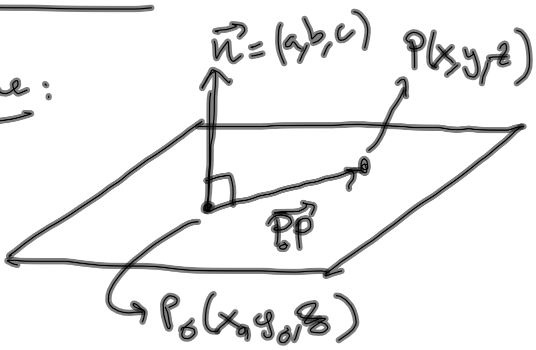
• POINT NORMAL EQ. FOR THE LINE AND PLANE :

Plane :

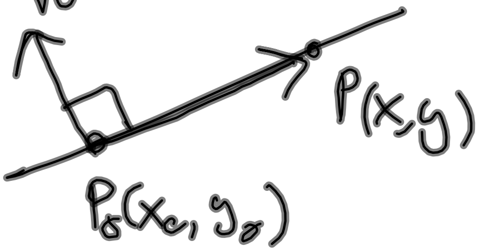


$$\vec{n}\cdot\overrightarrow{P_0P} = (a,b,c)\cdot(x-x_0,\ y-y_0,\ z-z_0)$$

$$= \boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0}$$

(point normal eq. for the plane)

Line: $\vec{n} = (a,b)$



(point normal eq. for the line)

$$(a,b)\cdot(x-x_0,\, y-y_0) = \boxed{a(x-x_0)+b(y-y_0)=0}$$

$$\Rightarrow ax+by+\underbrace{(-ax_0-by_0)}_{=d}=0$$

$$\Rightarrow \boxed{ax+by+d=0}$$

Ex: $P_0(1,1)$, $\vec{n}=(1,2)$

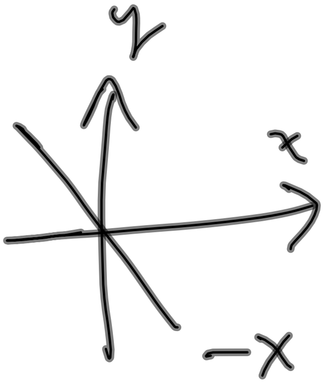
$$1\cdot(x-1)+2(y-1)=0$$

Ex: [Vector form of line through the origin $(a,b)\cdot(x,y)=0$]

$\vec{n}=(1,1)$

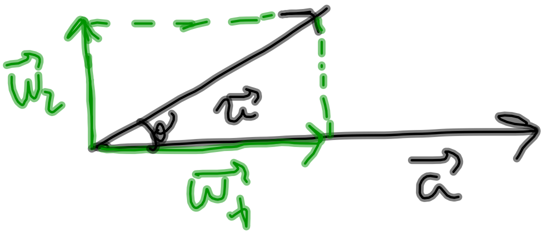
$\hookrightarrow (1,1)\cdot(x,y)=0$

$\hookrightarrow \boxed{y=-x}$



• DERIVATION OF $\vec{w}_1$ & $\vec{w}_2$:

$$\vec{u} = \vec{w}_1 + \vec{w}_2$$



Remember:

$$\vec{u}\cdot\vec{a} = \|\vec{u}\|\|\vec{a}\|\cos\theta$$

$c_1 = r\cdot\cos\theta$

$$\vec{u}\cdot\underbrace{\frac{\vec{a}}{\|\vec{a}\|}}_{\text{unit vec. associated to }\vec{a}} = \underbrace{\|\vec{u}\|\cos\theta}_{\text{norm of }\vec{w}_1}$$

$$[\![\, \vec{w}_1 = \|\vec{u}\|\cos\theta\,\frac{\vec{a}}{\|\vec{a}\|} = \left(u\cdot\frac{\vec{a}}{\|\vec{a}\|}\right)\frac{\vec{a}}{\|\vec{a}\|}$$
$$= (\vec{u}\cdot\vec{a})\frac{\vec{a}}{\|\vec{a}\|^2} \,]\!]$$

$$\boxed{\begin{aligned}\vec{w}_2 &= \vec{u} - \vec{w}_1 = \\ &= u - (\vec{u}\cdot\vec{a})\frac{\vec{a}}{\|\vec{a}\|^2}\end{aligned}}$$

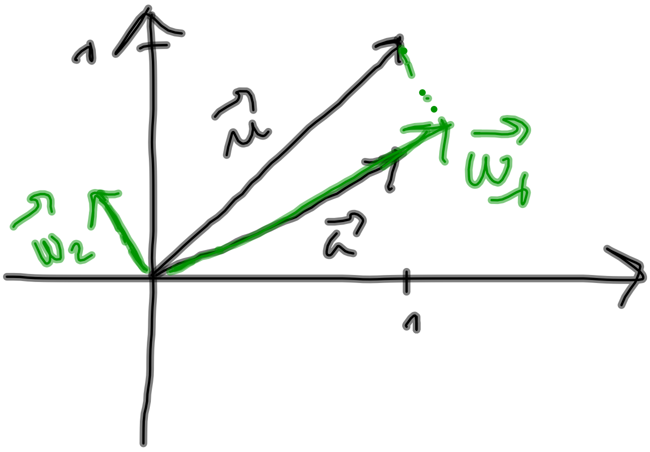
- Ex: [projection theorem]

$\vec{a} = (1, \frac{1}{2})$

$\vec{u} = (1, 1)$

$\|\vec{u}\| = \sqrt{2}$

$\|\vec{a}\| = \sqrt{5/4}$



$$\vec{w}_1 = (\vec{u}\cdot\vec{a})\frac{\vec{a}}{\|\vec{a}\|^2} = \left((1,1)\cdot(1,\tfrac{1}{2})\right)\cdot\frac{(1,\frac{1}{2})}{(\sqrt{5/4})^2}$$

$$= \frac{3}{2}\cdot\frac{4}{5}\left(1,\frac{1}{2}\right) = \frac{6}{5}\left(1,\tfrac{1}{2}\right)$$

$$\vec{w}_2 = \vec{u} - \vec{w}_1 = (1,1) - \frac{6}{5}\left(1,\tfrac{1}{2}\right)$$

$$= \left(-\frac{1}{5}, \frac{2}{5}\right)$$

Ex: [verification of P.T. in $\mathbb{R}^4$]

$\vec{u} = (0,1,1,-1)$ $\rightsquigarrow$ $\vec{u}\cdot\vec{v} = 0$ (check)

$\vec{v} = (1,0,1,1)$

$$\|\vec{u}+\vec{v}\|^2 = \|(0+1, 1+0, 1+1, -1+1)\|^2$$

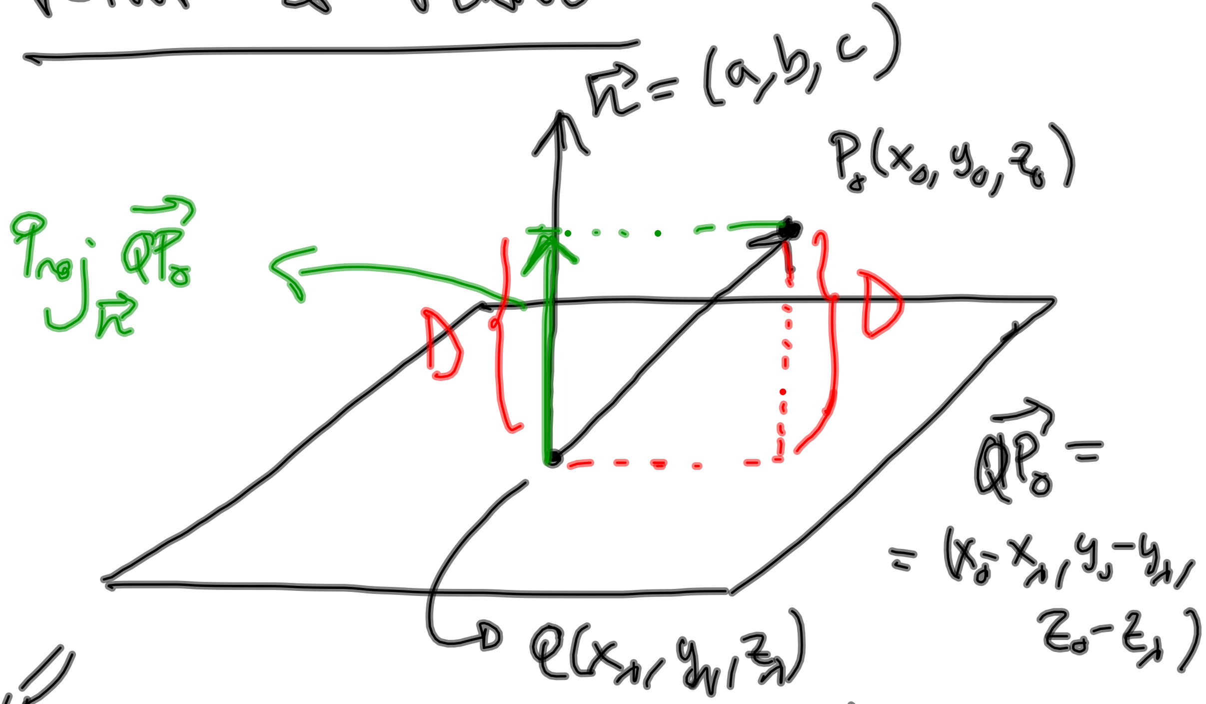
$$= \left(\sqrt{1^2+1^2+2^2+0^2}\right)^2 = 6$$

$$\|\vec{u}\|^2 = \left(\sqrt{0^2+1^2+1^2+(-1)^2}\right)^2 = 3$$

$$\|\vec{v}\|^2 = \left(\sqrt{1^2+0^2+1^2+1^2}\right)^2 = 3$$

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$$

• DERIVATION OF DISTANCE BETWEEN POINT & PLANE



$$\vec{QP_0} = (x_0 - x_1, y_0 - y_1, z_0 - z_1)$$

$$\left(\left(D = \| \text{Proj}_{\vec{n}} \vec{QP_0} \| = \frac{|\vec{QP_0} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \right)\right)$$

$\downarrow$

isolate "d" from $ax_1 + by_1 + cz_1 + d = 0$

∘ **Ex**: [DISTANCE BETWEEN 2 PARAL. PLANES]

Plane 1 → $x+y+z=1$
Plane 2 → $x+y+z=2$
} they are parallel because they have the same normal vector.

Choose arbitrary point in plane 1:

$$P_0(x_0, y_0, z_0) = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)$$

$$\hookrightarrow \frac{1}{3}+\frac{1}{3}+\frac{1}{3} = 1 \checkmark$$

Distance plane 2 to P_0:

$$D = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}} = \frac{\left|\frac{1}{3}+\frac{1}{3}+\frac{1}{3}-2\right|}{\sqrt{3}} = $$

$$= 1/\sqrt{3}$$

- COMMENT EQ. OF LINE AND PLANE:

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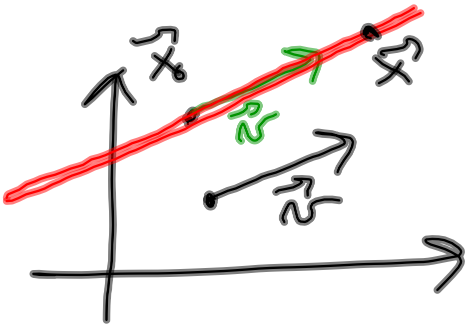
line:

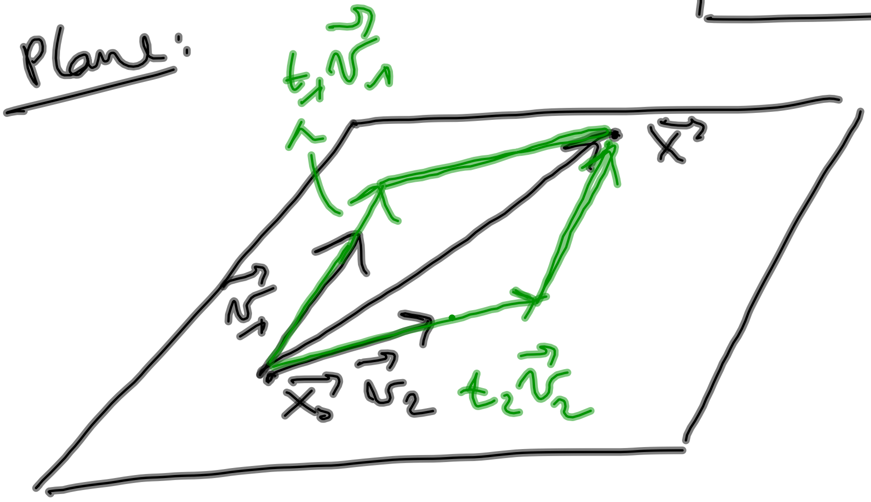


$\vec{x}-\vec{x}_0 = t\vec{v}$

where $t \in (-\infty, +\infty)$

$$\boxed{\vec{x} = \vec{x}_0 + t\vec{v}}$$

plane:



$\vec{x}-\vec{x}_0 = t_1\vec{v}_1 + t_2\vec{v}_2$

$$\hookrightarrow \boxed{\vec{x} = \vec{x}_0 + t_1\vec{v}_1 + t_2\vec{v}_2}$$

$t_1, t_2 \in (-\infty, +\infty)$

- **Ex**: [vector (or parametric) equation of lines and planes]

lines: Find line passing through origin in $\mathbb{R}^4$ and parallel to $\vec{v} = (1,0,0,1)$

$\vec{x} = \vec{x}_0 + t\vec{v}$, $x_0 = (0,0,0,0)$

$\vec{x} = (x_1, x_2, x_3, x_4)$

$$\boxed{(x_1, x_2, x_3, x_4) = t(1,0,0,1)}$$

(VECTOR FORM)

$$\boxed{x_1 = t,\ x_2 = 0,\ x_3 = 0,\ x_4 = t}$$

(PARAMETRIC FORM)

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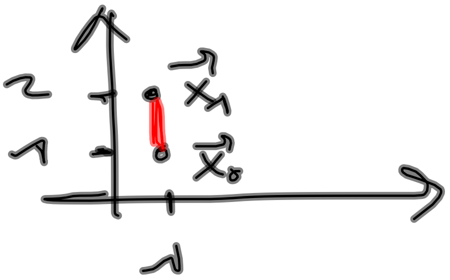
- **Ex**: [line segment in $\mathbb{R}^2$]

$\vec{x}_0 = (1,1)$

$\vec{x}_1 = (1,2)$



$$\vec{x} = \vec{x}_0 + t(\vec{x}_1 - \vec{x}_0) \quad , \quad (0 \leq t \leq 1)$$

$$\vec{x} = (1,1) + t(1-1, 2-1) =$$

$$= (1,1) + t(0,1)$$

$$(0 \leq t \leq 1)$$

- COMMENT: IF we allow $t \in (-\infty, +\infty)$, then

$$\vec{x} = \vec{x}_2 + t(\vec{x}_1 - \vec{x}_0)$$

is called two-point vector equation of a line in $\mathbb{R}^n$.

• **COMMENT** :

$$A = \begin{pmatrix} \overbrace{a_{11} \; a_{12} \; \cdots \; a_{1n}}^{= \vec{r}_1} \\ a_{21} \; a_{22} \\ \vdots \\ \underbrace{a_{m1} \; a_{m2} \; \cdots \; a_{mn}}_{= \vec{r}_m} \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Then $A\vec{x} = 0$ can be written as

$$\left.\begin{array}{l} \vec{r}_1 \cdot \vec{x} = 0 \\ \vdots \\ \vec{r}_m \cdot \vec{x} = 0 \end{array}\right\}$$

• **Ex** :

$$\left[A\vec{x} = 0 \longrightarrow \left.\begin{array}{l} \vec{r}_1 \cdot \vec{x} = 0 \\ \quad \ddots \\ \vec{r}_m \cdot \vec{x} = 0 \end{array}\right\} \right]$$

$$\begin{pmatrix} 4 & 4 & 4 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow (x, y, z) = (0, -t, t)$$

$\vec{r}_1 \cdot \vec{x} = 0$? $\longrightarrow (4,4,4) \cdot (0,-t,t) =$ ✓

$= 0 - 4t + 4t = 0$

$\vec{r}_2 \cdot \vec{x} = 0$ ✓

$\vec{r}_3 \cdot \vec{x} = 0$ ✓

- **Ex:** $[A\vec{x} = \vec{b}]$

$$\begin{pmatrix} 4 & 4 & 4 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \longrightarrow \begin{matrix} (x,y,z) \\ \parallel \\ (-1,1-s,s) \end{matrix}$$

Homog. $\longrightarrow (x,y,z) = (0,-t,t)$

Non H. $\longrightarrow (x,y,z) = (-1,1-s,s)$

Specific sol. of $A\vec{x} = \vec{b}$:

$s = 10 \longrightarrow (x,y,z) = (-1,-9,10)$

Add spec. solution to the sol. of $A\vec{x} = 0$:

$(-1,-9,10) + (0,-t,t) = (-1,-9-t,10+t)$

Redefine s: $s = 10+t$

$(-1,-9-t,10+t) = (-1,1-s,s)$

- **Ex**: [det form of cross product]

$\vec{u} = (1, 0, 1)$

$\vec{v} = (0, 1, 1)$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} =$$

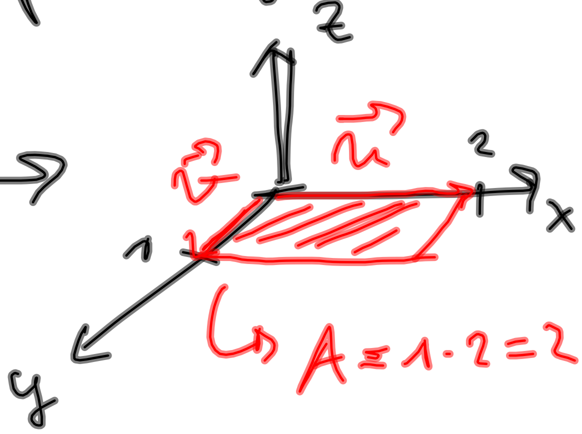
$$= \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \vec{k} =$$

$$= -1 \vec{i} - 1 \vec{j} + 1 \vec{k} = (-1, -1, 1)$$ //

∘ **Ex:** [Area of parallelogram]

$\vec{u} = (2,0,0)$

$\vec{v} = (0,1,0)$ $\longrightarrow$



↳ $A = 1\cdot 2 = 2$

$$\vec{u}\times\vec{v} = \begin{vmatrix} i & j & k \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 0\vec{i} - 0\vec{j} + 2\vec{k} = (0,0,2)$$

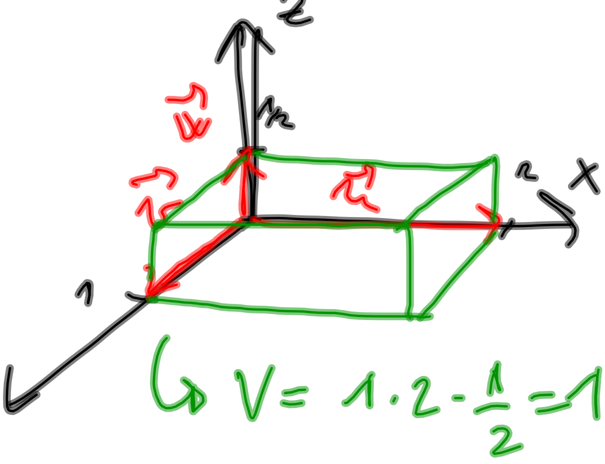
$$A = \|\vec{u}\times\vec{v}\| = \sqrt{0^2+0^2+2^2} = 2$$

• **Ex**: [Volume of parallelepiped]

$\vec{u} = (2,0,0)$

$\vec{v} = (0,1,0)$

$\vec{w} = (0,0,1/2)$



$\hookrightarrow V = 1 \cdot 2 \cdot \frac{1}{2} = 1$

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} =$$

$$= - \begin{vmatrix} w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} =$$

$$= \vec{w}\cdot(\underbrace{\vec{u}\times\vec{v}}) \rightarrow = (0,0,2)$$

(from previous example)

$$\left[\!\left[\, V = |\vec{u}\cdot(\vec{v}\times\vec{w})| = \right.\right.$$

$$= |\vec{w}\cdot(u\times\vec{v})| = |(0,0,\tfrac{1}{2})\cdot(0,0,2)|$$

$$\left.\left. = 1 \,\right]\!\right]$$

- **Ex**: $\left[\vec{u}\cdot(\vec{v}\times\vec{w}) = 0 \iff \begin{matrix} \vec{u},\vec{v},\vec{w} \text{ in} \\ \text{same} \\ \text{plane} \end{matrix} \right]$

In the previous example

if $\vec{w} = (1,1,0)$

$$\Big\langle\!\Big\langle\, V = \left| \vec{w}\cdot(\vec{u}\times\vec{v}| \right| =$$

$$= \left| (1,1,0)\cdot(0,0,2) \right| = 0 \,\Big\rangle\!\Big\rangle$$

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