MATHEMATICS 1LS3 TEST 2

Day Class
E. Clements, M. Lovrić, E. Miller
Duration of Examination: 60 minutes
McMaster University, 28 October 2019

First name (PLEASE PRINT): SOLUTIONS

Family name (PLEASE PRINT): ____________

Student No.: ____________

THIS TEST HAS 8 PAGES AND 6 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

Total number of points is 40. Marks are indicated next to the problem number. Calculator allowed: McMaster standard calculator Casio fx991MS or Casio fx991MS PLUS or lower Casio which has two lines of display and no graphing capabilities.

EXCEPT ON QUESTIONS 1 AND 2, you must show work to receive full credit.

Problem	Points	Mark
1	10	
2	6	
3	6	
4	7	
5	7	
6	4	
TOTAL	40	

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1. **Multiple choice questions: circle ONE answer. No justification is needed.**

(a)[2] If $f(x) = \ln(ax)\ln(bx)$ then $f'(1)$ is equal to

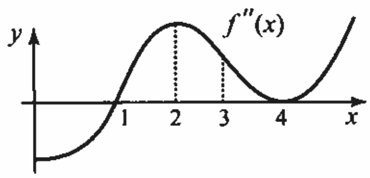
(A) $\ln a \ln b$ (B) $\ln(a+b)$ (C) $\ln(ab)$ (D) $\dfrac{\ln(a+b)}{a+b}$

(E) $\dfrac{\ln(ab)}{a+b}$ (F) $\dfrac{\ln a \ln b}{a+b}$ (G) $\dfrac{1}{ab}$ (H) $\dfrac{1}{a}+\dfrac{1}{b}$

$$f'(x) = \frac{1}{\cancel{a}x}\cdot\cancel{a}\cdot\ln(bx) + \ln(ax)\cdot\frac{1}{\cancel{b}x}\cdot\cancel{b}$$

$$f'(1) = \ln b + \ln a = \ln(ab)$$

(b) [2] The graph of the second derivative $f''(x)$ of a function $f(x)$ is given. Which statement(s) is/are true?



		1		4		x
f''	$\ominus$		$\oplus$		$\oplus$	
f	$\cap$		$\cup$		$\cup$	

(I) $x = 4$ is a point of inflection of $f(x)$ X

(II) The graph of $f(x)$ is concave up on $(1,4)$ ✓

(III) $x = 1$ is a point of inflection of $f(x)$ ✓

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

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(c)[2] It is known that the function $f(x)$ is defined for all real numbers, and its derivative is given by $f'(x) = \dfrac{(x-3)\,e^{-2x}}{(4-x)^{1/3}}$. Find all its critical points.

(A) no critical points (B) 0 only (C) 3 only (D) 4 only
(E) 0 and 3 (F) 0 and 4 (G) 3 and 4 (H) 0, 3 and 4

$f' = 0 \to x = 3$
f' dne $\to x = 4$

(d) [2] It is known that $f(3) = 4$ and $f'(3) = 0$ and $f''(3) = -2$. Which statements is/are true for all functions $f(x)$ which satisfy these three conditions?

(I) $f(3) = 4$ is a local (relative) maximum of $f(x)$ ✓ second derivative test

(II) the tangent line to the graph of $f(x)$ at $x = 3$ is $y = 0$ ✗

(III) the linear approximation of $f(x)$ at $x = 3$ is $L_3(x) = 4$. ✓

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

(e)[2] If $f(x) = 2^{\ln x} + (\ln x)^2 + \ln 2$, then $f'(1)$ is equal to

(A) $1 + \ln 2$ (B) $2^{\ln 2}$ (C) $2^{\ln 2} + (\ln 2)^2$ (D) 0
(E) 1 (F) $\ln 2$ (G) $4 \ln 2$ (H) $2 \ln 2$

$f'(x) = 2^{\ln x} \cdot \ln 2 \cdot \frac{1}{x} + 2 \cdot \ln x \cdot \frac{1}{x} + 0$
$f'(1) = 2^0 \cdot \ln 2 \cdot 1 + 2 \cdot 0 \cdot 1 = \ln 2$

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2. **True/false questions: circle ONE answer. No justification is needed.**

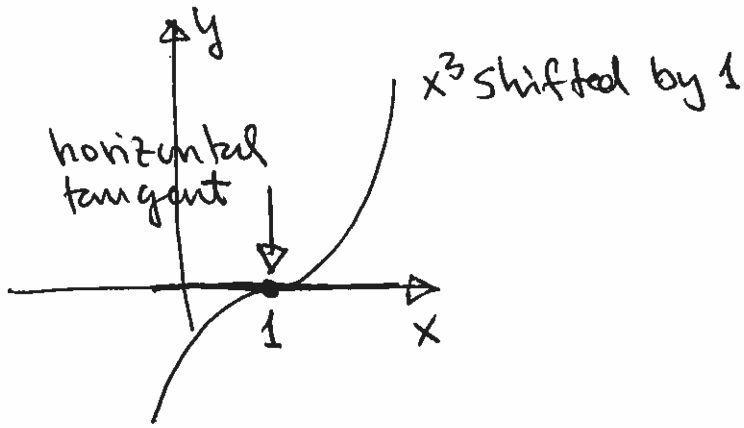
(a)[2] From $f''(x) = e^{-x}(3-x)$ we conclude that the graph of $f(x)$ is concave down on the interval $(0, 3)$.

TRUE FALSE

$$f''(x) = \underbrace{e^{-x}}_{\oplus} \underbrace{(3-x)}_{\oplus \text{ when } x \text{ is in } (0,3)} > 0$$

(b) [2] The function $f(x)$ has a horizontal tangent at $x = 1$. Therefore, it must have a local maximum or a local minimum at $x = 1$.

TRUE FALSE



(c)[2] If $f(x) = g(x)h(x)$, then by the product rule, $f''(x) = g''(x)h(x) + g(x)h''(x)$.

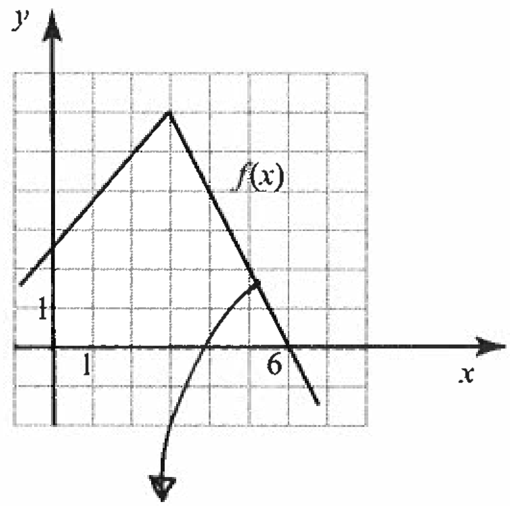
TRUE FALSE

$f' = g'h + gh'$

$\rightarrow f'' = g''h + g'h' + g'h' + gh'' \neq g''h + gh''$

Questions 3-6: You must show correct work to receive full credit.

3. (a)[3] Let $h(x) = \sin(2f(x))$. The graph of $f(x)$ is a line shown below. Find $h'(6)$.



slope of this line is -2

$h'(x) = \cos(2f(x)) \cdot 2f'(x)$

so

$$h'(6) = \underbrace{\cos(2 \cdot \underbrace{f(6)}_{0})}_{1} \cdot 2 \underbrace{f'(6)}_{-2}$$

so $h'(6) = -4$

(b)[3] Find $y'(0)$, if $\arcsin(xy) = x^3 + y^2 - 1$, and $y(0) = 1$.

$$\frac{1}{\sqrt{1-(xy)^2}} \cdot (y + xy') = 3x^2 + 2yy'$$

$y(0) = 1$ means $x=0, y=1$ $\rightarrow$ $\frac{1}{\sqrt{1-0}} \cdot (1+0) = 0 + 2y'$

$2y' = 1, \quad y' = \frac{1}{2}$

4. (a)[3] In the article *Migration behaviour of grizzly bears in Northern British Columbia: contribution to a modelling approach* we find the formula

$$P(t) = \arctan(1.7t) + 4.7$$

where t represents time. Next, we read "initially, $P(t) \approx 1.7t + 4.7$, which gives a linear relationship." Explain why this statement is correct. [Hint: Think in terms of the linear approximation at $t = 0$.]

$$L(t) = P(0) + P'(0) \cdot (t-0)$$

$$P(0) = \arctan(0) + 4.7 = 4.7$$

$$P'(t) = \frac{1}{1+(1.7t)^2} \cdot 1.7 \rightarrow P'(0) = \frac{1}{1+0} \cdot 1.7 = 1.7$$

thus, the lin. approx. is

$$L(t) = 4.7 + 1.7(t-0) = 4.7 + 1.7t$$

(b)[4] A simple model of diffusion states that the concentration of a substance diffusing in air is given by

$$c(x) = e^{-x^2+1}$$

where x is the distance from the source. This formula is sometimes simplified using a quadratic approximation near $x = 0$. Find that approximation.

$$T_2(x) = c(0) + c'(0)(x-0) + \frac{c''(0)}{2}(x-0)^2$$

$$c(0) = e^1$$

$$c'(x) = -2xe^{-x^2+1} \rightarrow c'(0) = 0$$

$$c''(x) = -2e^{-x^2+1} - 2x e^{-x^2+1}(-2x)$$

$$\rightarrow c''(0) = -2e - 0 = -2e$$

thus $T_2(x) = e^1 + 0 + \frac{-2e}{2}x^2 = \underline{e - ex^2}$

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5. The function $c(t) = t^2 e^{-6t}$ has been used to model the absorption of a drug (such as morphine); $c(t)$ is the concentration (in milligrams per millilitre, mg/mL) of the drug in the bloodstream, and $t \geq 0$ is time (in hours).

(a)[3] The function $c(t)$ has two critical points such that $t \geq 0$. Find them.

$$c'(t) = 2te^{-6t} + t^2e^{-6t}(-6)$$
$$= 2te^{-6t}(1-3t)$$

$c'(t) = 0 \rightarrow t = 0$ or $1-3t=0$, $t = 1/3$

$c'(t)$ dne $\rightarrow$ no such t

(b)[2] Give a statement of the Extreme Value Theorem. Make sure to clearly identify assumption(s) and conclusion(s).

IF $f(x)$ is continuous, $[a,b]$ closed interval

THEN $f(x)$ has an absolute max. and an absolute min. in $[a,b]$

(c)[2] Find the absolute maximum and the absolute minimum values that the concentration $c(t)$ reaches during the first hour after the drug is administered, i.e., over the interval $[0, 1]$.

t	$c(t) = t^2e^{-6t}$
0	0
1	$e^{-6} \approx 0{,}002$
$1/3$	$\frac{1}{9}e^{-2} \approx 0{,}015$

abs. min. $= 0$, at $t=0$

abs. max. $= \frac{1}{9}e^{-2}$, at $t = 1/3$

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6. Consider the function

$$f(x) = \begin{cases} \dfrac{x-1}{x^4 - x^2} & \text{if} \quad x < 1 \\ \dfrac{1}{4} & \text{if} \quad x = 1 \\ \dfrac{x}{2} & \text{if} \quad x > 1 \end{cases}$$

(a)[2] Find $\lim_{x \to 1} f(x)$.

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{x}{2} = \frac{1}{2}$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{x-1}{x^4 - x^2} = \lim_{x \to 1^-} \frac{\cancel{x-1}}{x^2\cancel{(x-1)}(x+1)}$$

$$= \frac{1}{1^2 \cdot 2} = \frac{1}{2}$$

thus $\lim_{x \to 1} f(x) = \frac{1}{2}$

(b)[2] Is $f(x)$ continuous at $x = 1$? Explain why or why not.

NO because

$\lim_{x \to 1} f(x) = \frac{1}{2}$, but $f(1) = \frac{1}{4}$

THE END