MATHEMATICS 1LS3 TEST 1

Day Class
Duration of Test: 60 minutes
McMaster University

E. Clements

4 February 2020

FIRST NAME (please print): Sol^Ns

FAMILY NAME (please print): ______

Student No.: ______

THIS TEST HAS 8 PAGES AND 7 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number in square brackets. You may use the McMaster standard calculator, Casio fx991 MS+, on this test.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL, YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You must show work to receive full credit.

Problem	Points	Mark
1	6	
2	8	
3	7	
4	5	
5	4	
6	4	
7	6	
TOTAL	40	

1. **True/false questions: circle ONE answer. No justification is needed.**

(a) [2] The graph of $f(x) = -x^2 + 8x - 17$ is obtained by reflecting the graph of $y = x^2$ in the x-axis, then shifting it right 4 units and down 1 unit.

TRUE FALSE

$x^2 \longrightarrow -x^2 \longrightarrow -(x-4)^2 - 1$

$-(x-4)^2 - 1 = -(x^2 - 8x + 16) - 1$
$= -x^2 + 8x - 17$

(b) [2] The semilog graph of $d(t) = 80e^{-0.71t}$ is linear.

TRUE FALSE

$\ln d = \ln 80e^{-0.71t}$
$= \ln 80 + \ln e^{-0.71t}$
$= \ln 80 - 0.71t$

(c) [2] The average rate of change of $f(x) = \arctan x$ from $x = -1$ to $x = 1$ is $\pi/4$.

TRUE FALSE

$\dfrac{f(1) - f(-1)}{1 - (-1)}$

$= \dfrac{\arctan 1 - \arctan(-1)}{2}$

$= \dfrac{\frac{\pi}{4} - (-\frac{\pi}{4})}{2}$

$= \dfrac{\pi}{2} \cdot \dfrac{1}{2}$

$= \dfrac{\pi}{4}$

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2. **Multiple choice questions: circle ONE answer. No justification is needed.**

(a) [2] Suppose that $f(x) = 1 - x^2$ and $g(x) = \ln x$. Then the domain of $(g \circ f)(x)$ is

$(g \circ f)(x) = \ln(1-x^2)$

domain $1 - x^2 > 0 \Rightarrow x^2 < 1 \Rightarrow -1 < x < 1$

(A) $x < -1$ (B) $x > -1$ (C) $x > 0$ (D) $x < 0$

(E) $-1 < x < 1$ (F) $0 < x < 1$ (G) $x > 1$ (H) $x < 1$

(b) [2] Turbidity T is a measure of cloudiness or haziness in water and is used to assess the quality of drinking water. It is known that turbidity is proportional to the natural logarithm of the number of phytoplankton N, proportional to the amount of sediment S, and inversely proportional to the square of the depth d. Which formula represents the turbidity? (k is a constant)

$$\left.\begin{array}{l} T \propto \ln N \\ T \propto S \\ T \propto \frac{1}{d^2} \end{array}\right\} \Rightarrow T = k \cdot \frac{S \ln N}{d^2}$$

(A) $T = k\frac{Sd^2}{N}$ (B) $T = k\frac{S\ln N}{d^2}$ (C) $T = k\frac{\ln N}{Sd^2}$ (D) $T = k\frac{\ln N}{d^2}$

(E) $T = k\frac{Sd}{N}$ (F) $T = k\frac{Sd^2}{\ln N}$ (G) $T = k\frac{Sd}{\ln N}$ (H) $T = k\frac{S\ln N}{d}$

(c) [2] Determine which of the following is/are true for the function $f(x) = \dfrac{2x^2 - 5x - 3}{x^2 - 3x}$.

(I) The graph of $f(x)$ has a vertical asymptote at $x = 0$ ✓
(II) The graph of $f(x)$ has a vertical asymptote at $x = 3$ ✗
(III) $y = 2$ is the horizontal asymptote for the graph of $f(x)$ ✓

$$f(x) = \frac{(2x+1)\cancel{(x-3)}}{x\cancel{(x-3)}} = \frac{2x+1}{x} = 2 + \frac{1}{x}$$

$$(\mathrm{I})\ \lim_{x\to 0} f(x) = 2 + \frac{1}{0} = \pm\infty$$

$$(\mathrm{II})\ \lim_{x\to 3} f(x) = 2 + \frac{1}{3} = \frac{7}{3}$$

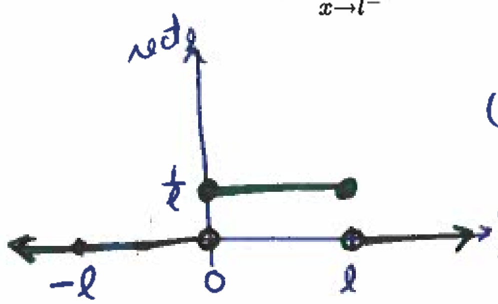
$$(\mathrm{III})\ \lim_{x\to \pm\infty} f(x) = 2 + \frac{1}{\pm\infty}^{\nearrow 0} = 2$$

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

(d) [2] Consider the function $rect_l(x) = \begin{cases} 1/l & \text{if} \quad 0 \le x \le l \\ 0 & otherwise \end{cases}$, where $l > 0$.

Determine which of the following is/are true.

(I) $\lim_{x\to l^-} rect_l = 1/l$ ✓ (II) $\lim_{x\to 0^+} rect_l = 0$ ✗ (III) $\lim_{x\to -l} rect_l = 0$ ✓



$$(\mathrm{I})\ \lim_{x\to l^-} rect_l = \frac{1}{l} \qquad (\mathrm{III})\ \lim_{x\to -l} rect_l = 0$$

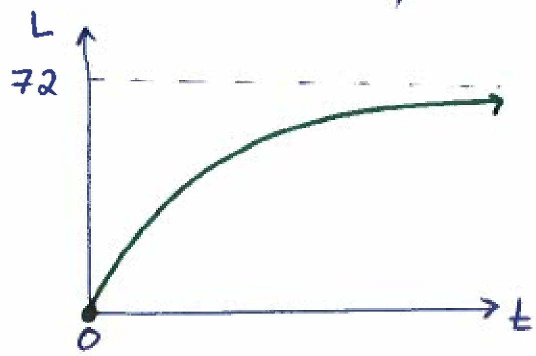
$$(\mathrm{II})\ \lim_{x\to 0^+} rect_l = \frac{1}{l}$$

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

3. The length of a certain species of fish can be modelled by $L(t) = 72(1 - e^{-t})$, where L is the length, in centimetres, and $t \geq 0$ is the time, in years.

(a) [3] Sketch the graph of $L(t)$, for $t \geq 0$. Make sure to clearly label any intercepts and/or asymptotes.

$e^{t} \rightarrow e^{-t} \rightarrow -e^{-t} \rightarrow -e^{-t}+1$



(b) [1] State (in one sentence) what question is answered by finding the inverse function.

The inverse will determine the age of a fish based on its length (ie time as a function of length).

(c) [3] Find the inverse function of $L(t)$.

$$L = 72(1 - e^{-t})$$

$$\frac{L}{72} = 1 - e^{-t}$$

$$e^{-t} = 1 - \frac{L}{72}$$

$$-t = \ln\left(1 - \frac{L}{72}\right)$$

$$t = -\ln\left(1 - \frac{L}{72}\right)$$

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4. Recall that body mass index is defined as $BMI = m/h^2$, where mass m is in kilograms and height h is in metres. In the following exercises, let $m = 70$ kg.

(a) [2] If the height of an individual increases by 5%, how will their body mass index change? Round your answer to two decimal places.

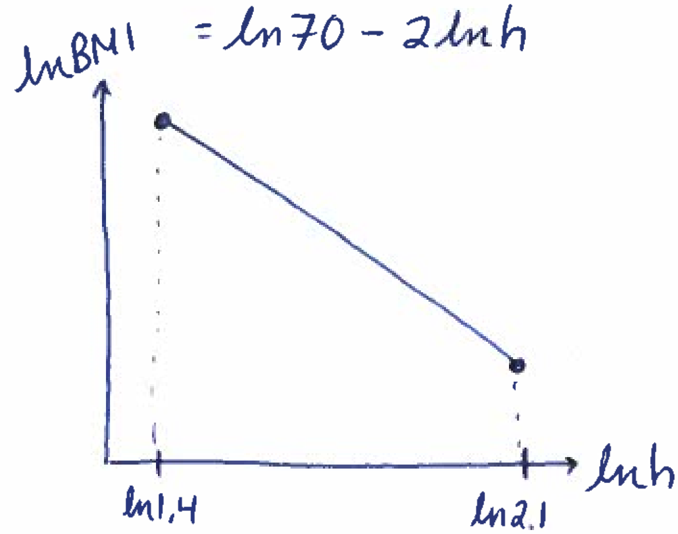
$h_{new} = 1.05 h_{old}$

$$BMI_{new} = \frac{70}{h_{new}^2} = \frac{70}{(1.05 h_{old})^2} = \frac{70}{1.05^2 \cdot h_{old}^2} \approx 0.91\, BMI_{old}$$

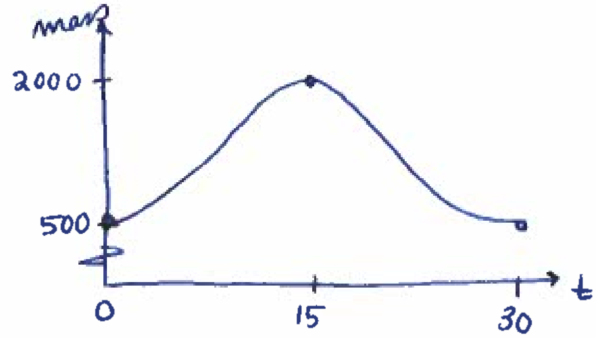
∴ If height increases by 5%, and mass remains the same, BMI will decrease by about 9%

(b) [3] Using "ln", sketch the double-log (or log-log) plot of $BMI = 70/h^2$ for $1.4 \le h \le 2.1$. Label the axes and any intercepts.

$$\ln BMI = \ln 70/h^2 = \ln 70 - \ln h^2 = \ln 70 - 2\ln h$$



5. [4] Certain aquatic plant varieties are known to exhibit periodic growth cycles between 25 and 35 days. ~~Suppose that a certain species has a growth~~ assume that the cycle of 30 days. Initially ($t = 0$), the species reaches its minimum of 500 g and 15 days later ($t = 15$), it reaches its maximum of 2000 g. Find a formula that describes how this species changes over time.



period $= 30 \Rightarrow \frac{2\pi}{B} = 30 \Rightarrow B = \frac{\pi}{15}$

avg. value $= \frac{500 + 2000}{2} = 1250$

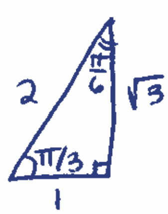
amplitude $= 2000 - 1250 = 750$

Let $P(t)$ = mass of species at time t, in days.

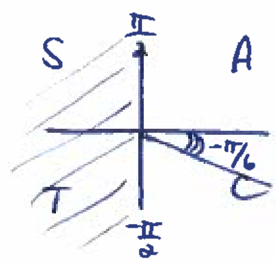
$P(t) = -750 \cos \frac{\pi}{15} + 1250$

6. (a) [2] Determine the exact value of arcsin(-0.5).

$\arcsin(-0.5) = \theta \Leftrightarrow \sin\theta = -\frac{1}{2}$, for $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$



$\sin \frac{\pi}{6} = \frac{1}{2}$



$\sin(-\frac{\pi}{6}) = -\frac{1}{2}$

$\therefore \arcsin(-0.5) = -\frac{\pi}{6}$

(b) [2] Find the range of $f(x) = \pi + 4\arctan(7 - x)$.

$-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{2} < \arctan(7-x) < \frac{\pi}{2}$

$\Rightarrow -2\pi < 4\arctan(7-x) < 2\pi$

$\Rightarrow -\pi < \pi + 4\arctan(7-x) < 3\pi$

$\therefore$ The range is $(-\pi, 3\pi)$.

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7. Evaluate each limit, if it exists.

(a) [2] $\lim_{x\to-1} \frac{1-|x|}{x+1}$ $= \lim_{x\to-1} \frac{1-(-x)}{x+1}$

$x<0$

$= \lim_{x\to-1} (1)$

$= 1$

(b) [2] $\lim_{N_i\to\infty} \frac{N_eV_e + N_iV_i}{N_i + N_e}$

$= \lim_{N_i\to\infty} \dfrac{\frac{N_eV_e}{N_i} + V_i}{1 + \frac{N_e}{N_i}}$

$= \dfrac{\cancel{\frac{N_eV_e}{\infty}} \to 0 + V_i}{1 + \cancel{\frac{N_e}{\infty}} \to 0}$

$= V_i$

(c) [2] $\lim_{x\to\infty} (\ln(2x+3) - \ln(0.1x-5))$

$= \lim_{x\to\infty} \ln\left(\frac{2x+3}{0.1x-5}\right)$

$= \lim_{x\to\infty} \ln\left(\frac{2+\frac{3}{x}}{0.1-\frac{5}{x}}\right)$

$= \ln\left(\frac{2+\cancel{\frac{3}{\infty}} \to 0}{0.1-\cancel{\frac{5}{\infty}} \to 0}\right)$

$= \ln 20$

THE END