MATHEMATICS 1LS3 FINAL EXAMINATION
version 1

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Day Class
Duration of Examination: 2.5 hours
McMaster University
19 December 2016

FIRST NAME (PRINT CLEARLY): SOLUTIONS

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Student No.: __________

THIS EXAMINATION PAPER HAS 15 PAGES AND 8 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF THE INVIGILATOR.

Total number of points is 75. Marks are indicated next to the problem number. McMaster Standard Calculator Casio fx991MS or Casio fx991MS+ is allowed. Write your answers in the space provided. EXCEPT ON QUESTIONS 1 AND 2, YOU MUST SHOW WORK TO GET FULL CREDIT. Good luck.

Problem	Points	Mark
1	20	
2	6	
3	12	
4	7	
5	8	
6	6	
7	7	
8	9	
TOTAL	75	

Question 1: Circle the correct answer. No justification is needed.

1. (a)[2] The resistance R of the flow of blood through a blood vessel (assumed to have the shape of a cylindrical tube) is given by

$$R = \frac{Kl\gamma^2}{d^4}$$

where l is the length of the tube, d is its diameter and $\gamma \geq 1$ is the curvature. The positive constant K represents the viscosity of the blood (viscosity is a measure of the resistance of fluid to stress; water has low viscosity, honey has high viscosity).
Which of the following statements is/are true?

(I) If the curvature γ doubles, then the resistance R doubles ✗

(II) If the viscosity K doubles, then the resistance R doubles ✓

(III) If the diameter d doubles, then the resistance R increases $2^4 = 16$ fold ✗

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

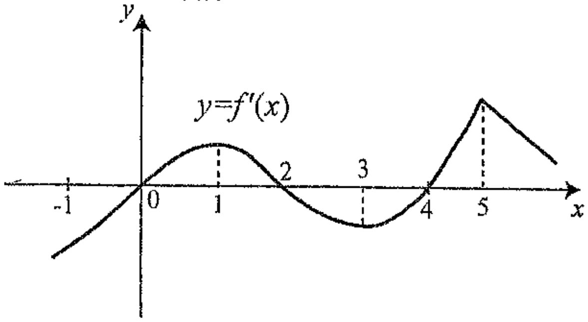
(b)[2] If $g(x) = \cos\left(e^{x^2+x}\right)$, then $g'(0)$ is equal to

(A) 0 (B) $\sin 1$ (C) $\cos 1$ (D) $\sin e$
(E) $-\sin 1$ (F) $\cos e$ (G) $-\cos e$ (H) $-\cos 1$

$g'(x) = -\sin\left(e^{x^2+x}\right) \cdot e^{x^2+x} \cdot (2x+1)$

$g'(0) = -\sin 1 \cdot 1 \cdot 1$

(c)[2] Determine which of the statements is/are true for the function $f(x)$ whose derivative $f'(x)$ is drawn below.



$f' = 0$ or f' dne

(I) $x = 3$ is a critical point (critical number) of $f(x)$

(II) $x = 4$ is a critical point (critical number) of $f(x)$

(III) $x = 5$ is a critical point (critical number) of $f(x)$

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

(d)[2] $\lim_{x\to 0+} \sqrt{x}\ln(3x) = \lim_{x \to 0+} \frac{\ln(3x)}{x^{-1/2}}$

(A) 3 (B) ∞ (C) $-\infty$ (D) -3

(E) 0 (F) 1/3 (G) $-1/3$ (H) 1/2

$\overset{LH}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}} = (-2)\lim_{x\to 0+} x^{1/2} = 0$

(e)[2] The formula (adapted from M. Benton and D. Harper, *Basic Palaeontology*)

$$\mathrm{Sk} = e^{0.08T}\,\mathrm{Sp}^{0.84}$$

relates the skull length, Sk, of a larger dinosaur to its spine length, Sp, at time of death T. Which of the following statements is/are true?

(I) The semilog graph of Sk as a function of Sp is a line

(II) The semilog graph of Sk as a function of T is a line

(III) The double log graph of Sk as a function of Sp is a line

(Note: it does not matter whether ln or log is used.)

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

$$\ln \mathrm{Sk} = 0.08T + 0.84 \ln \mathrm{Sp}$$

(f)[2] Identify all correct Taylor polynomials of the function $f(x) = \sin x$ at $x = 0$.

(I) $T_1(x) = x$ ✓

(II) $T_3(x) = x - x^2$ ✗

(III) $T_3(x) = x - \frac{x^3}{2}$ ✗ $\quad x - \frac{x^3}{6}$

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

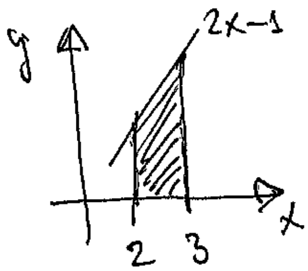
(g)[2] Identify all correct interpretations of the definite integral $\int_2^3 (2x - 1)\, dx$.

(I) Total change in the function $f(x) = 2x - 1$ from $x = 2$ to $x = 3$.

(II) Area of the region under the graph of $f(x) = 2x - 1$, above the x-axis, from $x = 2$ to $x = 3$.

(III) Average value of the function $f(x) = 2x - 1$ on the interval $[2, 3]$.

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three



(h)[2] Consider the differential equation

$$P'(t) = 1.1P(t)\left(1 - \frac{P(t)}{1400}\right)\left(1 - \frac{650}{P(t)}\right)$$

where $P(t)$ represents the number of elk in Douglas Provincial Park in Saskatchewan. The variable t represents time in years, with $t = 0$ representing 2013.
What does this model say about the population of elk? (Identify all true statements.)

(I) At the moment when there are 1250 elk, the population is increasing

(II) At the moment when there are 1450 elk, the population is decreasing

(III) At the moment when there are 600 elk, the population is increasing

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

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(i)[2] Start with the graph of $y = \cos x$. Scale (expand) the graph horizontally by a factor of 3 and then shift right the graph you obtained by 6 units. Finally, expand this graph vertically by a factor of 4. The graph you obtained is

(A) $y = \frac{1}{4}\cos\left(\frac{x+2}{6}\right)$ (B) $y = \frac{1}{4}\cos\left(\frac{x-2}{6}\right)$ (C) $y = 4\cos\left(\frac{x+6}{3}\right)$

(D) $y = 4\cos\left(\frac{x}{3}-2\right)$ (E) $y = 4\cos\left(\frac{x}{3}+6\right)$ (F) $y = 4\cos\left(\frac{x}{3}-\frac{2}{3}\right)$

(G) $y = \frac{1}{4}\cos\left(\frac{x+2}{3}\right)$ (H) $y = \frac{1}{4}\cos\left(\frac{x-6}{3}\right)$

$$\cos x \to \cos\frac{x}{3} \to \cos\frac{x-6}{3} \to 4\cos\frac{x-6}{3}$$
$$= 4\cos\left(\frac{x}{3}-2\right)$$

(j)[2] Which of the following improper integrals are convergent?

(I) $\int_0^1 x^{-2}\,dx$ (II) $\int_0^1 x^{-1}\,dx$ (III) $\int_0^1 x^{-0.5}\,dx$

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

$\int_0^1 \frac{1}{x^p}\,dx$ is conv. if $0<p<1$

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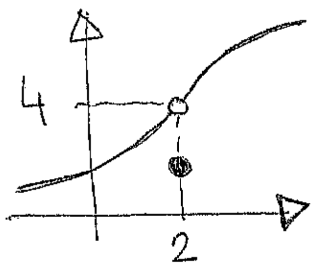
Question 2: Circle the correct answer. No justification is needed.

2. (a)[2] Based on knowing that $\lim_{x\to 2} f(x) = 4$, we can conclude that the function $f(x)$ is continuous at 2, and $f(2) = 4$.

TRUE FALSE



(b)[2] A population of monkeys on an island doubles every year; present count is 25 monkeys. Based on this data, we conclude that in 3 years there will be 75 monkeys on the island.

TRUE FALSE

$$25 \xrightarrow{1yr} 50 \xrightarrow{1yr} 100 \xrightarrow{1yr} 200$$

(c)[2] A population of bacteria triples every hour. Every hour, after reproduction, 800 bacteria are removed. The population starts with 1000 bacteria. The dynamical system which describes this population is given by $b_{t+1} = 3(b_t - 800)$, $b_0 = 1000$.

TRUE FALSE

$$b_{t+1} = 3b_t - 800$$
$$b_0 = 1000$$

Questions 3-8: You must show work to obtain full credit

3. (a)[3] Find $\int x \ln x dx$.

$$= \left\{ \begin{array}{l} u = \ln x \rightarrow u' = \frac{1}{x} \\ v' = x \rightarrow v = \frac{x^2}{2} \end{array} \right\}$$

$$= uv - \int v u' dx = \frac{x^2}{2} \ln x - \frac{1}{2} \int x dx$$

$$= \frac{x^2}{2} \ln x - \frac{x^2}{4} + C$$

(b)[3] Find $\int_1^9 \frac{(1+\sqrt{x})^2}{\sqrt{x}} dx$. $= \left\{ \begin{array}{l} u = 1 + \sqrt{x} \\ \frac{du}{dx} = \frac{1}{2\sqrt{x}} \rightarrow \frac{1}{\sqrt{x}} dx = 2du \end{array} \right\}$

$$= \int 2u^2 du = \frac{2}{3} u^3 = \frac{2}{3} (1+\sqrt{x})^3 \Big|_1^9$$

$$= \frac{2}{3}(4)^3 - \frac{2}{3}(2)^3 = \frac{112}{3} \approx 37.3$$

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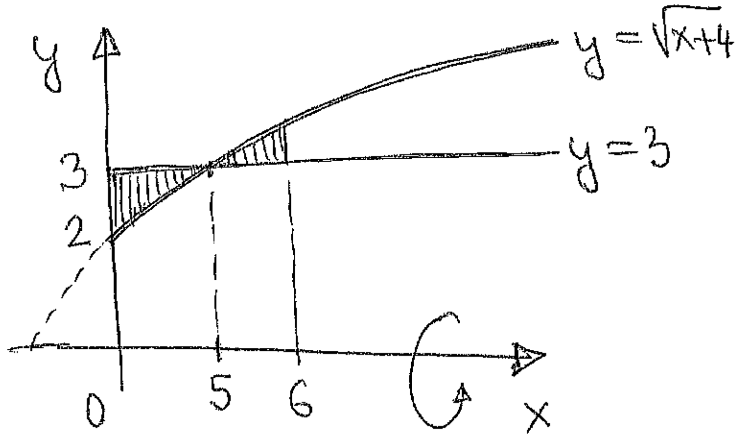
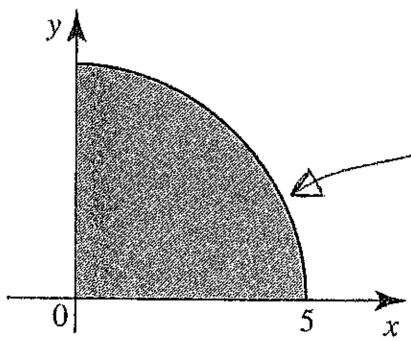
(c)[3] Consider the region R bounded by $y = \sqrt{x+4}$, $y = 3$, $x = 0$, and $x = 6$. Write a formula for the volume of the solid obtained by rotating the region R about the x-axis. Your answer must not contain absolute value. You do NOT need to compute the volume.



$$\sqrt{x+4} = 3 \rightarrow x = 5$$

$$V = \pi \int_0^5 \underbrace{(3)^2 - (\sqrt{x+4})^2}_{5-x} dx + \pi \int_5^6 \underbrace{(\sqrt{x+4})^2 - (3)^2}_{x-5} dx$$

(d)[3] Using a definite integral, write an expression for the area of the part of the circle of radius 5 centered at the origin lying in the first quadrant, as shown below. You do NOT need to compute the integral.



$$x^2 + y^2 = 25$$

$$y = \sqrt{25 - x^2}$$

$$A = \int_0^5 \sqrt{25 - x^2}\, dx$$

4. Consider the function $f(x) = x^{4/5}(2x - 9)$.

(a)[3] Find all critical points of $f(x)$.

$$f'(x) = \frac{4}{5} x^{-1/5} (2x-9) + x^{4/5} \cdot 2$$
$$= x^{-1/5}\left(\frac{8x}{5} - \frac{36}{5} + 2x \right)$$
$$= \frac{1}{\sqrt[5]{x}} \left(\frac{18x}{5} - \frac{36}{5} \right)$$

$f' = 0 \longrightarrow x = 2$

f' dne $\longrightarrow x = 0$

(b)[2] Give a precise statement of the Extreme Value Theorem.

IF $f(x)$ is continuous on a closed interval $[a,b]$

THEN $f(x)$ has an absolute max. and an absolute min. in $[a,b]$

(c)[2] Find the absolute maximum and the absolute minimum values of $f(x)$ on $[1, 5]$.

x	$f(x)$
1	-7
5	$5^{4/5} \approx 3.62$
2	$-5 \cdot 2^{4/5} \approx -8.71$

abs max $= 5^{4/5}$ at $x = 5$

abs min $= -5 \cdot 2^{4/5}$ at $x = 2$

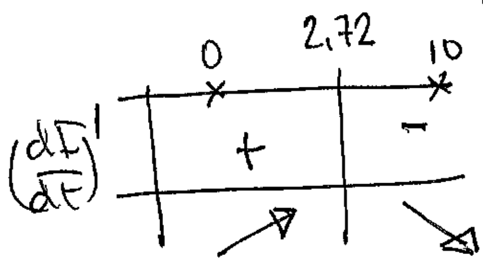
5. The rate at which new influenza cases occured in 2013 in Greater Edmonton Area followed the formula $dI/dt = 24e^{-0.2t} - 21e^{-0.3t}$. Units of $I(t)$ are *thousands* of people. By t we represent the time in days measured from 1 December 2013 (so $t = 0$ represents 1 December 2013). On 1 December 2013 there were 1250 cases of influenza.

(a) [4] On which day (state the date) did the rate $\underbrace{dI/dt}_{\text{max. of this}}$ reach its largest value?

$$\left(\frac{dI}{dt}\right)' = -4.8e^{-0.2t} + 6.3e^{-0.3t} = 0 \quad \Big| \cdot e^{0.3t}$$

$$-4.8e^{0.1t} + 6.3 = 0$$

$$e^{0.1t} = \frac{6.3}{48} \rightarrow t = \frac{\ln\left(\frac{6.3}{4.8}\right)}{0.1} \approx 2.72$$



$t = 0 \ldots$ 1 Dec
$1 \ldots$ 2 Dec
$2 \ldots$ 3 Dec
$3 \ldots$ 4 Dec

3 Dec or 4 Dec

(b)[1] Write the initial value problem for the number $I(t)$ of influenza cases. Keep in mind that the number of people is measured in thousands.

$$\frac{dI}{dt} = 24e^{-0.2t} - 21e^{-0.3t} \qquad I(0) = 1.25$$

(c)[3] Solve the initial value problem in (b) to find the formula for $I(t)$.

$$I(t) = \int \left(24e^{-0.2t} - 21e^{-0.3t}\right) dt$$

$$I(t) = -120e^{-0.2t} + 70e^{-0.3t} + C$$

$$I(0) = 1.25 \longrightarrow \quad 1.25 = -120 + 70 + C \rightarrow C = 51.25$$

$$I(t) = -120e^{-0.2t} + 70e^{-0.3t} + 51.25$$

6. Consider the alcohol consumption model $a_{t+1} = a_t - \frac{9.1a_t}{3.2 + 1.1a_t} + d$, where a_t is the amount of alcohol (in grams) and d is the constant amount that is consumed every hour.

(a)[1] What is the meaning of the term $\frac{9.1a_t}{3.2 + 1.1a_t}$?

amount of alcohol absorbed in an hour when the amount of alcohol present is a_t

(b)[2] Assume that $d = 5$. Find the equilibrium point of the given system.

$$\cancel{a^*} = \cancel{a^*} - \frac{9.1a^*}{3.2+1.1a^*} + 5$$

$$9.1a^* = 16 + 5.5a^*$$

$$3.6a^* = 16$$

$$a^* = \frac{16}{3.6} = \frac{4}{0.9} \approx 4.44 \text{ g}$$

(c)[3] Assume that $d = 5$. Determine whether the equilibrium you found in (b) is stable or unstable.

$$f(x) = x - \frac{9.1x}{3.2+1.1x} + 5$$

$$f'(x) = 1 - \frac{9.1(3.2+1.1x) - 9.1x \cdot (1.1)}{(3.2+1.1x)^2}$$

$$= 1 - \frac{9.1 \cdot 3.2}{(3.2+1.1x)^2}$$

$$|f'(4.44)| \approx 0.55 < 1 \rightarrow \text{stable}$$

7. Most human papillomavirus (HPV) infections in young women are temporary and have very little long-term effects. Assume that $P(t)$ is the proportion (or percent) of women initially infected with the HPV who no longer have the virus at time t (t is measured in years). The *rate of change* of $P(t)$ is modelled by the function

$$p(t) = 0.5 - 0.25t^{1.5} + 0.5e^{-1.2t}$$

where $0 \leq t \leq 2$.

(a)[2] Compute $p(1)$. What are the units of $p(1)$?

$p(1) = 0.5 - 0.25 + 0.5e^{-1.2} \approx 0.40$

Units are proportion/year or %/year

(b)[3] Find $\int_0^1 \left(0.5 - 0.25t^{1.5} + 0.5e^{-1.2t}\right) dt$.

$= 0.5t - \frac{0.25}{2.5}t^{2.5} + \frac{0.5}{-1.2}e^{-1.2t}\Big|_0^1$

$= 0.5t - 0.1t^{2.5} - \frac{5}{12}e^{-1.2t}\Big|_0^1$

$= \left(0.5 - 0.1 - \frac{5}{12}e^{-1.2}\right) - \left(-\frac{5}{12}\right) \approx 0.69$

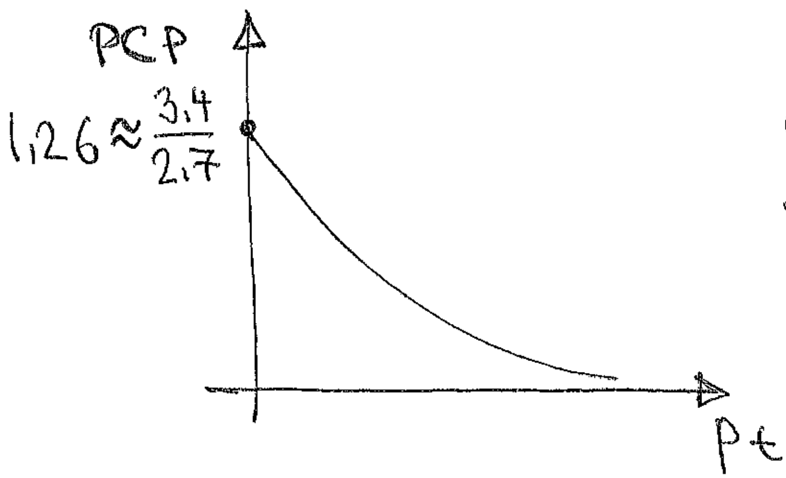
(c)[2] What does the number you obtained in (b) represent in the context of HPV infections?

At $t=1$ (so one year later) about 69% of women initially infected with the HPV will no longer have the virus

8. Consider the discrete-time dynamical system $p_{t+1} = \frac{3.4p_t}{2.7 + 0.001p_t}$, where p_t represents a number of frogs in thousands and time is in years.

(a)[3] Identify the per capita production function. and say what it represents Sketch its graph and label intercepts. Explain what it means, i.e., why it makes sense.

$$PCP = \frac{3{,}4}{2{,}7 + 0{,}001\,p_t} = \text{number of new individuals per individual}$$



as population increases the PCP (= number of new individuals per individual) decreases due to lack of space, resources (food)

(b)[2] Find all equilibrium points.

$$p^* = \frac{3{,}4\,p^*}{2{,}7 + 0.001\,p^*}$$

$$p^*\left(1 - \frac{3{,}4}{2{,}7 + 0{,}001\,p^*}\right) = 0$$

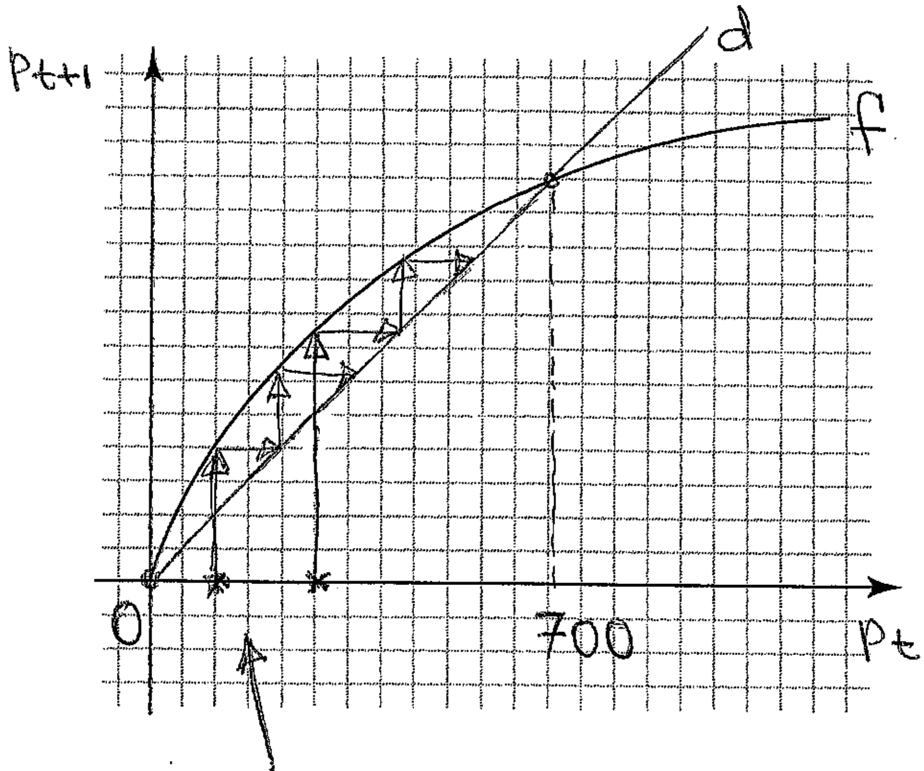
$p^* = 0$

$$3{,}4 = 2{,}7 + 0{,}001\,p^*$$

$$0{,}001\,p^* = 0.7$$

$$p^* = 700$$

(c)[2] The graph below shows the updating function of the given dynamical system. Label the coordinate axes and label all equilibrium points. Using cobwebbing, determine whether the smallest equilibrium point is stable or unstable.



start near $p^*=0$
solutions move away $\rightarrow$ $p^*=0$ is unstable

(d)[2] Explain what the stability or instability of the smallest equilibrium point means for the population of frogs.

even a small population of frogs will increase in numbers
(ie will not go extinct)