• **Ex:** [INFINITE & FINITE-DIM VECT. SPACES]

INFINITE	FINITE
$F(-\infty,\infty)$, R^∞, P_∞	R^n, M_{mn}, P_n

• **Ex:** [basis]

$R^n \longrightarrow \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$

$\vec{e}_1 = (1, 0, 0, \dots, 0)$

$\vec{e}_2 = (0, 1, 0, \dots, 0)$

$\vdots$

$\vec{e}_n = (0, 0, \dots, 1)$

R^3

$\hookrightarrow \{\vec{i}, \vec{j}, \vec{k}\}$

$$\mathbb{R}^3 \rightarrow S = \left\{ \begin{array}{l} \vec{v}_1 = (1,0,1) \\ \vec{v}_2 = (0,1,1) \\ \vec{v}_3 = (1,2,-1) \end{array} \right\}$$

basis for $\mathbb{R}^3$?

$\hookrightarrow$ (1) S is l-i?

$\hookrightarrow k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\hookrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\hookrightarrow k_1 = k_2 = k_3 = 0$ ✓

$\hookrightarrow$ S spans $\mathbb{R}^3$

$$\vec{k}_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \vec{b}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

EQUIVALENCE THEOREM:

$\det(A) \neq 0 \iff A\vec{x} = 0$ has only the trivial sol.

$\Updownarrow$

$A\vec{x} = \vec{b}$ has solution for every $\vec{b}$.

$\Rightarrow$ S is basis for $\mathbb{R}^3$ $\Longleftarrow$ $\det(A) = -4$

$P_n \rightarrow \{1, x, \ldots, x^n\}$ basis for P_n

Ex: [coordinates]

(i) $\mathbb{R}^3 \longrightarrow S = \{\vec{i}, \vec{j}, \vec{k}\}$

$\vec{v} = (a,b,c) \longrightarrow \vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$

$(\vec{v})_S = (a,b,c)$

(ii) $\mathbb{R}^3 \longrightarrow S' = \left\{ \begin{array}{l} (1,0,1) = \vec{v}_1 \\ (0,1,1) = \vec{v}_2 \\ (1,2,-1) = \vec{v}_3 \end{array} \right.$

Find the coordinate vector of $\vec{u} = (1,1,1)$ relative to S

$(1,1,1) = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 1/2 \\ 1/4 \end{pmatrix}$$

$\hookrightarrow (\vec{u})_{S'} = (3/4, 1/2, 1/4)$

(iii) M_{22}, $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$(B)_S = (a,b,c,d)$

Ex: [orthogonal $\Rightarrow$ l.i]

$\vec{v}_1 = (0,1,0)$, $\vec{v}_2 = (-1,0,1)$, $\vec{v}_3 = (1,0,1)$

$\langle \vec{v}_1, \vec{v}_2 \rangle = \vec{v}_1 \cdot \vec{v}_2 = 0$

$\vec{v}_2 \cdot \vec{v}_3 = 0$, $\vec{v}_1 \cdot \vec{v}_3 = 0$

$\Rightarrow$ $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is l.i.

$\hookrightarrow$ (orthogonal basis in $\mathbb{R}^3$)

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· **Ex**: [orthogonal basis]

↳ Represent $\vec{u} = (1,1,1)$ in terms of $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ (from previous example)

$$\vec{u} = \frac{\langle \vec{u}, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 + \frac{\langle \vec{u}, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 + \frac{\langle \vec{u}, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2} \vec{v}_3$$

$$\langle \vec{u}, \vec{v}_1 \rangle = (1,1,1) \cdot (0,1,0) = 1$$

$$\langle \vec{u}, \vec{v}_2 \rangle = (1,1,1) \cdot (-1,0,1) = 0$$

$$\langle \vec{u}, \vec{v}_3 \rangle = (1,1,1) \cdot (1,0,1) = 2$$

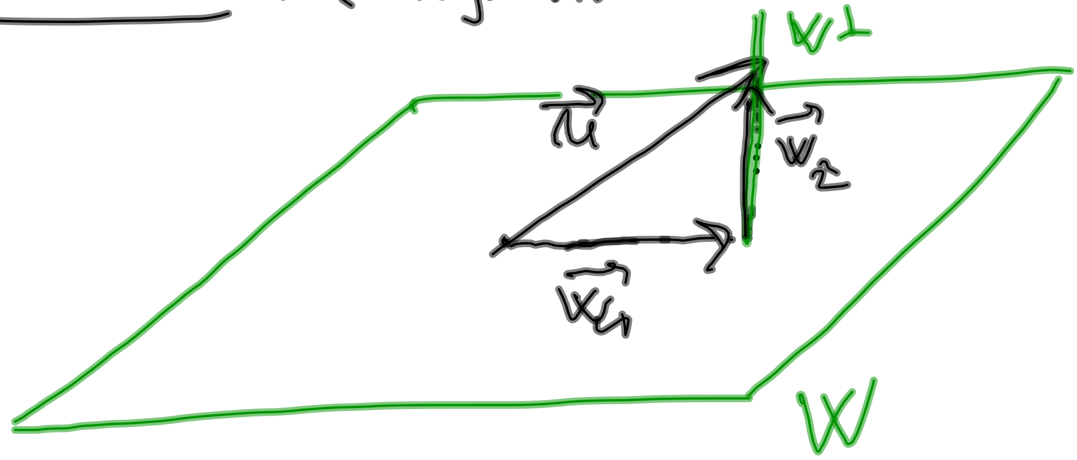
$$\|\vec{v}_1\|^2 = \left(\sqrt{0^2+1^2+0^2}\right)^2 = 1$$

$$\|\vec{v}_2\|^2 = 2, \quad \|\vec{v}_3\|^2 = 2$$

$$\vec{u} = \frac{1}{1}\vec{v}_1 + \frac{0}{2}\vec{v}_2 + \frac{2}{2}\vec{v}_3 = \vec{v}_1 + \vec{v}_3$$

$$(\vec{u})_S = (1,0,1)$$

Comment: (Projection theorem)



$\vec{w}_1 = \text{proj}_W \vec{u}$ (orthogonal projection of $\vec{u}$ on W)

$\vec{w}_2 = \text{proj}_{W\perp} \vec{u}$ (orthogonal projection of $\vec{u}$ on $W\perp$)

$$\vec{u} = \text{proj}_W \vec{u} + \text{proj}_{W\perp} \vec{u}$$

$$\vec{u} = \text{proj}_W \vec{u} + (\vec{u} - \text{proj}_W \vec{u})$$

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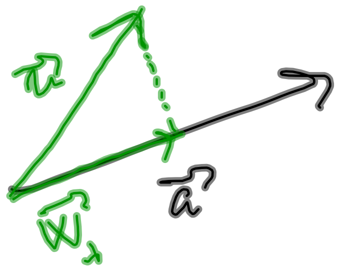
$$\vec{w}_1 = \text{proj}_{\vec{a}}\,\vec{u} = \frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2}\,\vec{a}$$



$$\langle \vec{u}, \vec{a} \rangle$$

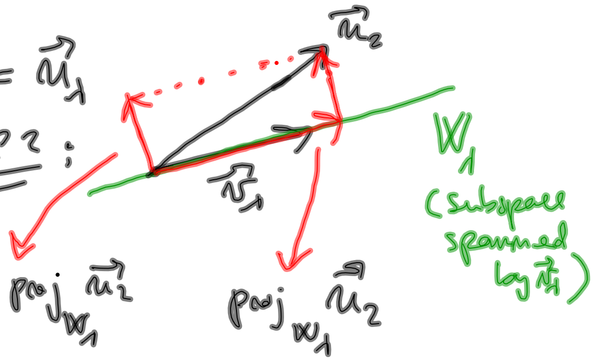
COMMENT : (GRAM-SCHMIDT THEOREM)

$\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_r\}$ not orthogonal basis of W $\longrightarrow$ find $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ which is orthogonal basis of W

· STEP 1

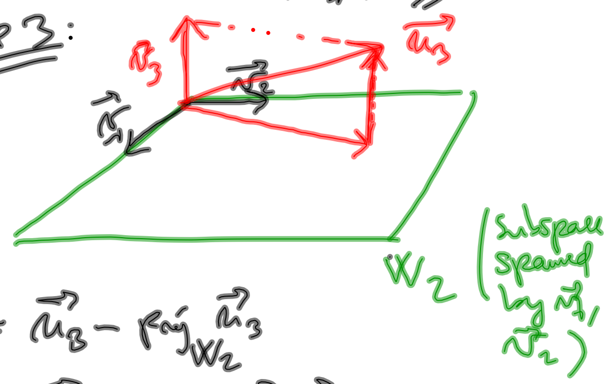
$\vec{v}_1 = \vec{u}_1$

· STEP 2 :



$$\vec{v}_2 = \vec{u}_2 - \text{proj}_{W_1} \vec{u}_2 =$$
$$= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$$

· STEP 3:



$$\vec{v}_3 = \vec{u}_3 - \text{proj}_{W_2} \vec{u}_3$$
$$= \vec{u}_3 - \left[\frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 + \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2 \right]$$
$$= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

⋮

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Ex: [Gram-Schmidt process]

$\vec{u}_1 = (1,2)$, $\vec{u}_2 = (-1,2)$

$\hookrightarrow$ not orthogonal basis of $\mathbb{R}^2$

$\hookrightarrow$ Find $\{\vec{v}_1, \vec{v}_2\}$ orthogonal

$\vec{v}_1 = \vec{u}_1 = (1,2)$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 =$$

$\|\vec{v}_1\|^2 = 5$

$\langle \vec{u}_2, \vec{v}_1 \rangle = 3$

$$= (-1,2) - \frac{3}{5}(1,2) =$$

$$= \left(-\frac{8}{5}, \frac{4}{5}\right) \qquad \vec{v}_1 \cdot \vec{v}_2 = 0 \ \checkmark$$

• **Ex**: $\dim(\mathbb{R}^n) = n \rightarrow \dim(\mathbb{R}^2)=2$

$\dim(\mathbb{P}_n) = n+1$

$\hookrightarrow \dim(\mathbb{P}_1) = 2$

$\hookrightarrow \{1, x\}$

$\dim(M_{mn}) = m \cdot n$

$\hookrightarrow \dim(M_{22}) = 2 \cdot 2 = 4$

$\hookrightarrow \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

• **Ex**. [dimension of span(S)]

If $S = \{\vec{v}_1, \vec{v}_1, \dots, \vec{v}_r\}$ is l.i. set

$\dim(\text{span}(S)) = r$

• **Ex**: Find the basis and dimension of the sol. space of the linear system:

$$\begin{pmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \text{Solution: } \quad x = -2s, \quad y = s, \quad z = s$$

$\Rightarrow$ Sol. space: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = s \underbrace{\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}}_{= \vec{v}}$

$\hookrightarrow$ the vector $\vec{v}$ spans the sol. space (and is a basis)

$\hookrightarrow$ $\dim(\operatorname{span}(\vec{v})) = 1$

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• **Ex**: [Application of plus/minus theorem]

Show that $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ are l.i.

$\left.\begin{array}{l} \hookrightarrow S = \{\vec{v}_1, \vec{v}_2\} \text{ l.i. set} \\ \hookrightarrow \vec{v}_3 \text{ is outside span}(S) \end{array}\right\} \Rightarrow$

$\Rightarrow$ The set $S \cup \{\vec{v}_3\}$ is l.i.

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Ex: i) Is the set

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

a basis for $\mathbb{R}^3$?

↳ $\dim(\mathbb{R}^3) = 3$

↳ # vectors we have $= 3$

↳ and they are l.i. (previous example)

$\Longleftarrow$

S is a basis for $\mathbb{R}^3$

ii) Is the set S formed by $\vec{p}_1 = 1$, $\vec{p}_2 = 1-x$, $\vec{p}_3 = 1-x^2$ a basis for $\mathbb{P}_2$?

↳ $\dim(\mathbb{P}_2) = 3$

↳ # vectors we have $= 3$

↳ are they l.i.?

$$k_1 \vec{p}_1 + k_2 \vec{p}_2 + k_3 \vec{p}_3 = 0$$

$$k_1 \cdot 1 + k_2(1-x) + k_3(1-x^2) = 0$$

$$(k_1 + k_2 + k_3)1 - k_2 x - k_3 x^2 = 0$$

$$k_1 = 0 \Leftarrow \Rightarrow k_2 = 0 \Rightarrow k_3 = 0$$

$\Rightarrow$ S is l.i. $\Rightarrow$

$\Rightarrow$ S is basis for $\mathbb{P}_2$ //

• **COMMENT** : (theorem row operations & null space)
(example)

We know sol. space of $A\vec{x} = 0$ is equal to $R\vec{x} = 0$, if R is row echelon form of A.

R is obtained by row ops on A $\Rightarrow$ row ops do not change null space of A.

• **Ex**: [row operations & row space]

$$A = \begin{pmatrix} 1 & 2 & -3 \\ -1 & 3 & 2 \\ 2 & 1 & -2 \end{pmatrix} \begin{matrix} =\vec{u}_1 \\ =\vec{u}_2 \\ =\vec{u}_3 \end{matrix} \xrightarrow[\text{elimination}]{\text{Gaussian}} R = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -1/5 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} =\vec{v}_1 \\ =\vec{v}_2 \\ =\vec{v}_3 \end{matrix}$$

(reduced row echelon form)

↳ Take row vectors of R and check l.i.

$$k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = 0$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1/5 & 1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$k_1 = 0$

$2k_1 + k_2 = 0 \Rightarrow k_2 = 0$

$\Downarrow$

$k_3 = 0$

So, $k_1 = k_2 = k_3 = 0$

$\Rightarrow \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are l.i and span $\mathbb{R}^3$

↳ Take row vectors of A and check l.i.

$$c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 = 0$$

$$\hookrightarrow \begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & 1 \\ -3 & 2 & -2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow c_1 = c_2 = c_3 = 0$$

$\Rightarrow \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is l.i. set and span $\mathbb{R}^3$

$\Rightarrow$ Elementary row operations on A did not change row space of A.

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• <u>Ex</u>:

$$R = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -1/5 \\ 0 & 0 & 1 \end{pmatrix}$$

basis for row space

basis for column space

$$R = \begin{pmatrix} 1 & 2 & -3 & 2 \\ 0 & 1 & -1/5 & 5 \\ 0 & 0 & 1 & 6 \end{pmatrix}$$

basis for column space

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