· Ex: [zero vector space]

let V consist of a single object that we denote by $\vec{0}$, and define 2 ops.:

$$\boxed{\begin{aligned} \vec{0}+\vec{0} &= \vec{0} \\ k\vec{0} &= \vec{0} \end{aligned}}$$

→ addition

→ mult. by scalar

STEP1: objects → $\vec{0}$

STEP2: operations

STEP 3 :

Axiom 1 [$\vec{u}+\vec{v}$ is in V]

$\vec{0}+\vec{0}=\vec{0}$ ✓ closure under addition.

Axiom 6 [$k\vec{u}$ is in V]

$k\vec{0}=\vec{0}$ ✓ closure under multip.

STEP 4 :

Axiom 2 : [$\vec{u}+\vec{v}=\vec{v}+\vec{u}$]

$\vec{0}+\vec{0}=\vec{0}+\vec{0}=\vec{0}$ ✓

Axioms 3, 4, 5

Axiom 7 : [$k(\vec{u}+\vec{v})=k\vec{u}+k\vec{v}$]

$$\left.\begin{aligned} &k(\vec{0}+\vec{0})=k\vec{0}=\vec{0} \\ &k\vec{0}+k\vec{0}=\vec{0}+\vec{0}=\vec{0} \end{aligned}\right\} ✓$$

Axioms 8, 9, 10 ..

$\Rightarrow$ Because Axioms 1 – 10 are satisfied $\Rightarrow$ V containing only $\vec{0}$ with the operations $\vec{0}+\vec{0}=\vec{0}$, $k\vec{0}=\vec{0}$ is a vector space!

Ex: [vector space $\mathbb{R}^n$]

STEP 1 (objects)

let $V = \mathbb{R}^n$ $\rightarrow$ n-tuples (ordered sequence of real numbers with n entries)

with ops

$$\vec{u}+\vec{v} = (u_1+v_1, \dots, u_n+v_n)$$

$$k\vec{u} = (ku_1, \dots, ku_n)$$

STEP 2

Ex: [vector space of 2x2 matrices]

let V be the set of 2x2 matrices with real entries.

operation:

$$\vec{u}+\vec{v} = \begin{pmatrix} u_{11}+v_{11} & \cdots \\ \vdots & u_{22}+v_{22} \end{pmatrix}$$

$$k\vec{u} = \begin{pmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{pmatrix}$$

• **Ex:** [not a vector space]

Let $V = \mathbb{R}^2$ with ops: (*)

$\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2)$

$k\vec{u} = (ku_1, 0)$ — (*)(*)

↳ Axioms 1–9 are satisfied

Let's check Axiom 10: $[1\vec{u} = \vec{u}]$

$1\vec{u} = (1u_1, 0) = (u_1, 0) \neq \vec{u}$

↳ $V = \mathbb{R}^2$ with ops. (*) and (*)(*) is not a vector space.

• Ex: [weird vector space]

let $V = \mathbb{R}^2$ with

$$\vec{u} + \vec{v} = (u_1 + v_1 + 1, u_2 + v_2 + 1) \quad (*)$$

$$k\vec{u} = (ku_1 + k - 1, ku_2 + k - 1) \quad (*)(*)$$

• Axiom 1: ✓ (closure under addition)

• Axiom 6: ✓ (closure under scalar mult.)

• Axiom 2 & 3 (easy to see)

• Axiom 4: [zero vector that $\vec{0} + \vec{u} = \vec{u} + \vec{0} = \vec{u}$]

$\boxed{\text{Take } \vec{0} = (-1, -1)}$ → find this by:

$(a, b) + (u_1, u_2) =$

$= (a + u_1 + 1, b + u_2 + 1)$

$= (u_1, u_2)$

$\begin{cases} a = -1 \\ b = -1 \end{cases}$

$$\vec{0} + \vec{u} = (-\cancel{1} + u_1 + \cancel{1}, -\cancel{1} + u_2 + \cancel{1}) =$$
$$= (u_1, u_2) = \vec{u} \quad ✓$$

Axiom 5: [negative of $\vec{u}$ such that $\vec{u} + (-\vec{u}) = (-\vec{u}) + \vec{u} = \vec{0}$]

First we see that

$$-\vec{u} = (-1)\vec{u} = ((-1)u_1 + (-1) - 1, (-1)u_2 + (-1) - 1)$$
$$= (-u_1 - 2, -u_2 - 2)$$

Now,

$$\vec{u} + (-\vec{u}) = (\cancel{u_1} + (-\cancel{u_1} - 2) + 1, \cancel{u_2} + (-\cancel{u_2} - 2) + 1)$$
$$= (-1, -1) = \vec{0}$$

↳ verify that the rest are satisfied.

- **Ex**: [Subspace]

↳ Lines through the origin in $\mathbb{R}^2$.

$\vec{x} = t\vec{v}$, $\vec{y} = s\vec{u}$

- **Addition**: $\vec{x} + \vec{y} = \underbrace{(tv_1 + su_1, tv_2 + su_2)}_{\text{also a line through the origin}}$ ✓

- **Multiplic. by scalar**: $k\vec{x} = \underbrace{(kt)\vec{v}}_{\text{also line through the origin}}$ ✓

$\boxed{\text{Closures are satisfied}}$

- **Ex:** [NOT A SUBSPACE]

The set of invertible 3x3 matrices is not a subspace of the vector space of 3x3 matrices (M_{33}).

Let, $U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \longrightarrow U^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$

$V = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \longrightarrow V^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$

$\Rightarrow$ $U + V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \rightarrow$ NOT INVERTIBLE

$\Rightarrow$ CLOSURE UNDER ADD. IS NOT SATISFIED $\Rightarrow$ set of invertible matrices in M_{33} is not a subspace.

$\Rightarrow$ Can be extended to M_{nn}.

• **PROOF**: [THEOREM SUBSPACE FROM SUBSPACES]

W_1, W_2 subsp. of V and $W = W_1 \cap W_2$

↳ Because W_1, W_2 are subspaces they have a $\vec{0}$ (the same as V)

↳ ($W_1 \cap W_2$ at least contains 1 object)

↳ If $\vec{u}$ and $\vec{v}$ belong to W, they belong to W_1 and W_2 $\Rightarrow$

$\Rightarrow$ $\vec{u}+\vec{v}$ is in W_1
$\vec{u}+\vec{v}$ is in W_2 $\Rightarrow$

$\Rightarrow$ $\vec{u}+\vec{v}$ is in W $\Rightarrow$

$\Rightarrow$ closure under addition ✓

↳ similarly for closure under scalar multiplication.

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• **Ex**: [standard unit vectors in $\mathbb{R}^3$ span $\mathbb{R}^3$]

The general vector $\vec{v} = (a, b, c)$ in $\mathbb{R}^3$ can be expressed as

$$\vec{v} = a(1,0,0) + b(0,1,0) + c(0,0,1)$$

$$= a\vec{i} + b\vec{j} + c\vec{k}$$

If $S = \{\vec{i}, \vec{j}, \vec{k}\}$ then

$$\mathbb{R}^3 = \text{span}\{S\}.$$

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• Ex: [Spanning set for $\boxed{P_2}$]

Polynomials of degree 2 $\vec{p} = a_0 + a_1x + a_2x^2$

$$P_2 = \text{span}\{1, x, x^2\}$$

↳ can be generalized to P_n

$$P_n = \text{span}\{1, x, \dots, x^n\}$$

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• **Ex:** [L.I. in $\mathbb{R}^3$] $\left\{ \begin{array}{l} \vec{v}_1 = (1,0,0) \\ \vec{v}_2 = (2,2,0) \\ \vec{v}_3 = (0,2,1) \end{array} \right\}$

$$k_1(1,0,0) + k_2(2,2,0) + k_3(0,2,1) = (0,0,0)$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} k_1 + 2k_2 + 0k_3 = 0 \\ 0k_1 + 2k_2 + 2k_3 = 0 \\ 0k_1 + 0k_2 + k_3 = 0 \end{cases}$$

Reminder: (Equivalence theorem)

$A\vec{x} = 0$ has only the trivial sol $\iff$ $\det(A) \neq 0$

$$\det(A) = 1 \cdot \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} \neq 0 \Rightarrow$$

$\Rightarrow \vec{v}_1, \vec{v}_2, \vec{v}_3$ are l.i.

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• <u>Ex</u> : [L.I. of polynomials in $\mathbb{P}_2$]

Let, $\vec{p}_1 = 1 - x$, $\vec{p}_2 = 1 - x^2$,

$$\vec{p}_3 = 1 + x + x^2$$

If $k_1\vec{p}_1 + k_2\vec{p}_2 + k_3\vec{p}_3 = 0$ is only satisfied when $k_1 = k_2 = k_3 = 0$

$\Rightarrow \vec{p}_1, \vec{p}_2, \vec{p}_3$ are l.i.

$$k_1(1-x) + k_2(1-x^2) + k_3(1+x+x^2) = 0$$

$$\underbrace{(k_1+k_2+k_3)}_{①} + \overbrace{(k_3-k_1)}^{②}x + \underbrace{(k_3-k_2)}_{③}x^2 = 0$$

①,②,③ must be zero for the eq. to be satisfied:

$$\left.\begin{aligned} k_1 + k_2 + k_3 &= 0 \\ -k_1 + 0k_2 + k_3 &= 0 \\ 0k_1 - k_2 + k_3 &= 0 \end{aligned}\right\} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\det(A) = \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} - \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}$$

$$= 1 + 1 + 1 = 3 \neq 0$$

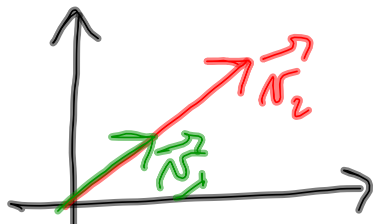
$\Rightarrow \vec{p}_1, \vec{p}_2, \vec{p}_3$ are l.i.

$\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \rightarrow \boxed{\vec{u} = 2\vec{v}}$

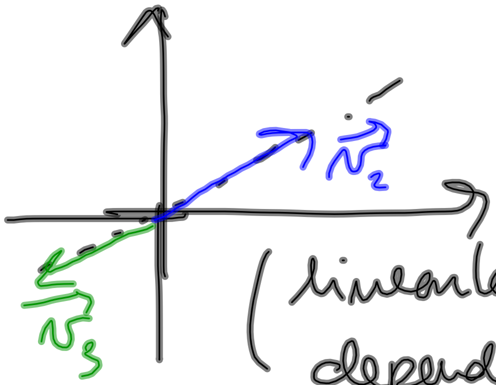
$k_1\vec{v} + k_2\vec{u} = 0 \rightarrow k_1\vec{v} + 2k_2\vec{v} = 0$

$(k_1 + 2k_2)\vec{v} = 0$

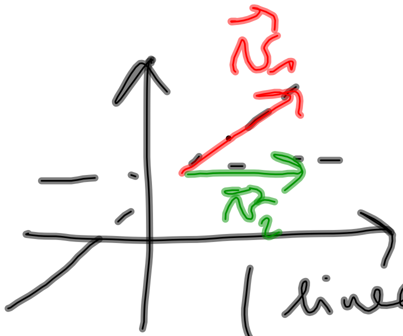
COMMENT [GEOMETRICAL INTERPRETATION OF L.I]



(linearly dependent)



(linearly dependent)



(linearly independent)

$$S = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \quad \mathbb{R}^2$$

$\hookrightarrow n = 2$

$\hookrightarrow r = 3$

$\hookrightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

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· **Ex** : [WRONSKIAN THEOREM]

$\vec{f}_1 = 1$, $\vec{f}_2 = e^{5x}$

$\hookrightarrow$ $\vec{f}_1{}' = 0$, $\vec{f}_2{}' = 5e^{5x}$

$$W = \begin{vmatrix} 1 & e^{5x} \\ 0 & 5e^{5x} \end{vmatrix} =$$

$$= 5e^{5x} - 0 = 5e^{5x} \neq 0 \text{ in } (-\infty, +\infty)$$

$\{\vec{f}_1, \vec{f}_2\}$ $\Longleftarrow$

l.i. set in $C^1(-\infty, +\infty)$.

$f_1(x) = 1$

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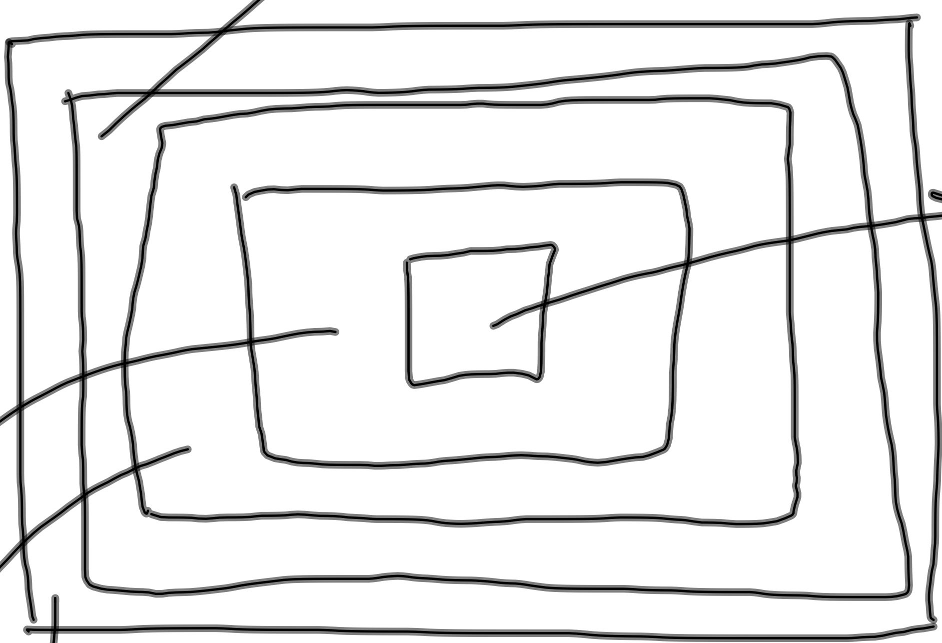
• <u>COMMENT</u>:

$C(-\infty,+\infty)$ v.s. of continuous functions in $(-\infty,+\infty)$



P_n

$F(-\infty,+\infty)$ vector space of real valued functions defined in $(-\infty,+\infty)$

$C^n(-\infty,+\infty)$ v.s. of continuous functions with $\underline{n}$ continuous derivatives.

$C^\infty(-\infty,+\infty)$

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Ex: [TEST SPANNING]

Do $\left\{ \begin{array}{l} \vec{v}_1 = (1,1,0) \\ \vec{v}_2 = (-1,-1,0) \\ \vec{v}_3 = (0,0,2) \end{array} \right\}$ span $\mathbb{R}^3$?

Let $\vec{b} = (b_1, b_2, b_3)$ a general vector in $\mathbb{R}^3$:

$$(b_1, b_2, b_3) = k_1 \vec{v}_1 + k_2 \vec{v}_2 + k_3 \vec{v}_3 =$$

$$= (k_1, k_1, 0) + (-k_1, -k_2, 0) + (0, 0, 2k_3)$$

$$\hookrightarrow \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ \boxed{0 \quad 0 \quad 2} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

If $A\vec{x} = \vec{b}$ is consistent $\Rightarrow$

$\Rightarrow \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ span $\mathbb{R}^3$

[Reminder: [eq. theorem]

$A\vec{x} = \vec{b}$ consistent $\Longleftrightarrow \det(A) \neq 0 \Longleftrightarrow$ $A\vec{x} = \vec{b}$ has only one solution]

In this case $\det(A) = 0 \Rightarrow$

$\Rightarrow \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ do not span $\mathbb{R}^3$