Assignment 2 Solutions, Math 712

1. Consider the class G_{nt} of all finite triangle-free graphs (nt for no triangles). Show that this is a Fraïssé class i.e. it is closed under isomorphisms, subgraphs, amalgamation, and for every n, there are, up to isomorphism, only finitely many triangle-free graphs of size n. Construct a generic countable graph H_{nt} as we did with the random graph with the property that it is universal for the class G_{nt} and is ultrahomogeneous. Show that there is only one countable graph with this property. Write out axioms for this class and conclude that these axioms are complete.

   Solution: To see that G_{nt} is a Fraïssé class, we really only have to check that it is closed under amalgamation. So suppose that G is a common subgraph of two triangle-free graphs H_1 and H_2. We can form an amalgamation of H_1 and H_2 over G which is triangle-free by, for instance, considering the disjoint union of H_1 and H_2 with the common G identified and then adding no new edges between vertices in $H_1 \setminus G$ and $H_2 \setminus G$. Since there were no triangles to begin with and we added no new edges, the resulting graph will be triangle-free.

   We can now construct H_{nt} as an increasing chain of finite triangle-free graphs
   $$H_0 \subset H_1 \subset H_2 \subset ...$$
   where at each stage in the construction we consider all the subgraphs G of H_i and all H which are one point triangle-free extensions of G. We promise for each such pair to consider an amalgamation that involves that pair at some future stage. Besides the standard bookkeeping, we are left, at stage i with H_i and some subgraph $G \subset H_i$ together with H, a one-point triangle-free extension of G. Let H_{i+1} be an amalgamation of H_i with H over G.

   Since the one point graph is triangle-free, it suffices to show that H_{nt} is ultrahomogeneous. That is, suppose that $A, B \subset H_{nt}$ are finite and isomorphic via some map f. We want to do a back and forth argument which shows that there is an automorphism of H_{nt} which extends f. We look at the forth argument; the back argument is similar. $A \subset H_n$ for some n. Pick $a \in H_{nt}$ which we wish to add to the domain of f. This represents a one point extension that we had to consider at some point in the construction of H_{nt}. $B \subset H_m$ for some m and hence we also had to consider the amalgamation problem involving B and a one point extension which is isomorphic to the pair A together with a. Hence, by construction of H_{nt} there is some $b \in H_{nt}$ such that $f \cup \{(a, b)\}$ is an isomorphism. Continuing like this inductively, we create an automorphism of H_{nt} which extends f.

Do some literature research and find out what you can about the almost sure theory of triangle-free graphs. Is it the same as the theory of H_{nt}? Is the theory of H_{nt} pseudo-finite? Hint: some of this is an open research question.

Comments: There is an almost sure theory of triangle-free graphs which is essentially the random bipartite graph. This is a result of Erdös, Kleitman and Rothschild. It is not the same as the theory of H_{nt} because, for instance, a 5-cycle is a subgraph of H_{nt} and it is not bipartite.

As far as I know the question of whether the generic triangle-free graph is pseudo-finite is wide open.

2. Prove the Łoś Theorem for metric spaces. That is, show that if $X = \prod_{\mathcal{U}} X_i$ where the X_i's are an I-indexed family of uniformly bounded metric spaces then whenever $\varphi(x_1, \ldots, x_n)$ is a formula in the language of metric spaces and $a^1, \ldots, a^n \in X$ then

$$\varphi^X(a^1, \ldots, a^n) = \lim_{\mathcal{U}} \varphi^{X_i}(a^1_i, \ldots, a^n_i).$$

Solution: We prove this by induction on the construction of the formula φ.

Case 1: The only atomic formula here is $d(x, y)$ where d is the metric symbol and so the result follows by the definition of the the metric on the ultraproduct. Note that the language provides a uniform bound on the value of d in any model.

Case 2: Suppose we have a continuous function $f : \mathbf{R}^n \to \mathbf{R}$ and formulas $\varphi_k(\bar{x})$ for $k = 1, \ldots, n$. We need to assume that each φ_k has some bound B_k and hence f restricted to $\prod_k [-B_k, B_k]$ is also bounded since f is continuous.

The essence of the rest of the proof is that

$$f(\lim_{\mathcal{U}} \varphi_1^X(\bar{a}), \ldots, \lim_{\mathcal{U}} \varphi_n^X(\bar{a}) = \lim_{\mathcal{U}} f(\varphi_1^{X_i}(\bar{a}_i), \ldots, \varphi_n^{X_i}(\bar{a}_i))$$

which follows from the continuity of f.

Case 3: Finally, assume that $\varphi(\bar{x}) = \inf_y \psi(\bar{x}, y)$. The bound on this formula will be the same as for ψ.

Now suppose $\varphi^X(\bar{a}) = r$ and $\epsilon > 0$. Then for some $b \in X$, $\psi^X(\bar{a}, b) < r + \epsilon$. By induction then, ultrafilter often we have $\psi^{X_i}(\bar{a}_i, b_i)$ which means that $\lim_{\mathcal{U}} \psi^{X_i}(\bar{a}_i, b_i) \leq r + \epsilon$. From this we conclude that $\lim_{\mathcal{U}} \varphi^{X_i}(\bar{a}_i) \leq \varphi^X(\bar{a})$.

Now if $\lim_{\mathcal{U}} \varphi^{X_i}(\bar{a}_i) = s < r$, ultrafilter often we could choose $b_i \in X_i$ such that $\psi(\bar{a}_i, b_i) < s + \epsilon$ where $\epsilon = \frac{r-s}{2}$. But then if we let $b = (b_i)$, we have $\lim_{\mathcal{U}} \psi^{X_i}(\bar{a}_i, b_i) \leq s + \epsilon < r$ which contradicts that $r = \inf_y \psi^X(\bar{a}, y)$.

3. Suppose that (X_i, d_i) for $i \in I$ is a uniformly bounded I-indexed family of metric spaces and f_i is a continuous function of one variable on X_i for each i (continuous with respect to d_i). Algebraically we can define $X' = \prod_I X_i$ and define f coordinate-wise on X' via the f_i's. If $\mathcal{U}$ is an ultrafilter on I, then $X = \prod_{\mathcal{U}} X_i$ is a quotient of X'. What condition do we need to put on the f_i's so that f is well-defined on this quotient?

   Solution: As I think I hinted at in class, assuming that the f_i's are uniformly uniformly continuous is enough. Suppose that for every $\epsilon > 0$ there is $\delta > 0$ such that for any i, whenever $d_i(x_i, y_i) < \delta$ for $x_i, y_i \in X_i$ then $d(f(x_i), f(y_i)) \leq \epsilon$. Now suppose that $\bar{x}, \bar{y} \in \prod_I X_i$ and $\lim_{\mathcal{U}} d_i(x_i, y_i) = 0$. Choose $\epsilon > 0$ and let δ be given by the uniform continuity. Then ultrafilter often $d_i(x_i, y_i) < \delta$ and hence $d_i(f(x_i, f(y_i)) \leq \epsilon$. So $\lim_{\mathcal{U}} d_i(f(x_i), f(y_i)) \leq \epsilon$. Since this is true for any ϵ, f is well-defined on the equivalence class of x modulo the metric on the ultraproduct.

4. Show that the Urysohn sphere, $\mathcal{U}$, is ultrahomogeneous. That is, suppose that $X \subset Y$ are both finite $[0, 1]$-metric spaces and $X \subset \mathcal{U}$. Then there is a Y', $X \subset Y' \subset \mathcal{U}$ with $Y \cong Y'$ with X fixed.

   Solution: I know that people found this to be a challenging problem so I will write a solution in two passes - first to get the basic idea down and then to come back and get the numbers right. We set the stage with some notation: $\mathcal{U}$ is Urysohn space which is the closure of U_0, a countable dense subset in which all the distances are rational and U_0 is both universal and ultrahomogeneous with respect to finite rational

$[0,1]$-metric spaces. Since we have $X \subset \mathcal{U}$ then by the density of U_0, we can find a sequence of subspaces $X_k \subset U_0$ such that X_k tends to X in the limit and moreover, we can assume that $\mathrm{Conf}_X(X_k)$ tends to 0 in the limit. That is,

(a) If $X = \{a_1, \ldots, a_n\}$ then we can find $X_k = \{a_1^k, \ldots, a_n^k\} \subset U_0$ such that $\lim_{k\to\infty} a_i^k = a_i$ for every $i = 1, \ldots, n$ and moreover,

(b) $\lim_{k\to\infty} d(a_i^k, a_j^k) = d(a_i, a_j)$ for all $i, j = 1, \ldots, n$.

We also have Y which we can assume is a one-point extension of X. We would like to choose Y_k, a one-point extension of X_k so that Y_k is a rational $[0,1]$-metric space and moreover, $\mathrm{Conf}_Y(Y_k)$ tends to 0 as k tends to infinity. Precisely, we mean that if $Y = \{a_1, \ldots, a_n, y\}$ and $Y_k = \{a_1^k, \ldots, a_n^k, y_k\}$ then $\lim_{k\to\infty} d(a_i^k, y_k) = d(a_i, y)$ for all $i = 1, \ldots, n$. Note we already have the convergence requirements for X_k. So how do we choose Y_k with this property? First of all, we can amalgamate Y with X_k over X. We do this in the minimal way that we did in class. That is, we let

$$d(y, a_i^k) = \min_j \Big(d(y, a_j) + d(a_j, a_i^k)) \Big)$$

for each i. This means that $|d(y, a_i^k) - d(y, a_i)| \leq d(a_i, a_i^k)$ for all i; this tells us that if X and X_k are close together then the configuration involving y is close to correct. The only remaining problem is that $d(y, a_i^k)$ might not be rational. Again, as we did in class, we can modify these distances. Note that all the distances in X_k are rational so we only have to modify $d(y, a_i^k)$. The conclusion from class was that we can increase only these values by any sufficiently small amount that makes them rational and still have a metric space.

Alright, so with all this preprocessing, how do we construct the necessary extension of X inside $\mathcal{U}$? We will produce it as the limit of a Cauchy sequence (b_k) which we create inductively as follows:

(a) b_0 realizes the extension Y_0 of X_0 in U_0. This is possible by the manner in which U_0 was constructed.

(b) In general we will have b_k realizing Y_k over X_k and $b_k \in U_0$. The trick will be how to construct b_{k+1} so that it isn't too far from b_k.

Toward this end, we can assume that we have X_k and b_k by induction as well as X_{k+1} and Y_{k+1}. The idea is to amalgamate Y_{k+1} with X_k and b_k over X_{k+1}. Notice that by induction, all the distances between X_k, X_{k+1} and b_k are known as these are elements of U_0. We want to construct a one point extension so that we realize the metric space described by Y_{k+1}. We do this in as minimal way as possible subject to the triangle inequality. We define the distance from y_{k+1} to any point z in X_k or b_k by

$$d(z, y) = \max_a |d(z, a) - d(a, y)|$$

where a ranges over X_{k+1}. It is an exercise to show that this function defines a metric on X_k, X_{k+1}, b_k and y. As all the distances are rational, this extension is realized in U_0 by some b_{k+1}. Let's establish some bounds. Suppose that $d(b_k, b_{k+1}) = |d(b_k, a_i^{k+1}) - d(a_i^{k+1}, b_{k+1})|$. That is, a_i^{k+1} realizes the maximum in this case. There are two cases:

(a) Case 1: $d(b_k, a_i^{k+1}) \geq d(a_i^{k+1}, b_{k+1})$. In this case, notice that $d(b_k, a_i^{k+1}) \leq d(b_k, a_i^k) + d(a_i^k, a_i^{k+1})$ which means that

$$d(b_k, b_{k+1}) \leq d(a^k, a_i^{k+1}) + d(b_k, a_i^k) - d(b_{k+1}, a_i^{k+1}).$$

(b) Case 2: $d(b_k, a_i^{k+1}) \leq d(a_i^{k+1}, b_{k+1})$. In this case, notice that $d(b_k, a_i^{k+1}) \geq d(b_k, a_i^k) - d(a_i^k, a_i^{k+1})$. Once again, we have

$$d(b_k, b_{k+1}) \leq d(a_i^{k+1}, a_i^k) + d(b_k, a_i^k) - d(b_{k+1}, a_i^{k+1}).$$

The takeaway from all this is that if Y_k and Y_{k+1} are close to the configuration of Y, and the Cauchy sequence a_i^k is close to a_i then $d(b_k, b_{k+1})$ can't be too big.

Now to get the numbers correct, we can choose a subsequence of the (a_i^k)'s so that $d(a_i^k, a_i^{k+1}) < \frac{1}{2^k}$ for all i and that $\mathrm{Conf}_Y(Y_k) < \frac{1}{2^k}$. In this way, $d(b_k, b_{k+1})$ is no larger than $\frac{3}{2^k}$. This implies that (b_k) is a Cauchy sequence and if we let $b \in \mathcal{U}$ be its limit than we have X together with b realizing the one-point extension Y.

5. I made the claim in class that there is no effective difference between pseudo-compact and pseudo-finite in continuous logic. Let me try to justify that claim. Suppose that X is a compact metric space. For each

n, choose $a_1, \ldots, a_{k_n} \in X$ such that the open $\frac{1}{n}$-balls centered at the a_i's cover X; we can assume that the a_i's are at least $\frac{1}{n}$ apart (Why?). Let X_n be the subspace consisting of $\{a_i : i \leq k_n\}$. Let $\mathcal{U}$ be any non-principal ultrafilter on $\mathbf{N}$ and prove that $X \cong \prod_{\mathcal{U}} X_n$.

Solution: My original solution made use of the requirement that the centers of the balls were $\frac{1}{n}$ apart. In the end, I didn't need this although you can assume this by choosing a maximal set of balls with this property -it must be a cover - and then taking a finite subcover. Here is a solution that doesn't use this property.

Let's introduce some additional notation: for each n, let's refer to the given open cover as C_n and the centers as

$$\{a_i^n : i \leq k_n\}.$$

Consider the map $i : x \mapsto (x_n)$ where x_n is a center of an open ball in C_n containing x. This map is well-defined since if (x_n) and (x'_n) are two such sequences then $d(x_n, x'_n) < 2/n$ and so the two sequences are identified in the ultraproduct. A similar argument shows this map is injective.

To see that the map is surjective, pick any sequence $x = (x_n) \in \prod_n X_n$. We now create a descending chain Y_k of elements of $\mathcal{U}$ such that:

(a) $Y_0 = \mathbf{N}$,

(b) if $k > 0$ then choose i least such that

$$Y_k = \{n \in Y_{k-1} : n > \min Y_{k-1} \text{ and } d(x_n, a_i^k) < \frac{1}{n}\} \in \mathcal{U}.$$

Since C_k is a cover, such an i exists. Let's call it i_k. Notice that $\bigcap_k Y_k = \emptyset$.

Now define a sequence $y_k = a_{i_n}^n$ for $n = \min Y_k$. This sequence is a Cauchy sequence since $y_l \in Y_k$ for all $l > k$ and hence if $l, m > k$ then $d(y_l, y_m) < \frac{2}{k}$. Let $y = \lim y_k \in X$. Now a representation of $i(y)$ where the n^{th} entry is y_k if n is the least element of Y_k. For all $m \in Y_k$, we $d(x_m, y_k) < \frac{1}{k}$ and so $d(x, i(y))$ in the ultraproduct is 0 hence i is onto.