MATHEMATICS 1LS3 TEST 3

Day Class
Duration of Examination: 60 minutes
McMaster University, 25 November 2019

E. Clements, M. Lovrić, E. Miller

First name (PLEASE PRINT): SOLUTIONS

Family name (PLEASE PRINT): __________

Student No.: __________

THIS TEST HAS 8 PAGES AND 6 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

Total number of points is 40. Marks are indicated next to the problem number. Calculator allowed: McMaster standard calculator Casio fx991MS or Casio fx991MS PLUS or lower Casio which has two lines of display and no graphing capabilities.

EXCEPT ON QUESTIONS 1 AND 2, you must show work to receive full credit.

Problem	Points	Mark
1	10	
2	6	
3	6	
4	8	
5	5	
6	5	
TOTAL	40	

Continued on next page

Name:________________
Student No.: ________________

1. **Multiple choice questions: circle ONE answer. No justification is needed.**

(a) [2] Which of the following improper integrals are convergent?

(I) $\int_1^\infty x^{-1.5}\,dx$ ✓ (II) $\int_1^\infty x^{-1}\,dx$ ✗ (III) $\int_1^\infty x^{-0.5}\,dx$ ✗

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

$$\int_1^\infty x^{-p}\,dx = \int_1^\infty \frac{1}{x^p}\,dx \quad \text{is conv. if } \underline{p>1}$$

(b) [2] Find $\lim_{x\to 0} \frac{e^{x^2}-1-x^2}{x^4} = \frac{0}{0}$

(A) 0 (B) 1/5 (C) 1/3 (D) 1/2

(E) 1 (F) ∞ (G) 1/6 (H) 5/6

$$\overset{LH}{=} \lim_{x\to 0} \frac{e^{x^2}\cdot 2x - 2x}{4x^3} = \lim_{x\to 0} \frac{e^{x^2}-1}{2x^2} = \frac{0}{0}$$

$$\overset{LH}{=} \lim_{x\to 0} \frac{e^{x^2}\cdot 2x}{4x} = \lim_{x\to 0} \frac{e^{x^2}}{2} = \frac{1}{2}$$

Continued on next page

(c) [2] It is known that $\left(\dfrac{3x-1}{2x+1}\right)' = \dfrac{5}{(2x+1)^2}$. What is the value of $\displaystyle\int_0^{1/3} \frac{5}{(2x+1)^2}\,dx$?

(A) 0 (B) 1/5 (C) 1/3 (D) 1/2
(E) 1 (F) 2 (G) 3 (H) 5

$$= \frac{3x-1}{2x+1}\Big|_0^{1/3} = 0-(-1) = 1$$

(d) [2] The value of $\displaystyle\int_0^1 \frac{4}{1+x^2}\,dx$ is

(A) 0 (B) 1 (C) $\pi/4$ (D) $\pi/2$
(E) 2π (F) π (G) $3\pi/4$ (H) $3\pi/2$

$$= 4\arctan x\Big|_0^1 = 4\underbrace{\arctan 1}_{\pi/4} - 4\underbrace{\arctan 0}_{0}$$

(e) [2] Which of the following numbers is/are positive?

(I) $\displaystyle\int_{-1}^{1} \underbrace{e^{-3x}}_{\oplus}\,dx$ ✓ (II) $\displaystyle\int_{0.1}^{1} \underbrace{\ln x}_{\ominus}\,dx$ (III) $\displaystyle\int_{-4}^{-2} \underbrace{|x|}_{\oplus}\,dx$ ✓

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

Continued on next page

Name: ____________

Student No.: ____________

2. **True/false questions: circle ONE answer. No justification is needed.**

(a)[2] The temperature of a bottle of water placed in a fridge decreases proportionally to the difference between the temperature of the bottle and the fridge. A differential equation modelling this event would be described as a *pure-time* differential equation.

TRUE FALSE

$$\frac{d\,\text{temperature}}{dt} = \text{in terms of temperature} \rightarrow \text{autonomous}$$

(b) [2] The improper integral $\int_0^\infty e^{-2x}\,dx$ is convergent.

TRUE FALSE

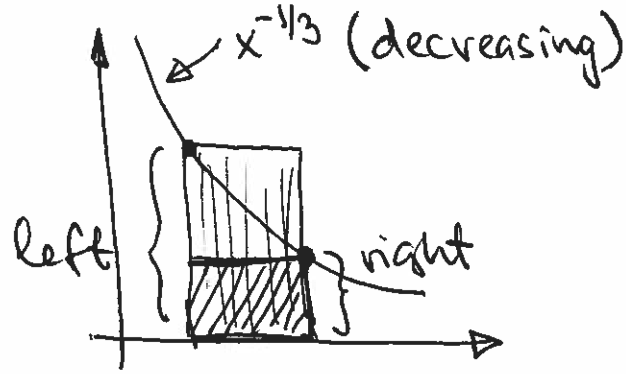
$$= \lim_{T\to\infty} \int_0^T e^{-2x}\,dx$$

$$= \lim_{T\to\infty} \left(-\frac{1}{2}\right)e^{-2x}\Big|_0^T$$

$$= \lim_{T\to\infty} \left(\cancel{-\frac{1}{2}e^{-2T}}\right)_{\to 0} - \left(-\frac{1}{2}e^{0}\right) = \frac{1}{2}$$

(c)[2] The right and the left Riemann sums of $f(x) = x^{-1/3}$ on $[2, 12]$ satisfy $R_{15} < L_{15}$.

TRUE FALSE



Continued on next page

Name:

Student No.:

Questions 3-6: You must show correct work to receive full credit.

3. (a) [3] Find an antiderivative of $f(x) = \frac{\sqrt{\ln x}}{x}$.

$$\int \frac{\sqrt{\ln x}}{x} dx = \left\{ \begin{array}{l} u = \ln x \\ \frac{du}{dx} = \frac{1}{x} \to \frac{1}{x} dx = du \end{array} \right\}$$

$$= \int \sqrt{u} \cdot du = \frac{2}{3} u^{3/2} + C = \underline{\frac{2}{3} (\ln x)^{3/2} + C}$$

(b)[3] In *Wolf Science, 16 (7), 2018,* we find a model for the change in the number of Eastern wolves in Algonquin Provincial Park

$$W'(t) = 2.2\sqrt{t} + 2e^{-0.1t}, \quad W(0) = 140$$

Time t is measured in years, and $t = 0$ represents 1 July 2018.

Using Euler's method with the step size $\Delta t = 1$ estimate the number of wolves on 1 July 2020.

$t_0 = 0$

$W_0 = 140$

$$\boxed{\begin{array}{l} t_{n+1} = t_n + \Delta t = t_n + 1 \\ W_{n+1} = W_n + (2.2\sqrt{t_n} + 2e^{-0.1 t_n}) \underbrace{\Delta t}_{1} \end{array}}$$

$t_1 = 0 + 1 = 1 \to W_1 = 140 + (2.2 \cdot \underbrace{\cancel{\sqrt{0}}}_{0} + 2 \underbrace{e^0}_{1}) \cdot 1$

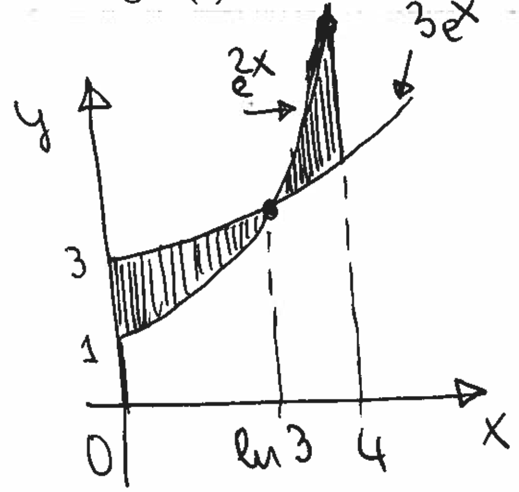
$= \underline{142}$

$t_2 = 1 + 1 = 2$ $\quad W_2 = 142 + (2.2 \cdot \sqrt{1} + 2e^{-0.1}) \cdot 1$

July 2020 $\quad = 142 + 4 = \underline{146}$

Continued on next page

4. (a)[4] Sketch (shade) the region bounded by the graphs of $y = 3e^x$ and $y = e^{2x}$ on $[0, 4]$. Write a formula for its area. Your answer should not contain absolute value. **Do not evaluate the integral(s) involved.**



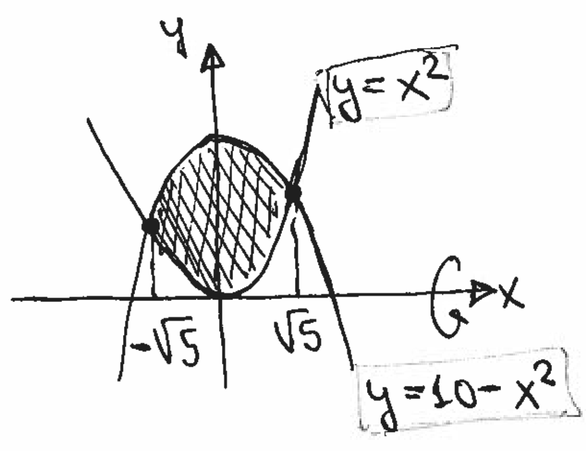
$e^{2x} = 3e^x \quad | \div e^x$

$e^x = 3$

$x = \ln 3$

$$A = \int_0^{\ln 3} (3e^x - e^{2x})\,dx + \int_{\ln 3}^{4} (e^{2x} - 3e^x)\,dx$$

(b)[4] Consider the region bounded by the graphs of $y = 10 - x^2$ and $y = x^2$. Write a formula for the volume of the solid obtained by revolving this region about the x-axis. **Do not evaluate the integral(s) involved.**



$10 - x^2 = x^2$

$\rightarrow x^2 = 5 \rightarrow x = \pm\sqrt{5}$

$$V = \pi \int_{-\sqrt{5}}^{\sqrt{5}} \left((10 - x^2)^2 - (x^2)^2\right)dx$$

Continued on next page

Name:

Student No.:

5. (a)[2] Find the Taylor polynomial $T_2(x)$ for $f(x) = e^x$ at $x = 0$.

$f, f', \ldots$	at 0
$f = e^x$	1
$f' = e^x$	1
$\vdots$	1
	$\vdots$

$$T_2(x) = 1 + x + \frac{1}{2}x^2$$

(b)[1] Use your answer in (a) to find a polynomial approximation of e^{-x^2}.

$$e^{-x^2} \approx T_2(-x^2) = 1 - x^2 + \frac{1}{2}x^4$$

(c)[2] Use the polynomial from (b) to find an approximation of $\int_0^1 e^{-x^2}\,dx$. Leave you answer as a fraction or round off to two decimal places.

$$\int_0^1 e^{-x^2}dx \approx \int_0^1 \left(1 - x^2 + \frac{1}{2}x^4\right)dx$$

$$= x - \frac{1}{3}x^3 + \frac{1}{10}x^5 \Big|_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{10} = \frac{23}{30} \approx 0.77$$

Continued on next page

6. The *rate of change* of the number of new individuals infected by a strain H2T1 of influenza virus in Hamilton in January 2018 has been modelled by the function $p(t) = 120te^{-0.1t}$. The variable t is time in days; the time $t = 0$ represents 15 January 2018.

(a)[3] Find $\int_0^4 120\boxed{te^{-0.1t}\,dt}$ $= \left\{ \begin{array}{l} u = t \longrightarrow u' = 1 \\ v' = e^{-0.1t} \longrightarrow v = -\frac{1}{0.1}e^{-0.1t} = -10e^{-0.1t} \end{array} \right\}$

$$= uv - \int vu'dx = -10te^{-0.1t} + \int 10\underbrace{e^{-0.1t}dt}_{-10e^{-0.1t}}$$

$$= -10te^{-0.1t} - 100e^{-0.1t}$$

thus

$$\int_0^4 120\, te^{-0.1t}dt = 120\left(-10te^{-0.1t} - 100e^{-0.1t}\right)\Big|_0^4$$

$$= 120(-40e^{-0.4} - 100e^{-0.4}) - 120(0 - 100)$$

$$\approx 738.62$$

(b)[2] What does your answer in (a) represent?

total number of new individuals infected between $\underbrace{\text{15 Jan 2018}}_{t=0}$ and $\underbrace{\text{19 Jan 2019}}_{t=4}$ is 738 or 739

either is OK

THE END