MATHEMATICS 1LS3 TEST 3

Day Class
Duration of Test: 24 hours
McMaster University

E. Clements

31 March 2020

FIRST NAME (please print): Sol^{N}s

FAMILY NAME (please print): ______

Student No.: ______

THIS TEST HAS 8 PAGES AND 7 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number in square brackets. You may use the McMaster standard calculator, Casio fx991 MS+, on this test.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL, YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You must show work to receive full credit.

Problem	Points	Mark
1	6	
2	10	
3	3	
4	4	
5	6	
6	4	
7	7	
TOTAL	40	

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1. State whether each statement is **true or false**. **Explain** your reasoning.

(a) [2] The following use of L'Hopital's rule is correct: $\lim_{x\to 0} \frac{\sin x}{x^2} = \lim_{x\to 0} \frac{\cos x}{2x} = \lim_{x\to 0} \frac{-\sin x}{2} = 0$

$\lim_{x\to 0} \frac{\sin x}{x^2} = \frac{0}{0}$ (L'Hopital's Rule applies)

$\overset{LH}{=} \lim_{x\to 0} \frac{\cos x}{2x} = \frac{1}{0}$ (L'Hopital's Rule does NOT apply)

$= \pm\infty$

$\therefore$ FALSE

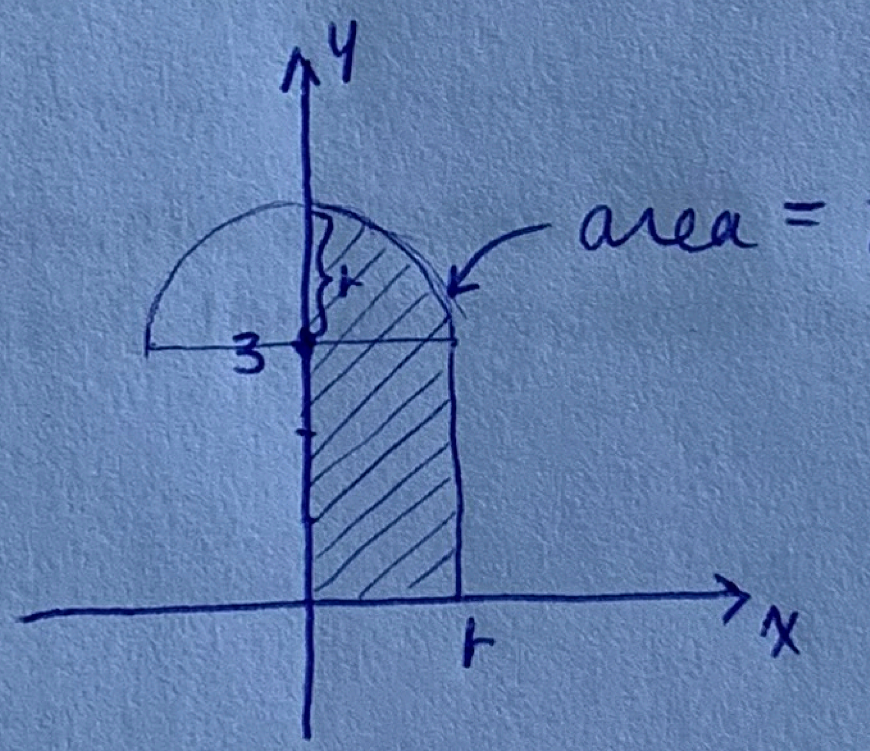
(b) [2] $\int_0^r \left(\sqrt{r^2 - x^2} + 3\right) dx = 6r + \frac{\pi r^2}{4}$, where $r > 0$.

$y = \sqrt{r^2 - x^2} + 3$

$y - 3 = \sqrt{r^2 - x^2}$

$x^2 + (y-3)^2 = r^2$

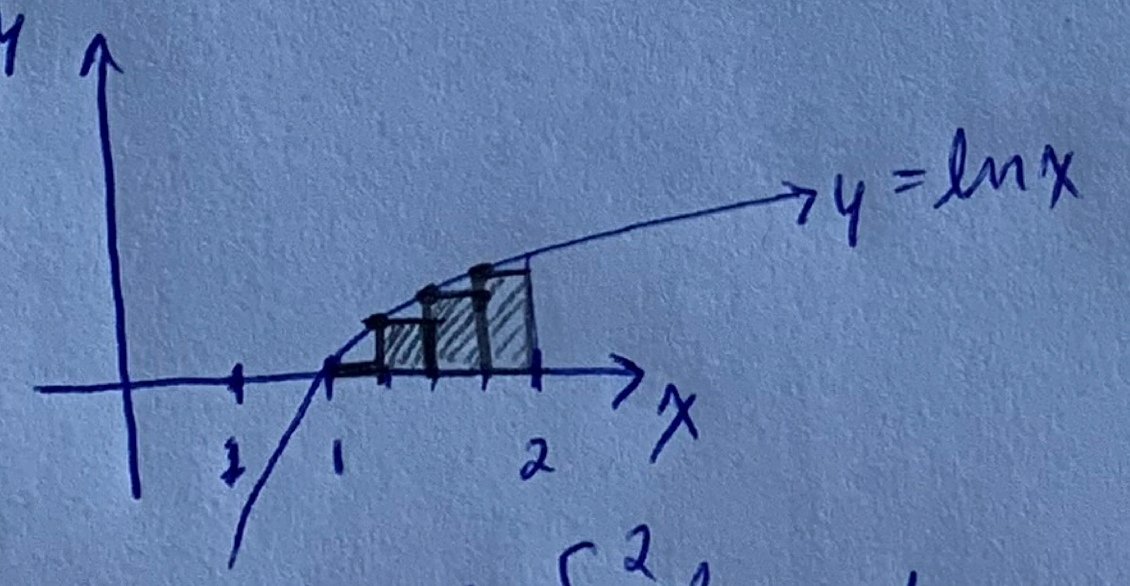
circle w/ centre $(0,3)$, radius r



area $= \frac{1}{4}\pi r^2 + 3r$

$\therefore$ FALSE

(c) [2] $\int_1^2 \ln x \, dx < L_4$, where L_4 is the left Riemann sum with four rectangles used to estimate the area under $y = \ln x$ on $[1, 2]$.



$L_4 < \int_1^2 \ln x \, dx$

$y = \ln x$ is increasing on $(1,2)$ (and on $x > 0$ for that matter)

$\therefore$ any estimate using a left Riemann sum will produce an underestimate.

2. **Multiple choice questions: circle ONE answer. No justification is needed.**

(a) [2] Which of the following differential equations are autonomous?

(I) $dy/dx = y^2 - x$ X "mixed"   (II) $dy/dx = x^2 - x$ X pure-time   (III) $dy/dx = y^2 - y$ ✓

$\frac{dy}{dx} = f(y)$

(A) none   (B) I only   (C) II only   (D) III only
(E) I and II   (F) I and III   (G) II and III   (H) all three

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(b) [2] Consider the differential equation $dP/dt = t + P$, where $P(0) = 120$. Using Euler's Method with step size $h = 5$, the approximate value of $P(10)$ is

$t_0 = 0$
$P_0 = 120$

$t_1 = t_0 + h = 0 + 5 = 5$
$P_1 = P_0 + (t_0 + P_0)h = 120 + (0+120)(5) = 720$

$t_2 = t_1 + h = 5 + 5 = 10$
$P_2 = P_1 + (t_1 + P_1)h = 720 + (5+720)5 = 4345$

$\therefore P(10) \simeq 4345$

(A) 1230   (B) 167   (C) 2019   (D) 320
(E) 482   (F) 2386   (G) 4345   (H) none of these

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(c) [2] Which of the following formulas is/are correct?

(I) $\int \sec^2 x\,dx = \tan x + C$ ✓ (II) $\int \frac{1}{x}\,dx = \ln|x| + C$ ✓ (III) $\int \frac{1}{1+x^2}\,dx = \arctan x + C$ ✓

$(\tan x)' = \sec^2 x$ $\quad (\ln|x|)' = \frac{1}{x}$ $\quad (\arctan x)' = \frac{1}{1+x^2}$

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

(d) [2] Suppose that $f(x)$ is a continuous function with antiderivative $F(x)$. If $F(4) = 12$ and $F(0) = -1$, then $\int_0^4 (2f(x) - 5)\,dx =$

$$\int_0^4 (2f(x)-5)\,dx = 2\int_0^4 f(x)\,dx - \int_0^4 5\,dx$$
$$= 2[F(4)-F(0)] - 5(4-0)$$
$$= 2[12-(-1)] - 20$$
$$= 6$$

(A) 6 (B) 5 (C) -1 (D) 3
(E) -7 (F) 19 (G) 0 (H) none of these

(e) [2] Using $T_2(x) = 1 - x^2$ as an approximation of $f(x) = e^{-x^2}$ near 0, we estimate that $\int_0^1 e^{-x^2}\,dx \approx$

$$\int_0^1 e^{-x^2}dx \simeq \int_0^1 (1-x^2)\,dx$$
$$\simeq \left[x - \frac{x^3}{3}\right]_0^1$$
$$\simeq 1 - \frac{1}{3}$$
$$\simeq 0.667$$

(A) -1 (B) 0.333 (C) 1 (D) 0.667
(E) 0.664 (F) 0.822 (G) 0.749 (H) 0.743

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3. [3] Evaluate $\lim_{x\to 0} \frac{\sin^2 x}{3x^2}$.

$$\lim_{x\to 0} \frac{\sin^2 x}{3x^2} \left(= \frac{0}{0}\right)$$

$$\overset{LH}{=} \lim_{x\to 0} \frac{2\sin x \cdot \cos x}{6x} \left(= \frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{\sin 2x}{6x}$$

$$\overset{LH}{=} \lim_{x\to 0} \frac{\cos 2x \cdot 2}{6}$$

$$= \frac{1}{3}$$

4. A pie, initially at the temperature of 20°C, is put into an 300°C oven. Let $T(t)$ be the temperature of the pie at time t. The temperature of the pie changes proportionally to the difference between the temperature of the oven and the temperature of the pie. Assume that the proportionality constant is 0.02.

(a) [2] Describe this event as an initial value problem (i.e., write down a differential equation and an initial condition).

Newton's Law of Cooling

$T(0) = 20$

$$\frac{dT}{dt} \propto (300 - T(t)) \Rightarrow \frac{dT}{dt} = 0.02(300 - T)$$

(autonomous)

(b) [2] Show that $T(t) = 300 - 280e^{-0.02t}$ (*) is the solution to your initial value problem in part (a).

$$\frac{dT}{dt} = -280e^{-0.02t}(-0.02) = (0.02)280e^{-0.02t}$$

$T(0) = 300 - 280e^0 = 20$ ✓ (*) satisfies the initial condition

DE: $LS = \frac{dT}{dt} = (0.02)280e^{-0.02t}$

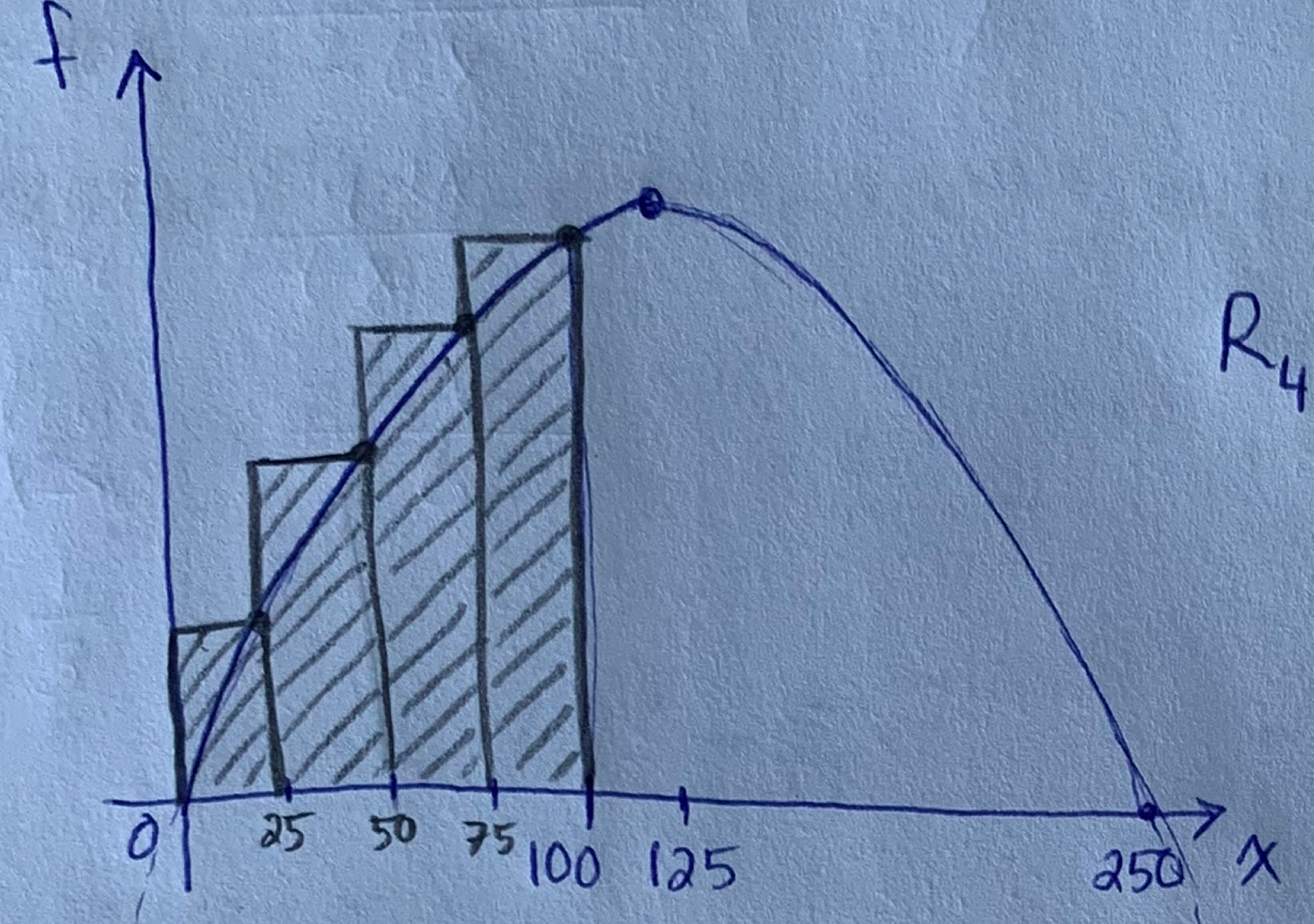
$RS = 0.02(300 - T) = 0.02(\cancel{300} - (\cancel{300} - 280e^{-0.02t})) = 0.02(280e^{-0.02t})$

$LS = RS \Rightarrow$ (*) satisfies the DE

$\therefore$ (*) is a solN to the IVP in part (a).

5. The density of monkeys in Kruger National Park in South Africa is given by the function $f(x) = 0.004x(250 - x)$ monkeys per kilometre, where x is the distance in km from the main entrance into the park.

(a) [3] Approximate the area of the region below the graph of $f(x) = 0.004x(250 - x)$ and over the interval $[0, 100]$, using R_4 (i.e., right sum with four rectangles). Sketch the function and the four rectangles involved.



$$\Delta x = \frac{100 - 0}{4} = 25$$

$$R_4 = (f(100) + f(75) + f(50) + f(25))\Delta x$$
$$= 4375$$

(b) [3] Evaluate $\int_0^{100} 0.004x(250 - x)\, dx$ algebraically. What does this number represent?

$$\int_0^{100} 0.004(250x - x^2)\,dx = 0.004\left(250\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^{100}$$

$$= 0.004\left(250\frac{(100)^2}{2} - \frac{(100)^3}{3}\right)$$

$$\simeq 3667 \text{ monkeys}$$

This number represents the # of monkeys located within 100 m of the front entrance.

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6. Most human papillomavirus (HPV) infections in young women are temporary and have very little long-term effects. Assume that $P(t)$ is the proportion (or percent) of women initially infected with the HPV who no longer have the virus at time t (t is measured in years). The *rate of change* of $P(t)$ is modelled by the function

$$p(t) = 0.5 - 0.25t^{1.5} + 0.5e^{-1.2t}$$

where $0 \leq t \leq 2$.

(a) [3] Find $\int_0^1 \left(0.5 - 0.25t^{1.5} + 0.5e^{-1.2t}\right) dt$. Round off to two decimal places.

$$= \left[0.5t - \frac{0.25t^{2.5}}{2.5} + \frac{0.5e^{-1.2t}}{-1.2} \right]_0^1$$

$$= \left[0.5 - 0.1(1)^{2.5} - \frac{5}{12}e^{-1.2(1)} \right] - \left[-\frac{5}{12}e^{0} \right]$$

$$\approx 0.69$$

(b) [1] What does the number you obtained in (a) represent in the context of HPV infections?

This represents the proportion of women who were initially infected w/ HPV, but who no longer have it one year later.

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7. Evaluate the following integrals algebraically.

(a) [3] $\int_0^1 4xe^{-0.2x^2}\,dx$

$u = -0.2x^2$

$\frac{du}{dx} = -0.4x$

$dx = \frac{du}{-0.4x}$

$\int 4xe^{-0.2x^2}dx = \int 4xe^{u}\frac{du}{-0.4x}$

$= -10\int e^{u}du$

$= -10e^{u}$

$= -10e^{-0.2x^2}$

$\therefore \int_0^1 4xe^{-0.2x^2} = -10e^{-0.2x^2}\Big|_0^1$

$= -10e^{-0.2} - (-10)$

$= 10 - e^{-0.2}$

≈ 1.813

(b) [4] $\int \ln(x+1)\,dx$

$u = \ln(x+1)$ $\quad dv = dx$

$du = \frac{1}{x+1}dx$ $\quad v = x$

$= x\ln(x+1) - \int \frac{x}{x+1}dx$

$$\begin{array}{r} 1 \\ x+1 \overline{)\, x+0} \\ -\,x+1 \\ \hline -1 \end{array}$$

$= x\ln(x+1) - \int (1 - \frac{1}{x+1})\,dx$

$= x\ln(x+1) - x + \ln|x+1| + C$

$(= (x+1)\ln(x+1) - x + C)$

THE END