MATHEMATICS 1LS3 TEST 2

Day Class
Duration of Examination: 60 minutes
McMaster University, 29 October 2018

E. Clements, J. Hofscheier, M. Lovrić

First name (PLEASE PRINT): SOLUTIONS

Family name (PLEASE PRINT): ______________________

Student No.: ______________________

THIS TEST HAS 8 PAGES AND 6 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. USE A PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

Total number of points is 40. Marks are indicated next to the problem number. Calculator allowed: McMaster standard calculator Casio fx991MS or Casio fx991MS PLUS or lower Casio which has two lines of display and no graphing capabilities.

EXCEPT ON QUESTIONS 1 AND 2, you must show work to receive full credit.

Problem	Points	Mark
1	10	
2	6	
3	6 ~~4~~	
4	6 ~~7~~	
5	6 ~~7~~	
6	6	
TOTAL	40	

Continued on next page

1. **Multiple choice questions: circle ONE answer. No justification is needed.**

(a)[2] It is known that $f(4) = 0$, $f'(4) = 0$ and $f''(4) = 0$. Which statement(s) is/are true for all functions $f(x)$ which satisfy these two conditions?

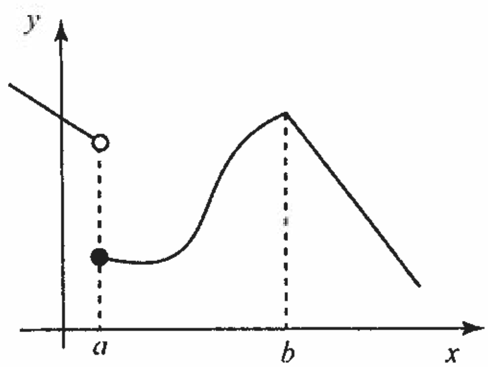
(I) $f(4) = 0$ is a local (relative) minimum of $f(x)$

(II) the tangent line to the graph of $f(x)$ at $x = 4$ is $y = 0$

(III) $f(4) = 0$ is a point of inflection of the graph of $f(x)$

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

(b)[2] Identify all correct statements for the function $f(x)$ whose graph is given below.



(I) $f(x)$ is differentiable at a

(II) $f(x)$ is continuous at b

(III) $f(x)$ is differentiable at b

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

Continued on next page

Name:________________

Student No.: ________________

(c)[2] The slope of the tangent to the curve given implicitly by $x^2y^4 = 1$ at the point $(1, 1)$ is

(A) 2	(B) -2	(C) 1	(D) -1
(E) $-1/4$	(F) $-1/2$	(G) $1/2$	(H) $1/4$

$$2xy^4 + x^2 \cdot 4y^3 \cdot y' = 0$$

$$2 + 4y' = 0 \longrightarrow y' = -1/2$$

(d)[2] If $f(x) = Ax\ln(B + x)$, then $f'(0)$ is equal to

(A) A	(B) B	(C) AB	(D) $B\ln B$
(E) $AB\ln B$	(F) $AB\ln A$	(G) $A\ln B$	(H) $B\ln A$

$$f'(x) = A\ln(B+x) + Ax\frac{1}{B+x}$$

$$f'(0) = A\ln B + 0$$

(e)[2] Identify all correct Taylor polynomials of the function $f(x) = \sin 2x$ at $x = 0$.

(I) $T_1(x) = x$ ✗

(II) $T_3(x) = 2x - \dfrac{x^3}{3}$ ✗

(III) $T_3(x) = 2x - \dfrac{4x^3}{3}$ ✓

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

$f = \sin 2x \ldots 0$

$f' = 2\cos 2x \ldots 2$

$f'' = -4\sin 2x \ldots 0$

$f''' = -8\cos 2x \ldots -8$

$$\cancel{f(0)} + f'(0)x + \frac{f''(0)}{\cancel{2}}\cancel{x^2} + \frac{f'''(0)}{6}x^3$$

$$= 2x - \frac{8}{6}x^3$$

Continued on next page

Name:

Student No.:

2. **True/false questions: circle ONE answer. No justification is needed.**

(a)[2] Knowing that $g''(x) = x \ln x$, we conclude that the function $g(x)$ is concave down on $(0, 1)$.

$\oplus$ $\ominus$

TRUE FALSE

So $g''(x) < 0$

(b)[2] The linear-quadratic model for the percent S of cancer cells surviving radiation treatment states that

$$S(d) = e^{-d^2 - 0.1d - 0.2}$$

where $d \geq 0$ is the dose (in Gray) per treatment of radiation.

$S(d)$ is an increasing function.

TRUE FALSE

$$S'(d) = \underbrace{e^{-d^2-0.1d-0.2}}_{\oplus} \underbrace{(-2d-0.1)}_{\ominus}$$

(c)[2] Let $m(t)$ represent the mass of melting snow in kilograms, where t is the time in days. The units of $m'(t)$ are kilograms.

TRUE FALSE

$$m' = \lim \frac{\Delta m}{\Delta t} \therefore \frac{kg}{day}$$

Continued on next page

Questions 3-6: You must show correct work to receive full credit.

3. (a)[3] Find $\lim_{x\to 0} \frac{\sin x - x}{x^3}$ $= \frac{0}{0}$

$$\overset{LH}{=} \lim_{x\to 0} \frac{\cos x - 1}{3x^2} = \frac{0}{0}$$

$$\overset{LH}{=} \lim_{x\to 0} \frac{-\sin x}{6x} = \frac{0}{0}$$

$$\overset{LH}{=} \lim_{x\to 0} \frac{-\cos x}{6} = -\frac{1}{6}$$

(b)[3] Find $\lim_{x\to 0^+} x^4 \ln x$ $= 0 \cdot (-\infty)$

$$= \lim_{x\to 0^+} \frac{\ln x}{x^{-4}} \overset{LH}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-4x^{-5}}$$

$$= \lim_{x\to 0^+} \left(-\frac{1}{4} \cdot \frac{1}{x} \cdot x^5\right) = \lim_{x\to 0^+} \left(-\frac{1}{4}x^4\right) = 0$$

Continued on next page

4. (a)[3] The resistance R of the flow of blood through a blood vessel (assumed to have the shape of a cylindrical tube) is given by

$$R = \frac{K^{0.96} L(\gamma+1)^2}{d^4} = K^{0.96} L(\gamma+1)^2 \cdot d^{-4}$$

where L is the length of the tube, d is its diameter and $\gamma \geq 0$ is the curvature. The positive constant K represents the viscosity of the blood.

Find the derivative of R with respect to d and interpret your answer, i.e., explain what your answer implies for the dependence of R on the diameter of a blood vessel.

$$R'(d) = \underbrace{K^{0.96}}_{\oplus} \cdot \underbrace{L}_{\oplus} \cdot \underbrace{(\gamma+1)^2}_{\oplus} \cdot (-4) \underbrace{d^{-5}}_{\oplus} < 0$$

the resistance decreases as the diameter increases (but the change becomes smaller and smaller)

(b)[3] In the article *Phenomenological Theory of World Population Growth* by S. Kapitza, Physics-Uspekhi (39)1, we find the formula

$$P(t) = 4.43\left(\frac{\pi}{2} + \arctan\frac{t}{42}\right)$$

where t is the time in years, with $t = 0$ representing 2007.

Find the linear approximation of $P(t)$ at $t = 0$. Round off all numbers to two decimal places.

$$P(0) = 4.43\left(\frac{\pi}{2} + 0\right) = 6.96$$

$$P'(t) = 4.43\left(0 + \frac{1}{1+(t/42)^2} \cdot \frac{1}{42}\right)$$

$$P'(0) = 4.43/42 = 0.11 \qquad (0.105476)$$

$$T_1 = P(0) + P'(0)t = 6.96 + 0.11t$$

or $6.96 + 0.10t$

Continued on next page

Name:________________
Student No.: ______________

5. (a)[3] In *Hybrid equation/agent-based model of ischemia-induced hyperemia and pressure ulcer formation* by Alexey Solovyev et al., PLoS Computational Biology 9.5 (May 2013), the authors analyze the function

$$I(t) = I_{rest}\left(1 + ae^{-2t} + be^{-3t}\right)$$

where I_{rest} and a are positive constants, and the parameter b is negative. Find all critical numbers (t values only) of $I(t)$.

$$I'(t) = I_{rest}\left(-2ae^{-2t} - 3be^{-3t}\right)$$

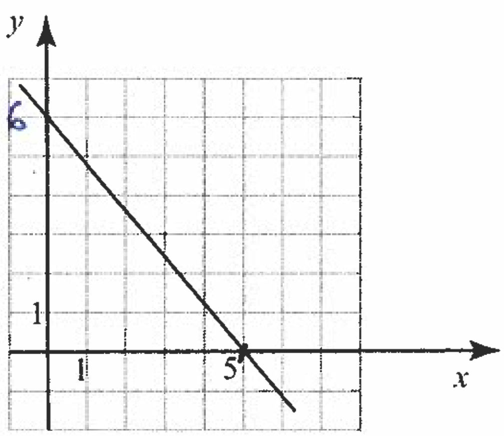
$I'(t)$ dne $\rightarrow$ no t

$I'(t) = 0 \rightarrow -2ae^{-2t} - 3be^{-3t} = 0 \;\Big|\cdot e^{3t}$

$-2ae^{t} - 3b = 0$

$e^{t} = -\frac{3b}{2a} \rightarrow t = \ln\Big(\underbrace{\frac{-3b}{2a}}_{\oplus ve}\Big)$

(b)[3] Let $h(x) = x\sin(f(x))$. The graph of $f(x)$ is a line shown below. Find $h'(5)$.



$h' = \sin(f(x))$
$\quad + x\cdot\cos(f(x))\cdot f'(x)$

$h'(5) = \sin(f(5))$
$\quad + 5\cdot\cos(f(5))\cdot f'(5)$

$h'(5) = 0 + 5\cdot 1\cdot\left(-\frac{6}{5}\right) = \underline{\underline{-6}}$

or: $f(x) = -\frac{6}{5}x + 6$ and $h(x) = x\cdot\sin\left(-\frac{6}{5}x+6\right)$

compute $h'(x)$ then $h'(5)$

Continued on next page

6. (a)[2] Show that $f(x) = (x^2 - 1)e^{-x^2}$ has three critical points 0, $-\sqrt{2}$, and $\sqrt{2}$.

$$f' = 2xe^{-x^2} + (x^2-1)e^{-x^2}(-2x)$$
$$= 2xe^{-x^2}(1-(x^2-1)) = 2x(2-x^2)e^{-x^2}$$

f' dne ... no x

$f' = 0 \rightarrow \underline{x=0}$ or $2-x^2=0$, ie $x^2=2$, $\underline{x=\pm\sqrt{2}}$

(b)[2] State the Extreme Value Theorem. Make sure to clearly identify assumption(s) and conclusion(s).

assume: $f(x)$ is continuous on a closed interval $[a,b]$

conclude: $f(x)$ has an abs. max and abs. min. in $[a,b]$

(c)[2] Find the absolute maximum and the absolute minimum of the function $f(x)$ from (a) on the interval $[0, 2]$.

x	$f=(x^2-1)e^{-x^2}$	
0	-1	$\rightarrow$ abs. min.
2	$3e^{-4} \approx 0.0549$	
$\sqrt{2}$	$e^{-2} \approx 0.135$	$\rightarrow$ abs. max.

THE END