SOLUTIONS

Section 2.2 (geese) 1.2 (elephants): Semilog and double-log plots

1. The amount of carbon-14 (^{14}C) left t years after the death of an organism is given by

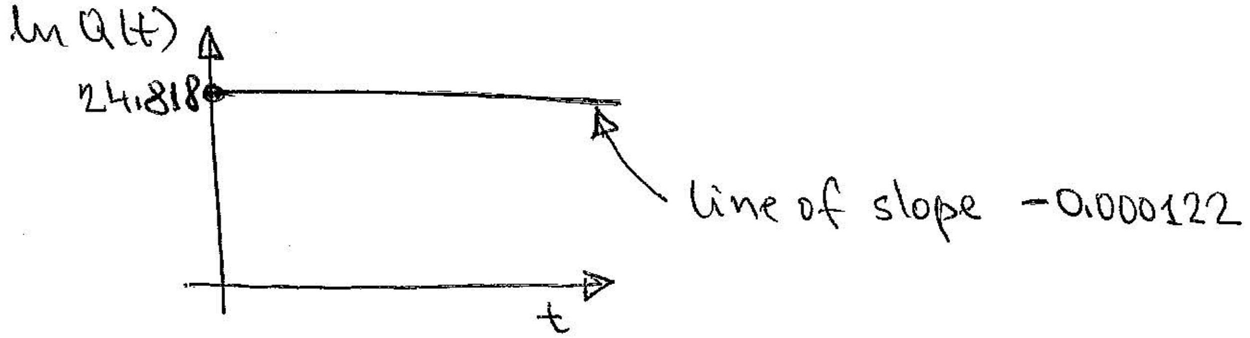
$$Q(t) = 6 \cdot 10^{10} e^{-0.000122t}$$

where $Q(t)$ counts the number of ^{14}C atoms.

(a) Sketch a semilog graph of $Q(t)$, using ln.

$$\ln Q(t) = \underbrace{\ln(6 \cdot 10^{10})}_{= \ln 6 + 10 \ln 10 \approx 24.818} + \ln(e^{-0.000122t})$$

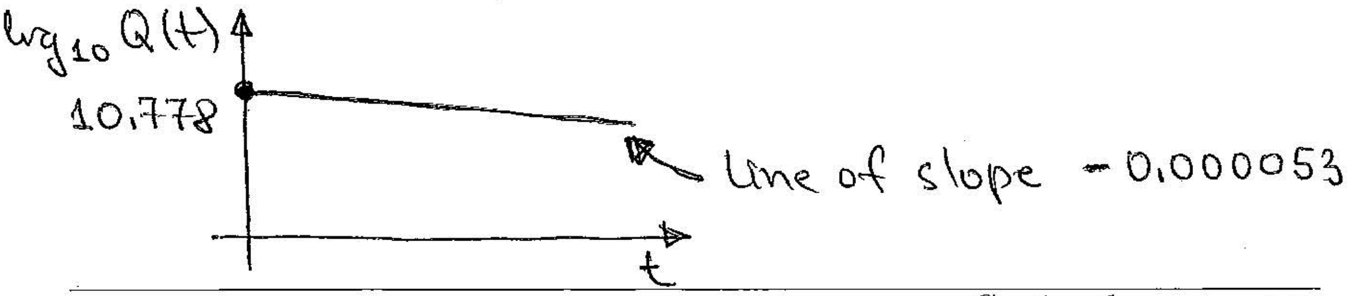
$$= 24.818 - 0.000122t$$



(b) Sketch a semilog graph of $Q(t)$, using $\log_{10}$.

$$\log_{10} Q(t) = \log_{10}(6 \cdot 10^{10}) + \underbrace{\log_{10}(e^{-0.000122t})}_{= -0.000122t \cdot \log_{10}(e) \approx -0.000053t}$$

$$\log_{10}(6 \cdot 10^{10}) = \log_{10} 6 + \underbrace{10 \log_{10} 10}_{10} \approx 10.778$$



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2. Consider the population of bacteria growing according to $P(t) = P(0)e^{kt}$. Assume that $P(0) = 100$ and $k = 0.56$.

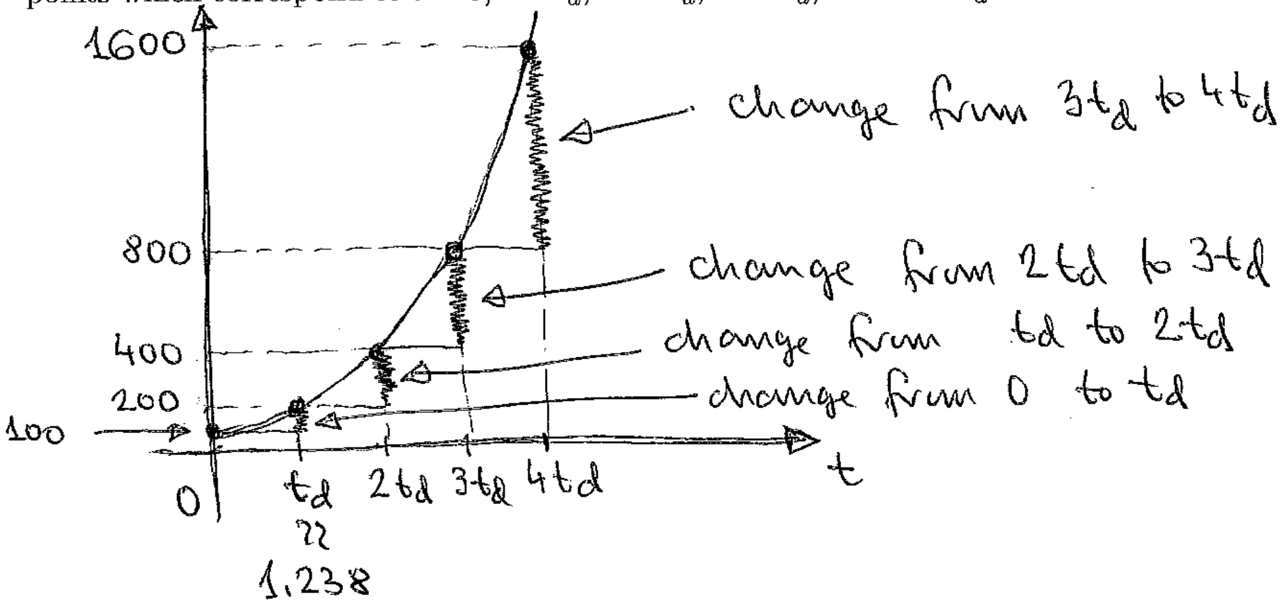
(a) Find the doubling time t_d of $P(t)$.

$$\underset{200}{P(t)} = 100\, e^{0{,}56t}$$

$$2 = e^{0{,}56t}$$

$$\ln 2 = 0{,}56t \longrightarrow t = \frac{\ln 2}{0{,}56} \approx 1{,}238$$

(b) Sketch the graph of $P(t)$ is the usual coordinate system (i.e., $P(t)$ vs. t) and label the points which correspond to $t = 0$, $t = t_d$, $t = 2t_d$, $t = 3t_d$, and $t = 4t_d$.



(c) In your graph in (b), indicate the changes in the values of $P(t)$ as t changes from 0 to t_d, then from t_d to $2t_d$, and so on. What pattern of growth do these changes show?

changes are: $200 - 100 = 100$
$400 - 200 = 200$
$800 - 400 = 400$
$1600 - 800 = 800$

they show the same exponential pattern as $P(t)$

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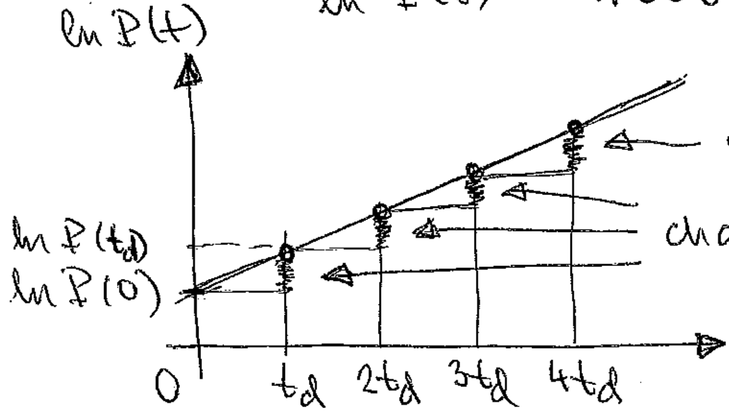
(d) Sketch the semilog graph of $P(t)$ (i.e., $\ln P(t)$ vs. t) and label the points which correspond to $t = 0$, $t = t_d$, $t = 2t_d$, $t = 3t_d$, and $t = 4t_d$.

$$\ln P(t) = \ln P(0) + kt = \ln 100 + 0.56t$$

$$\ln P(t) \approx 4.605 + 0.56t$$



(e) In your graph in (b), indicate the changes in the values of $P(t)$ as t changes from 0 to t_d, then from t_d to $2t_d$, and so on. What pattern of growth do these changes show?

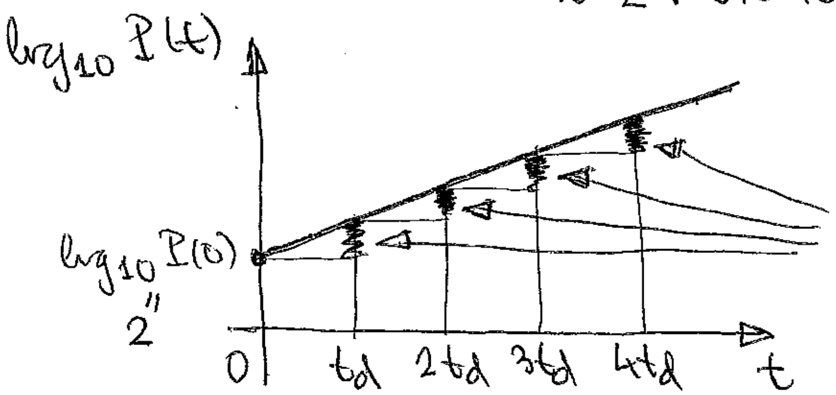
semi-log graph is a line → thus all changes are equal!

constant slope

(f) Repeat (d) and (e) the semilog graph of $P(t)$ where $\log_{10}$ is used instead of ln.

$$\log_{10} P(t) = \log_{10} 100 + 0.56t \cdot \log_{10} e$$

$$\approx 2 + 0.243t$$



as in (e), semi-log graph is a line, so all changes are equal

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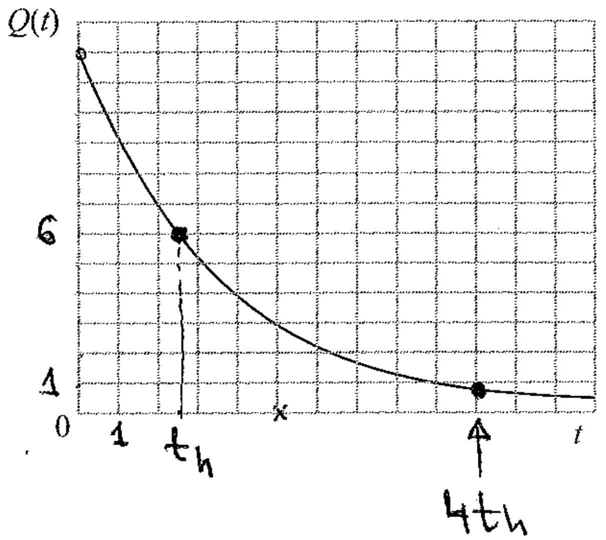
3. (a) The graph below shows an exponentially decreasing quantity $Q(t)$. Identify the point on the t-axis which represents the half-life t_h of $Q(t)$. Identify the point on the t-axis where $Q(t)$ decreases 16-fold.



$Q(0) = 12$

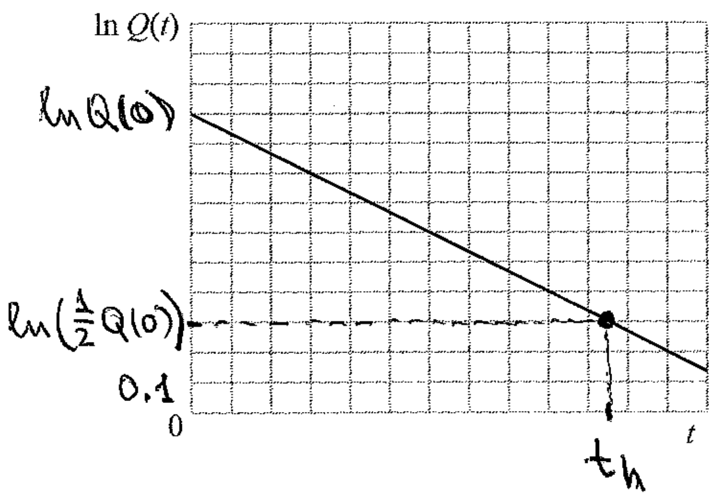
need t so that $Q(t) = 6$

16-fold decrease

$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}\, Q(0)$

$\to$ four half-lives

(b) The semilog graph below shows an exponentially decreasing quantity $Q(t)$. Identify the point on the t axis which represents the half-life t_h of $Q(t)$.



$Q(0) \xrightarrow{\text{half-life}} \frac{1}{2} Q(0)$

$\ln Q(0) \qquad \ln(\frac{1}{2} Q(0))$

$= \ln \frac{1}{2} + \ln Q(0)$

$\approx \ln Q(0) - 0.693$

i.e.: half life: $\underline{t}$ needed for $Q(0) \to \frac{1}{2} Q(0)$; in a semilog graph, it is $\underline{t}$ needed for $\ln(Q(0)) \to \ln(\frac{1}{2} Q(0))$

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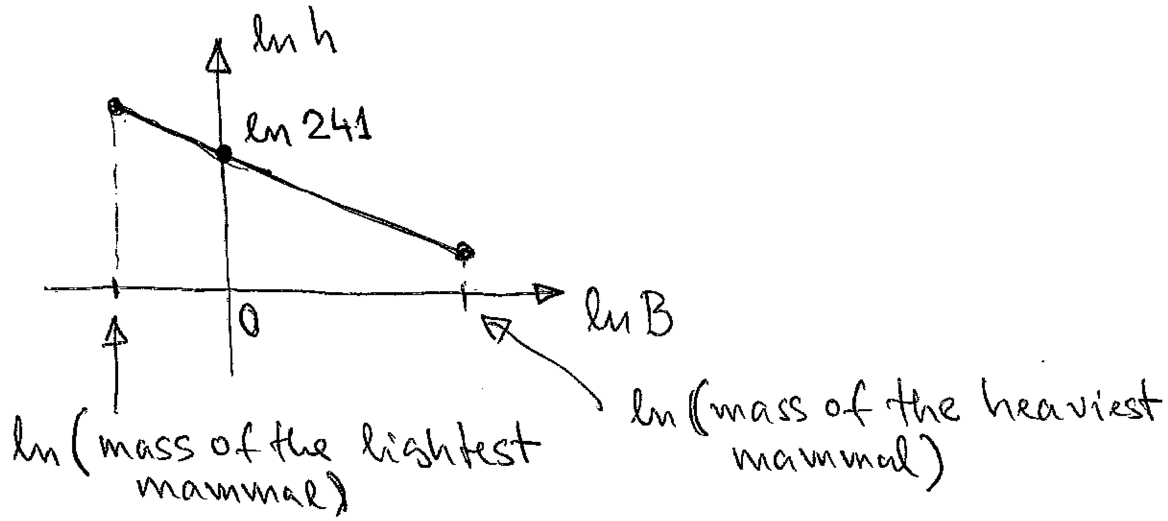
4. Consider the formula $h = 241B^{-0.25}$ for the dependence of the heartbeat frequency h on body mass B of a mammal. B is measured in kilograms and units of h are 1/min.

(a) Use ln to sketch a double-log plot of h. Label the axes and indicate a reasonable domain.

$\ln h = \ln(241B^{-0.25}) = \ln 241 - 0.25 \ln B$

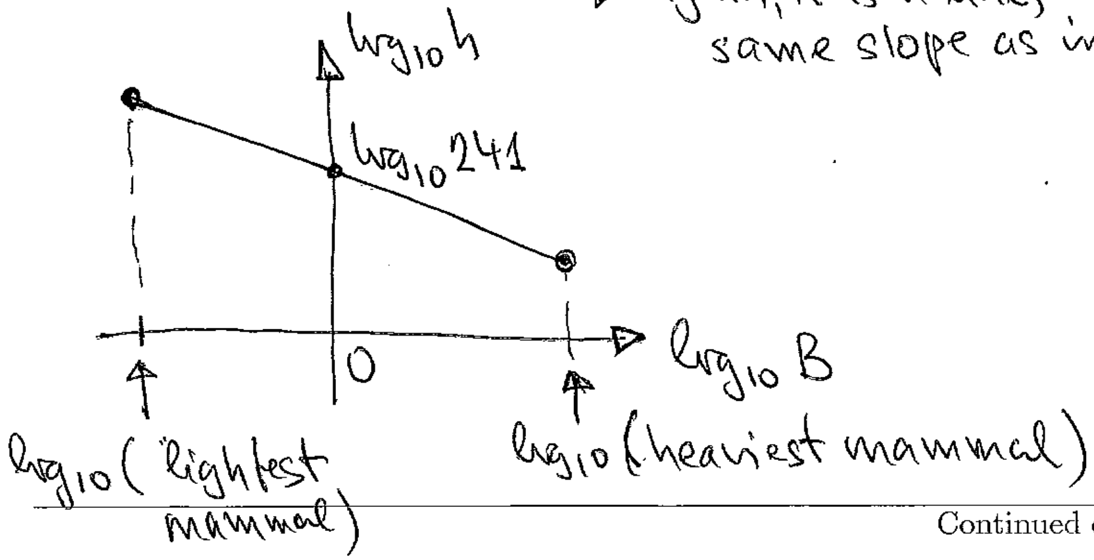
→ so the graph is a line



(b) Use $\log_{10}$ to sketch a double-log plot of h. Label the axes and indicate a reasonable domain.

$\log_{10} h = \log_{10}(241B^{-0.25}) = \log_{10}(241) - 0.25 \log_{10} B$

→ again, it is a line, same slope as in (a)



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5. The resistance R of the flow of blood through a blood vessel (assumed to have the shape of a cylindrical tube) is given by

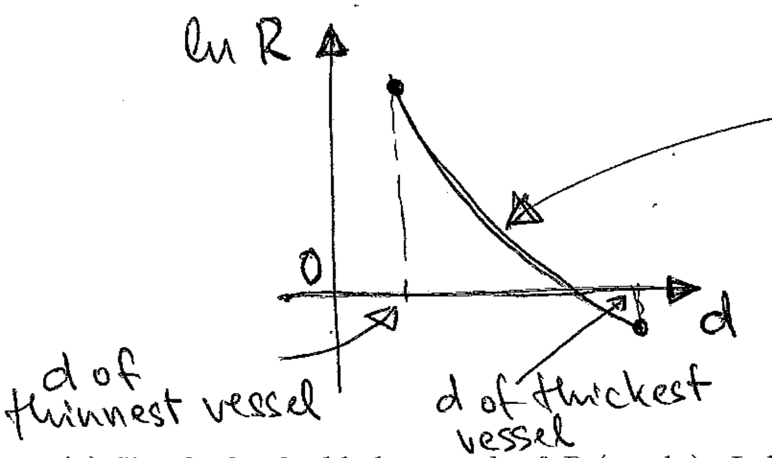
$$R = \frac{Kl(\gamma+1)^2}{d^4}$$

where l is the length of the vessel, d is its diameter and $\gamma \geq 0$ is the curvature. The positive constant K represents the viscosity of the blood (viscosity is a measure of the resistance of fluid to stress; water has low viscosity, honey has high viscosity).
In this exercise we view R as a function of d.

(a) Sketch the semilog graph of R (use ln). Label the axes and identify a reasonable domain for d.

$$\ln R = \ln \frac{Kl(\gamma+1)^2}{d^4} = \underbrace{\ln Kl(\gamma+1)^2}_{\text{constant}} - 4 \ln d$$

constant could be positive or negative

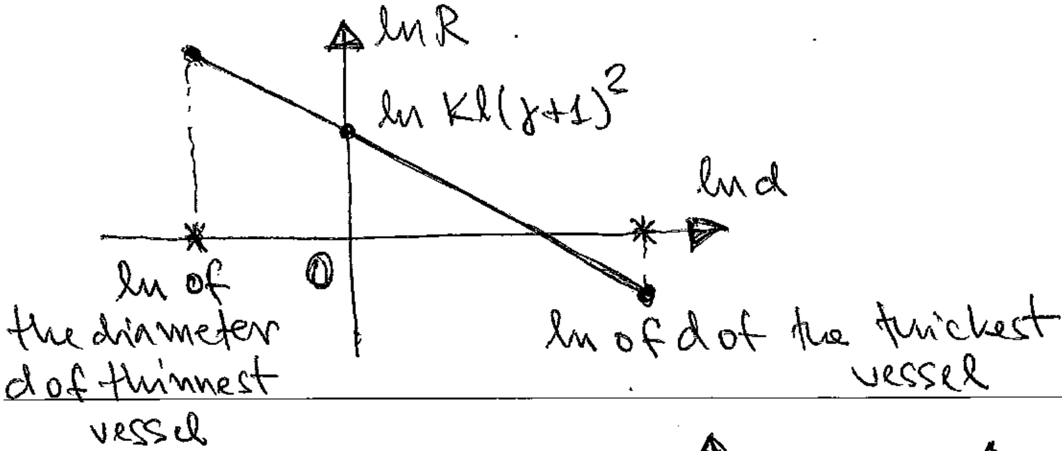


ln d reflected about d axis, stretched vertically by a factor of 4 and moved up or down for $\ln Kl(\gamma+1)^2$

See next page for details

(b) Sketch the double-log graph of R (use ln). Label the axes and identify a reasonable domain for d.

$$\ln R = \ln Kl(\gamma+1)^2 - 4 \ln d \longrightarrow \text{line of slope } -4$$



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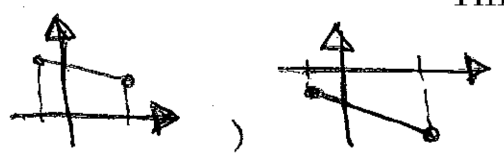
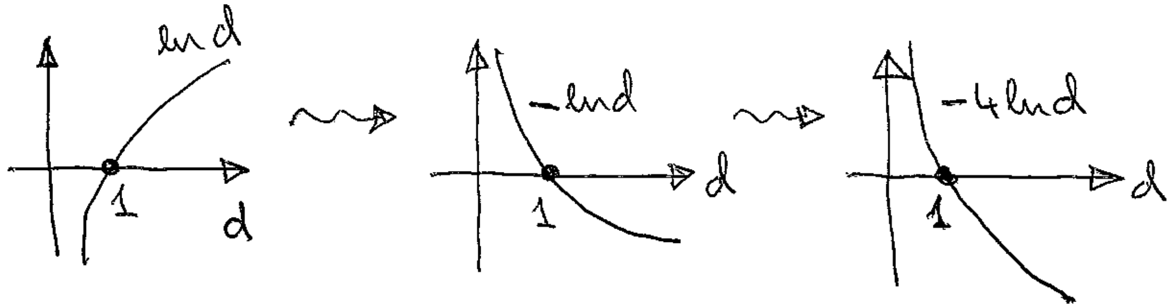
(note, as in (a), the graph could be different: ).

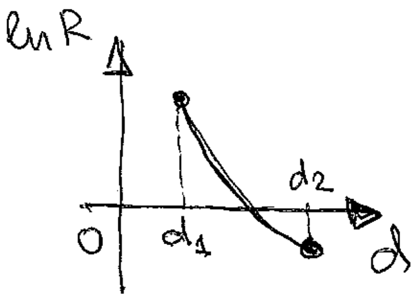
(a) 

if $K\ell(\gamma+1)^2 = 1 \rightarrow \ln K\ell(\gamma+1)^2 = 0$, so no shift

$K\ell(\gamma+1)^2 > 1 \rightarrow \ln K\ell(\gamma+1)^2 > 0$, so shift up

$K\ell(\gamma+1)^2 < 1 \rightarrow \ln K\ell(\gamma+1)^2 < 0$, so shift down

shifted graph could look like



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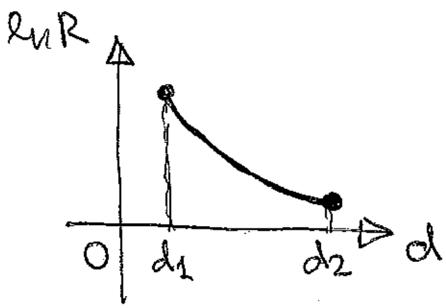

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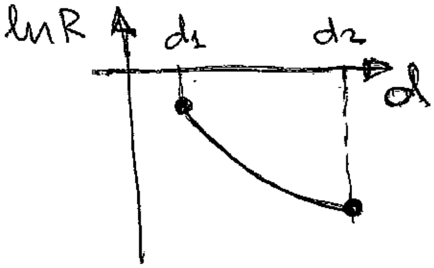

d_1 = d of thinnest vessel
d_2 = d of thickest vessel