MATHEMATICS 1LS3 TEST 1

Day Class
Duration of Examination: 60 minutes
McMaster University, 30 September 2019

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First name (PLEASE PRINT): SOLUTIONS

Family name (PLEASE PRINT): ____________

Student No.: ____________

THIS TEST HAS 8 PAGES AND 6 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

Total number of points is 40. Marks are indicated next to the problem number. Calculator allowed: McMaster standard calculator Casio fx991MS or Casio fx991MS PLUS or lower Casio which has two lines of display and no graphing capabilities.

EXCEPT ON QUESTIONS 1 AND 2, you must show work to receive full credit.

Problem	Points	Mark
1	10	
2	6	
3	4	
4	8	
5	8	
6	4	
TOTAL	40	

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1. **Multiple choice questions: circle ONE answer. No justification is needed.**

(a)[2] Start with the graph of $y = \cos x$. Scale (expand) the graph horizontally by a factor of 3 and then shift right the graph you obtained by 6 units. Finally, expand this graph vertically by a factor of 4. The graph you obtained is

(A) $y = \frac{1}{4}\cos\left(\frac{x+2}{6}\right)$ (B) $y = \frac{1}{4}\cos\left(\frac{x-2}{6}\right)$ (C) $y = 4\cos\left(\frac{x+6}{3}\right)$

(D) $y = 4\cos\left(\frac{x}{3} - 2\right)$ (E) $y = 4\cos\left(\frac{x}{3} + 6\right)$ (F) $y = 4\cos\left(\frac{x}{3} - \frac{2}{3}\right)$

(G) $y = \frac{1}{4}\cos\left(\frac{x+2}{3}\right)$ (H) $y = \frac{1}{4}\cos\left(\frac{x-6}{3}\right)$

$$\cos x \to \cos\left(\frac{x}{3}\right) \to \cos\left(\frac{x-6}{3}\right) \to 4\cdot\cos\left(\frac{x-6}{3}\right)$$

$$= 4\cdot\cos\left(\frac{x}{3} - 2\right)$$

(b)[2] The turbidity (denoted by T) is a measure of cloudiness or haziness in water, and is used to assess the quality of drinking water. It is calculated from the formula

$$T = k\frac{S\ln N}{d^2}$$

where N is the number of phytoplankton, S is the amount of sediment and d is the depth (k is a positive constant). If S and d double (and k and N do not change), then the turbidity T will change by a factor of

(A) 1/2 (B) 2 (C) 1/4 (D) 4

(E) 1/8 (F) 8 (G) 1/16 (H) 16

$$T = k\cdot\frac{2S\cdot\ln N}{(2d)^2} = \frac{2}{2^2}\,k\cdot\frac{S\ln N}{d^2}$$

$$\frac{1}{2}$$

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(c)[2] The domain of the function $f(x) = \sqrt{e^{1-x} - 4}$ is:

(A) all real numbers (B) $(0, 1 - \ln 4)$ (C) $(-1, \ln 4)$ (D) $(-\infty, 1 - \ln 4]$
(E) $(0, 1 - \ln 4]$ (F) $(-1, \ln 4]$ (G) $(-\infty, 1 - \ln 4)$ (H) $(-\infty, 1)$

domain: $e^{1-x} - 4 \geq 0$

$e^{1-x} \geq 4$

$1 - x \geq \ln 4$

$x \leq 1 - \ln 4$

(d)[2] Identify all correct statements about the function $f(x) = \dfrac{x^2+1}{x^2-1}$.

(I) $f(x)$ is not defined when $x = 0$. ✗ $f(0) = -1$
(II) $x = -1$ is a vertical asymptote of the graph of $f(x)$. ✓ ← $\dfrac{(-1)^2+1}{(-1)^2-1} = \dfrac{2}{0}$
(III) $y = 1$ is a horizontal asymptote of the graph of $f(x)$. ✓

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

(e)[2] Find the range of the function $y = 1 + 4\arcsin(12x)$.

(A) $[-1, 1]$ (B) $[0, 2\pi]$ (C) $[-4, 4]$ (D) $[-3, 5]$
(E) $[-\pi - 1, \pi + 1]$ (F) $[-2\pi, 2\pi]$ (G) $[-2\pi + 1, 2\pi + 1]$ (H) $[-4\pi, 4\pi]$

$-\frac{\pi}{2} \leq \arcsin(12x) \leq \frac{\pi}{2} \quad | \cdot 4$

$-2\pi \leq 4\arcsin(12x) \leq 2\pi \quad | +1$

$-2\pi + 1 \leq 1 + 4\arcsin(12x) \leq 2\pi + 1$

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2. True/false questions: circle ONE answer. No justification is needed.

(a)[2] An oscillatory input (intensity function of a group of spiking neurons) is given by the formula $\hat{\lambda}_1(t) = v_0 + a\cos(2\pi f_m(t+d))$. The period of $\hat{\lambda}_1(t)$ is $2\pi f_m$.

TRUE FALSE

period of $\cos(ax)$ is $2\pi/a$

period of $\cos(2\pi f_m t)$ is $\dfrac{2\pi}{2\pi f_m} = \dfrac{1}{f_m}$

(b)[2] The line $y = \pi$ is a horizontal asymptote of the graph of $f(x) = 2\arctan(x^3+1)$.

TRUE FALSE

$$\lim_{x\to\infty} 2\underbrace{\arctan(x^3+1)}_{\substack{\text{approaches} \\ \arctan(\infty) = \frac{\pi}{2}}} = 2\cdot\frac{\pi}{2} = \pi$$

(c)[2] A substance started decaying exponentially in year 2000, and reached 50% of its original amount in 2010. By 2030, it will decay to 25% of its original amount.

TRUE FALSE

2000 2010 2030

100% 50%

half-life is 10 years

(so it will be 25% in 2020)

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Questions 3-6: You must show correct work to receive full credit.

3. Based on the density of a soil sample taken from a forest floor, scientists can determine the depth it came from, by using the formula

$$d = f(\rho) = -5\ln\left(\frac{0.7}{\rho} - 0.8\right)$$

In this formula, ρ is the density of a soil sample and d is the depth in metres (so $d = 0$ labels the surface, and $d = 3$ is 3 m below the surface).

(a)[1] In the above formula, d is a function of ρ. State (in one sentence) what question is answered by finding the inverse function of d.

what is the density of a soil sample at depth d?

or: if we know the depth, what is the density?

(b)[3] Find a formula for the inverse function of d.

$$d = -5\ln\left(\frac{0.7}{\rho} - 0.8\right)$$

$$-\frac{d}{5} = \ln\left(\frac{0.7}{\rho} - 0.8\right)$$

$$e^{-d/5} = \frac{0.7}{\rho} - 0.8$$

$$\frac{0.7}{\rho} = e^{-d/5} + 0.8$$

$$\rho = 0.7 \cdot \frac{1}{e^{-d/5} + 0.8} = \frac{0.7}{0.8 + e^{-d/5}}$$

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4. Find each limit (or else say that the limit does not exist, and explain why).

(a)[2] $\lim_{x\to\pi^+} \frac{13x+1}{\sin x}$ $= \frac{13\pi+1}{0} = -\infty$

$\frac{13x+1}{\sin x}$ $\frac{\oplus}{\ominus}$ (as $x \to \pi^+$, $\sin x \to 0^-$)

(b)[3] $\lim_{x\to\infty} \left(\ln(2x^3+4) - \ln(x^3-x+2)\right)$ $= \lim_{x\to\infty} \ln \frac{2x^3+4}{x^3-x+2} = \ln 2,$

because $\lim_{x\to\infty} \frac{2x^3+4}{x^3-x+2} = \lim_{x\to\infty} \frac{2+4/x^3}{1-1/x^2+2/x^2} = 2$

(c)[3] Consider the function

$$f(x) = \begin{cases} \frac{x-1}{x^3-x} & \text{if } x < 1 \\ \frac{x}{4} & \text{if } x \geq 1 \end{cases}$$

Then $\lim_{x\to 1} f(x) =$ DNE

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x-1}{x^3-x} = \lim_{x\to 1^-} \frac{\cancel{x-1}}{x(\cancel{x-1})(x+1)} = \frac{1}{2}$

because

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x}{4} = \frac{1}{4}$

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5. The survival rate (i.e., the chance of survival) $S(D)$ of clonogenic cells (cancer cells) exposed to a radiation treatment can be modelled by

$$S(D) = e^{-0.5D^2 - D}$$

where $D \geq 0$ represents the applied radiation dose (measured in grays, Gy).

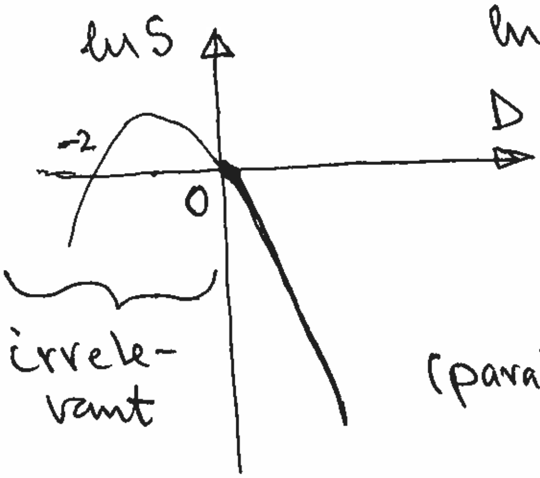
(a)[2] What is $S(0)$? Does it make sense?

$S(0) = e^0 = 1 \longrightarrow$ meaning 100% survival rate

YES: if radiation dose is zero (ie, no radiation) all cells will survive

(b)[3] Sketch the semilog graph (use ln) of the survival rate for $D \geq 0$. Label the axes.

$\ln S = -0.5D^2 - D$



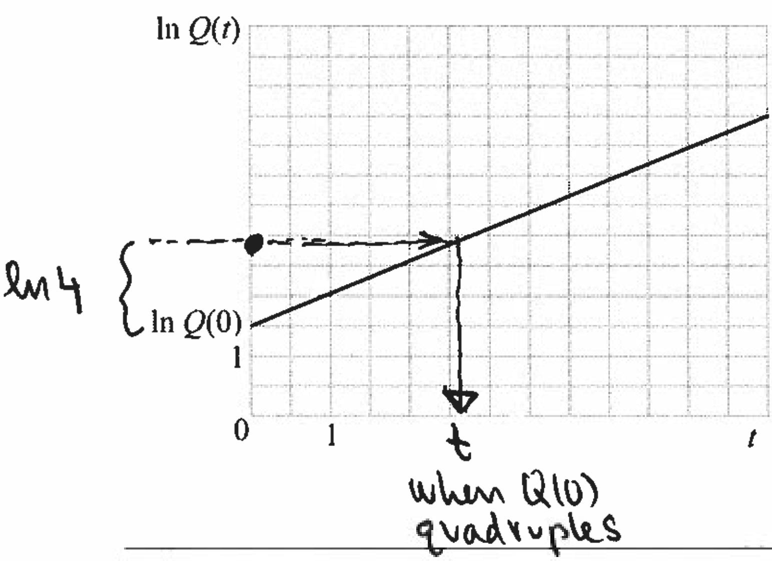
$-0.5D^2 - D$

$= -D(0.5D+1)$

$\downarrow \quad \downarrow$

$0 \quad -2$

(c)[3] The semilog graph below shows an exponentially increasing quantity $Q(t)$. Identify the point on the t axis which represents the time when the quantity quadruples (i.e., is four times larger than initially). Justify your answer.



$\ln(4Q(0))$

$= \ln 4 + \ln(Q(0))$

$= \ln(Q(0)) + 1.39$

$\ln 4 \approx 1.39$

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6. The following excerpt is taken from *The laminar cortex model: a new continuum cortex model incorporating laminar architecture.* J. Du, V. Vegh, and D.C. Reutens. PLoS Computational Biology. 8.10 (Oct. 2012).

> the average of membrane potentials of neurons in the element, that is
>
> $$V = \frac{N_e V_e + N_i V_i}{N_e + N_i}$$
>
> where N_e, N_i are the numbers of excitatory and inhibitory neurons and V_e and V_i are the (average) membrane potentials of excitatory and inhibitory neuron populations respectively.

(a)[2] View V as a function of V_i. Describe its graph in words.

$$V(V_i) = \frac{N_e V_e}{N_e + N_i} + \frac{N_i}{N_e + N_i} V_i$$

<u>line</u>, slope $= \frac{N_i}{N_e + N_i}$

vertical (V) intercept $= \frac{N_e V_e}{N_e + N_i}$

(b)[2] View V as a function of N_e. What is the limit of V as N_e increases beyond any bounds (i.e., as it approaches ∞)?

$$\lim_{N_e \to \infty} V = \lim_{N_e \to \infty} \frac{N_e V_e + N_i V_i \quad / N_e}{N_e + N_i \quad / N_e}$$

$$= \lim_{N_e \to \infty} \frac{V_e + \cancel{N_i V_i / N_e} \to 0}{1 + \cancel{N_i / N_e} \to 0} = V_e$$

THE END