MATHEMATICS 1LS3 TEST 2

Day Class
Duration of Test: 60 minutes
McMaster University

E. Clements

3 March 2020

FIRST NAME (please print): Sol^{N}s

FAMILY NAME (please print): ____________

Student No.: ____________

THIS TEST HAS 8 PAGES AND 8 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number in square brackets. You may use the McMaster standard calculator, Casio fx991 MS+, on this test.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL, YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You must show work to receive full credit.

Problem	Points	Mark
1	6	
2	10	
3	3	
4	3	
5	3	
6	3	
7	6	
8	6	
TOTAL	40	

MATH 1LS3 * Test 2 * 3 March 2020

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1. **True/false questions: circle ONE answer. No justification is needed.**

(a) [2] The formula $\underbrace{\lim_{h\to 0} \frac{\arcsin(0.5+h)-\arcsin(0.5)}{h}}_{= f'(0.5) \text{ for } f(x)=\arcsin x} = \frac{2}{\sqrt{3}}$ is correct.

TRUE FALSE

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$f'(0.5) = \frac{1}{\sqrt{1-(0.5)^2}} = \frac{1}{\sqrt{\frac{3}{4}}} = \frac{2}{\sqrt{3}}$$

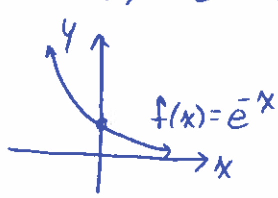
(b) [2] If a function $f(x)$ is positive, i.e., $f(x) > 0$ for all x, then its derivative $f'(x)$ is positive.

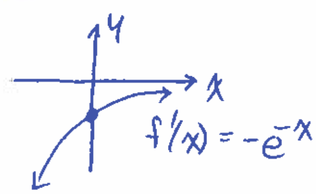
Not necessarily!

Consider $f(x) = e^{-x}$

$f(x) > 0 \ \forall x$ but $f'(x) = -e^{-x} < 0 \ \forall x$

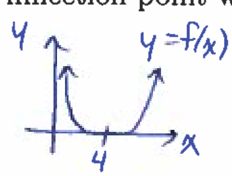
TRUE FALSE





(c) [2] If $f''(4) = 0$, then the graph of $f(x)$ has an inflection point when $x = 4$.

Not necessarily!



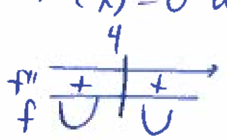
TRUE FALSE

Consider $f(x) = (x-4)^4$.

$f'(x) = 4(x-4)^3$

$f''(x) = 12(x-4)^2$

$f''(x) = 0$ when $x = 4$, but f'' does not change sign at 4



$\therefore$ f does not have an IP at $x = 4$

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2. **Multiple choice questions: circle ONE answer. No justification is needed.**

(a) [2] Which of the following is/are true for the function $f(x) = \begin{cases} x^2 - 1 & \text{if } x \le 2 \\ 2.5 + 1/x & \text{if } x > 2 \end{cases}$?

(I) f is continuous at $x = 0$ ✓ (II) f is continuous at $x = 2$ ✓ (III) $\lim_{x\to 2.5} f(x) = f(2.5)$ ✓

(I) $\lim_{x\to 0} f(x) = \lim_{x\to 0}(x^2-1) = 0^2 - 1 = -1 = f(0)$

(II) $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(x^2-1) = 2^2-1 = 3$
$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}\left(2.5+\frac{1}{x}\right) = 2.5 + \frac{1}{2} = 3$
$\Big\}\ \lim_{x\to 2} f(x) = f(2)$

(III) true, since f is continuous at 2.5

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

(b) [2] If $f(x) = \dfrac{x}{x+3}$, then $f'(1) = \lim_{h\to 0} \dfrac{\frac{1+h}{(1+h)+3} - \frac{1}{1+3}}{h}$

$= \lim_{h\to 0} \dfrac{\frac{h+1}{h+4} - \frac{1}{4}}{h}$

(A) $\lim_{h\to 0} \dfrac{\frac{1}{4}+h}{h}$ (B) $\lim_{h\to 0} \dfrac{\frac{h}{h+3}-\frac{1}{4}}{h}$ (C) $\lim_{h\to 0} \dfrac{\frac{h+1}{h+4}-\frac{1}{4}}{h}$ (D) $\lim_{h\to 0} \dfrac{\frac{h+1}{h+3}-\frac{1}{2}}{h}$

(E) $\lim_{h\to 0} \dfrac{\frac{h+1}{h+4}-1}{h}$ (F) $\lim_{h\to 0} \dfrac{\frac{h}{h+4}-\frac{1}{2}}{h}$ (G) $\lim_{h\to 0} \dfrac{\frac{h}{h+3}-1}{h}$ (H) none of these

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(c) [2] The resistance R of the blood flow through a vessel is given by $R = Kl(\gamma+1)^2 D^{-4}$, where l is the length of the vessel, D is its diameter, $\gamma \geq 0$ is the curvature, and $K > 0$ represents the viscosity of the blood. In this exercise, we treat R as a function of D, where $D > 0$. Which of the following is/are true?

(I) R is decreasing ✓ (II) dR/dD is increasing ✓ (III) R has no critical numbers ✓

(I) $\frac{dR}{dD} = \underbrace{-4}_{\ominus}\underbrace{K\ell(\gamma+1)^2 D^{-5}}_{\oplus} < 0 \Rightarrow R$ is decreasing

(II) $\frac{d^2R}{dD^2} = 20K\ell(\gamma+1)^2 \cdot D^{-6} > 0 \Rightarrow \frac{dR}{dD}$ is increasing

(III) $\frac{dR}{dD} = 0$ has no sol^{ns} and $\frac{dR}{dD}$ DNE for no $D>0$

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

(d) [2] The instantaneous rate of change of $f(x) = e^{\tan(\pi x)}$ at $x = 0$ is

$f'(x) = e^{\tan(\pi x)} \cdot \sec^2(\pi x) \cdot \pi$

$f'(0) = \underbrace{e^{\tan 0}}_{=1} \cdot \underbrace{\sec^2(0)}_{=1} \cdot \pi = \pi$

(A) 0 (B) 1 (C) -1 (D) π
(E) $\pi/4$ (F) $-\pi$ (G) $-\pi/4$ (H) none of these

~~(e)~~ [2] Let $g(x) = x^2\sqrt{f(x)}$, where f is a differentiable function such that $f(1) = 4$ and $f'(1) = 1$. Find $g'(1)$.

CUT

$g'(x) = 2x \cdot \sqrt{f(x)} + x^2 \cdot \frac{1}{2\sqrt{f(x)}} \cdot f'(x)$

$g'(1) = \underbrace{2(1)}_{=2} \cdot \underbrace{\sqrt{f(1)}}_{=2} + 1^2 \cdot \frac{1}{\underbrace{2\sqrt{\underbrace{f(1)}_{=4}}}_{=4}} \cdot \underbrace{f'(1)}_{=1} = 4 + \frac{1}{4} = \frac{17}{4}$

(A) 0 (B) 1/4 (C) 3/2 (D) 17/4
(E) 9/4 (F) 4 (G) 1/2 (H) 5/2

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3. [3] Find the critical numbers of $g(x) = x^{4/5}(2x-1)^2$.

domain: $x \in \mathbb{R}$

$$g' = \frac{4}{5}x^{-1/5}(2x-1)^2 + x^{4/5} \cdot 2(2x-1)^1 \cdot (2)$$
$$= (2x-1)\left[\frac{4(2x-1)}{5x^{1/5}} + 4x^{4/5} \cdot \frac{5x^{1/5}}{5x^{1/5}}\right]$$
$$= \frac{(2x-1)(28x-4)}{5x^{1/5}}$$

$g' = 0$ when $(2x-1)(28x-4) = 0 \Rightarrow x = \frac{1}{2}$ or $x = \frac{1}{7}$

g' DNE when $5x^{1/5} = 0 \Rightarrow x = 0$

$\therefore$ The critical #s are $0, \frac{1}{7}$, and $\frac{1}{2}$.

4. [3] Find an equation of the tangent line to the curve $y = x^2\sin(1/x)$ at $x = 1/\pi$.

$$y\left(\tfrac{1}{\pi}\right) = \left(\tfrac{1}{\pi}\right)^2 \sin\left(\frac{1}{\frac{1}{\pi}}\right) = 0$$

$$y' = 2x \cdot \sin\left(\tfrac{1}{x}\right) + \cancel{x^2} \cdot \cos\left(\tfrac{1}{x}\right) \cdot \left(-\frac{1}{\cancel{x^2}}\right)$$
$$= 2x \cdot \sin\left(\tfrac{1}{x}\right) - \cos\left(\tfrac{1}{x}\right)$$

$$y'\left(\tfrac{1}{\pi}\right) = 2\left(\tfrac{1}{\pi}\right) \cdot \sin\left(\frac{1}{\frac{1}{\pi}}\right) - \cos\left(\frac{1}{\frac{1}{\pi}}\right) = 1$$

$\therefore$ The equation of the tangent is

$$y - 0 = 1\left(x - \tfrac{1}{\pi}\right)$$
$$\Rightarrow y = x - \frac{1}{\pi}$$

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5. [3] Consider the curve defined implicitly by $\ln(x^2 + y^2) - x^3 - y^4 = 0$. Compute y' when $x = 1$ and $y = 1$.

$$\frac{d}{dx}\left(\ln(x^2+y^2) - x^3 - y^4\right) = \frac{d}{dx}(0)$$

$$\frac{1}{x^2+y^2}(2x + 2y \cdot y') - 3x^2 - 4y^3 \cdot y' = 0 \quad \Big|_{\substack{x=1 \\ y=1}}$$

$$\frac{1}{2}(2 + 2y') - 3 - 4y' = 0$$

$$y' = -\frac{2}{3}$$

6. [3] Determine the interval(s) on which the graph of $f(x) = e^{-0.1x^2}$ is concave down.

$\leftarrow x \in \mathbb{R}$

$$f' = e^{-0.1x^2}(-0.2x)$$

$$f'' = e^{-0.1x^2}(-0.2x)\cdot(-0.2x) + e^{-0.1x^2}(-0.2)$$

$$= e^{-0.1x^2}(0.04x^2 - 0.2)$$

$f'' = 0$ when $0.04x^2 - 0.2 = 0$

$$x^2 = 5$$

$$x = \pm\sqrt{5}$$

	$-\sqrt{5}$		$\sqrt{5}$	
f''	+	$-$	+	
f	$\cup$	$\cap$	$\cup$	

$\therefore$ f is concave down on $(-\sqrt{5}, \sqrt{5})$.

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7. (a) [3] In the article *Phenomenological Theory of World Population Growth* by S. Kapitza, Physics-Uspekhi (39)1, we find the formula

$$P(t) = 4.43\left(\frac{\pi}{2} + \arctan\frac{t}{42}\right)$$

where t is the time in years, with $t = 0$ representing 2007. Find the linear approximation of $P(t)$ at $t = 0$. Round off all numbers to two decimal places.

$$L(t) = P(0) + P'(0)(t-0)$$

$$P(0) = 4.43\left(\frac{\pi}{2} + \arctan 0\right) = \frac{4.43\pi}{2} \approx 6.96$$

$$P' = 4.43 \cdot \frac{1}{1+\left(\frac{t}{42}\right)^2} \cdot \frac{1}{42} \quad \ldots \quad P'(0) = \frac{4.43}{42} \approx 0.11$$

$$\therefore \; L(t) = 6.96 + 0.11t$$

(b) [3] Estimate the value of $\sqrt{15}$ using a quadratic approximation of $f(x) = \sqrt{x}$ at a suitable base point. Round your answer to four decimal places.

use $a = 16$

$$T_2(x) = f(16) + f'(16)(x-16) + \frac{f''(16)}{2}(x-16)^2$$

$$f(16) = \sqrt{16} = 4$$

$$f' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \quad \ldots \quad f'(16) = \frac{1}{2\sqrt{16}} = \frac{1}{8}$$

$$f'' = -\frac{1}{4}x^{-3/2} = \frac{-1}{4\sqrt{x}^3} \quad \ldots \quad f''(16) = \frac{-1}{4\sqrt{16}^3} = -\frac{1}{256}$$

$$\therefore \; T_2(x) = 4 + \frac{1}{8}(x-16) - \frac{1}{512}(x-16)^2$$

$$\sqrt{15} \approx 4 + \frac{1}{8}(15-16) - \frac{1}{512}(15-16)^2$$

$$\approx 3.8730$$

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8. The function $c(t) = \dfrac{10t}{0.16 + t^2}$ has been used to model the absorption of a drug (such as morphine); where $c(t)$ is the concentration (in milligrams per millilitre, mg/mL) of the drug in the bloodstream, and $t \geq 0$ is the time (in hours).

(a) [2] Find the critical number(s) of $c(t)$.

$$c' = \frac{10(0.16+t^2) - 10t(2t)}{(0.16+t^2)^2} = \frac{10(0.16-t^2)}{(0.16+t^2)^2}$$

$c' = 0$ when $0.16 - t^2 = 0 \Rightarrow t = \pm\sqrt{0.16} \Rightarrow t = \pm 0.4$

c' DNE ... no t

when $t \geq 0$, $c(t)$ has one critical #: $t = 0.4$

(b) [2] State the assumption(s) and conclusion(s) of the Extreme Value Theorem.

If $c(t)$ is continuous on a closed, finite interval $[a,b]$, then $c(t)$ has an absolute max and an absolute min. on $[a,b]$.

(c) [2] Find the absolute maximum and the absolute minimum values that the concentration $c(t)$ reaches during the first hour after the drug is administered, i.e., over the interval [0, 1].

t	$c(t)$
0	0
0.4	$\frac{10(0.4)}{0.16+(0.4)^2} = 12.5$
1	$\frac{10}{0.16+1^2} = 8.62$

THE END