{Q}}$ and suppose that $\mathbb{X}$ is the following set of points:

$$\mathbb{X} = \{P_1 \times Q_1, P_1 \times Q_2, P_1 \times Q_3, P_2 \times Q_1, P_3 \times Q_4\}.$$

It follows that $|\pi_1(\mathbb{X})| = 3$ and $|\pi_2(\mathbb{X})| = 4$. From Corollary 3.1.7, we need to compute those $(i,j) \leq (3-1, 4-1) = (2,3)$ to determine all the values of the Hilbert function. Our computation of $H_{\mathbb{X}}$ using `BiHilbert` shows that this is indeed the case.

```
W:=Mat([[1,1,1,1],
        [1,1,0,0],
        [0,0,1,1]]);
Use R::=Q[x[1..2],y[1..2]],Weights(W);
I_P1xQ1:=Ideal(x[1]-x[2],y[1]-y[2]);
I_P1xQ2:=Ideal(x[1]-x[2],2y[1]-y[2]);
I_P1xQ3:=Ideal(x[1]-x[2],3y[1]-y[2]);
I_P2xQ1:=Ideal(2x[1]-x[2],y[1]-y[2]);
I_P3xQ4:=Ideal(3x[1]-x[2],4y[1]-y[2]);
I:=Intersection(I_P1xQ1,I_P1xQ2,I_P1xQ3,I_P2xQ1,I_P3xQ4);
BiHilbert(I,4,5);
Mat[
  [1, 2, 3, 4, 4, 4],
  [2, 4, 5, 5, 5, 5],
  [3, 4, 5, 5, 5, 5],
  [3, 4, 5, 5, 5, 5],
  [3, 4, 5, 5, 5, 5]
```

```
]
-------------------------------
```

We are now able to read off the *border* (see Definition 3.2.8) of the Hilbert function. For this example, the border $B_{\mathbb{X}} = ((3,4,5,5),(4,5,5))$.

Example A.3.3. Let $R = \mathbb{Q}[x,y]$ with $\deg x = (1,0)$ and $\deg y = (0,1)$, and let $I = (x^3, x^2y, xy^4, y^5)$. We use `BiHilbert` to compute the Hilbert function of R/I:

```
W:=Mat([[1,1],
        [1,0],
        [0,1]]);
Use R::=Q[x,y],Weights(W);
I:=Ideal(x^3,x^2y,xy^4,y^5);
BiHilbert(I,5,5);
BiHilbert(I,5,5);
Mat[
  [1, 1, 1, 1, 1, 0],
  [1, 1, 1, 1, 0, 0],
  [1, 0, 0, 0, 0, 0],
  [0, 0, 0, 0, 0, 0],
  [0, 0, 0, 0, 0, 0],
  [0, 0, 0, 0, 0, 0]
]
```

From Corollary 4.4.14, it follows that R/I is a bigraded artinian quotient. Thus, by Corollary 4.4.15 there exists an ACM set of points in $\mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ with Hilbert function H such that ΔH is equal to the above Hilbert function. We claim that the set of points

$$\mathbb{X} = \left\{ \begin{array}{l} [1:1]\times[1:1],[1:1]\times[1:2],[1:1]\times[1:3],[1:1]\times[1:4],[1:1]\times[1:5] \\ [1:2]\times[1:1],[1:2]\times[1:2],[1:2]\times[1:3],[1:2]\times[1:4] \\ [1:3]\times[1:1] \end{array} \right\}$$

is such a set. From Theorem 5.4.4, the set $\mathbb{X}$ is ACM. Using `BiHilbert` to compute $H_{\mathbb{X}}$ we find

```
Use R::=Q[x[1..2],y[1..2]],Weights(Mat[[1,1,1,1],[1,1,0,0],[0,0,1,1]]);
I1:=Ideal(x[1]-x[2],y[1]-y[2]);
I2:=Ideal(x[1]-x[2],2y[1]-y[2]);
I3:=Ideal(x[1]-x[2],3y[1]-y[2]);
```

```
I4:=Ideal(x[1]-x[2],4y[1]-y[2]);
I5:=Ideal(x[1]-x[2],5y[1]-y[2]);
I6:=Ideal(2x[1]-x[2],y[1]-y[2]);
I7:=Ideal(2x[1]-x[2],2y[1]-y[2]);
I8:=Ideal(2x[1]-x[2],3y[1]-y[2]);
I9:=Ideal(2x[1]-x[2],4y[1]-y[2]);
I10:=Ideal(3x[1]-x[2],y[1]-y[2]);
I:=Intersection(I1,I2,I3,I4,I5,I6,I7,I8,I9,I10);
BiHilbert(I,5,5);
Mat[
  [1, 2, 3, 4, 5, 5],
  [2, 4, 6, 8, 9, 9],
  [3, 5, 7, 9, 10, 10],
  [3, 5, 7, 9, 10, 10],
  [3, 5, 7, 9, 10, 10],
  [3, 5, 7, 9, 10, 10]
]
-------------------------------
```

A routine calculation will now verify that the first difference function of this Hilbert function is equal to Hilbert function of the above bigraded artinian ring.

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