QUESTION FROM LAST WEEK:

In $(\lambda I - A)\vec{x} = 0$

why don't we just do $(\lambda I - A) = 0$ and solve for $\lambda$ instead of $\det(\lambda I - A) = 0$ ?

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$\lambda I - A = 0$

ANSWER

(2x2)

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

unless $b = c = 0$

$\lambda - a = 0 \Rightarrow \lambda = a$

$0 - b = 0 \rightarrow b = 0$

<u>Ex</u>: [FIND EIGENVALUES]

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \longrightarrow \det(\lambda I - A) = 0$$

$$\left| \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \right| =$$

$$= \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 1 \end{vmatrix} =$$

$$= (\lambda - 2)(\lambda - 1) - 1 = 0$$

$$\Leftrightarrow \lambda^2 - 3\lambda + 1 = 0$$

$$\lambda = \frac{3 \pm \sqrt{9-4\cdot 1\cdot 1}}{2} = \begin{cases} \lambda_1 = \frac{3+\sqrt{5}}{2} \\ \lambda_2 = \frac{3-\sqrt{5}}{2} \end{cases}$$

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<u>Ex</u>:
$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 3 & 1 & 8 \end{pmatrix}$$

$\to \det(\lambda I - A) = 0$

$$\left| \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 3 & 1 & 8 \end{pmatrix} \right| =$$

$$= \begin{vmatrix} \lambda-1 & -2 & 0 \\ 0 & (\lambda-4) & 0 \\ -3 & -1 & (\lambda-8) \end{vmatrix} =$$

$$= (\lambda-8)(-1)^{3+3} \begin{vmatrix} \lambda-1 & -2 \\ 0 & \lambda-4 \end{vmatrix}$$

$$= (\lambda-8)(\lambda-1)(\lambda-4) = 0$$

$$\Rightarrow \lambda_1 = 8, \lambda_2 = 1$$

$$\lambda_3 = 4$$

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Ex : [$\lambda$'s of a triangular matrix ]

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

$$\det(\lambda I - A) = 0$$

$$\parallel$$

$$\left| \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \right| =$$

$$= \begin{vmatrix} \lambda-1 & 0 & 0 & 0 \\ 0 & \lambda-2 & 0 & 0 \\ 0 & 0 & \lambda-3 & 0 \\ 0 & 0 & 0 & \lambda-4 \end{vmatrix} =$$

$$= (\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4) = 0$$

$$\hookrightarrow \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3$$
$$\lambda_4 = 4$$

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Ex :

$$A = \begin{pmatrix} 1 & \pi & 2\pi & 1 \\ 0 & 2 & 5 & 10 \\ 0 & 0 & 3 & 10^3 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & \pi & 2\pi & 1 \\ 0 & \lambda-2 & -5 & -10 \\ 0 & 0 & \lambda-3 & 10^3 \\ 0 & 0 & 0 & \lambda-4 \end{vmatrix}$$

$$= (\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4) = 0$$

$\lambda_1 = 1,\ \lambda_2 = 2$

$\lambda_3 = 3,\ \lambda_4 = 4$

If $A$ triangular

$\Rightarrow \det(A) = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$

# LINEAR COMBINATION & BASE

$$\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \vec{v} = a\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\vec{P}_1} + b\underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\vec{P}_2}$$

- We say $\vec{v}$ is a linear combination of $\vec{P}_1$ and $\vec{P}_2$.
- $\vec{P}_1$, $\vec{P}_2$ are a base.
- $\vec{P}_1$ & $\vec{P}_2$ are linearly independent (they cannot be generated from each other)

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<u>Ex</u> : [Find eigenvectors and bases of eigenspace]

$$A = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$$

<u>Eigenvalues</u> :

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -1 \\ -3 & \lambda + 1 \end{vmatrix} =$$

$$= (\lambda - 1)(\lambda + 1) - 3 = 0$$

$$\Rightarrow \quad \lambda^2 \cancel{+ \lambda - \lambda} - 1 - 3 = 0$$

$$\lambda^2 - 4 = 0 \Rightarrow \begin{cases} \lambda_1 = 2 \\ \lambda_2 = -2 \end{cases}$$

Eigenvector for $\lambda_1 = 2$ : $\left[(\lambda_1 I - A)\vec{x} = 0\right]$

$$\left[2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}\right]\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\hookrightarrow$ We see that we can eliminate row 2. So,

$x - y = 0 \Rightarrow \boxed{x = y}$

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Let $y = t$ where $\underline{t}$ is a real number. So,

$$x = y = t \quad \Rightarrow$$

$$\Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ t \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The family of solutions is the eigenspace of $\lambda_1 = 2$.

base for the eigenspace of $\lambda_1 = 2$

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Eigenvectors for $\lambda_1 = -2$ :

$$\left[-2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}1 & 1\\ 3 & -1\end{pmatrix}\right]\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}-3 & -1\\ -3 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\hookrightarrow -3x - y = 0 \Rightarrow$$

$$\Rightarrow \boxed{x = -\frac{1}{3}y}$$

Let, $y = t$ ($t$ any real value)

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Say $x = -\frac{y}{3}t$

eigenspace

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}t \\ t \end{pmatrix} =$$

$$= t \begin{pmatrix} -1/3 \\ 1 \end{pmatrix}$$

base

Ex :

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{vmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{vmatrix} =$$

$$= (\lambda-2) \begin{vmatrix} \lambda & 2 \\ -1 & \lambda-3 \end{vmatrix} =$$

$$= (\lambda-2) \underline{\left[\lambda(\lambda-3)+2\right]} = 0$$

$$\boxed{\lambda_1 = 2}$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\hookrightarrow \lambda = \frac{3 \pm \sqrt{9 - 4 \cdot 1 \cdot 2}}{2} =$$

$$= \begin{cases} \boxed{\lambda_2 = 2} \\ \lambda_3 = 1 \end{cases}$$

<u>Eigenvectors for $\lambda_1 = \lambda_2 = 2$</u>

$$\left[ 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \right] \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{G-J}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x + z = 0 \Longrightarrow \boxed{x = -z}$$

Let, $y = t$ and $z = s$

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$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s \\ t \\ s \end{pmatrix} = \begin{pmatrix} -s \\ 0 \\ s \end{pmatrix} + \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix}$$

eigenspace

$$= s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{base} = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

<u>Eigenvector for $\lambda_3$</u> $\longrightarrow$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2s \\ s \\ s \end{pmatrix} = s\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

Ex : [A invertible $\Leftrightarrow$ $\lambda = 0$ not an eigenvalue]

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{vmatrix} \lambda - a & 0 & 0 \\ 0 & \lambda - b & 0 \\ 0 & 0 & \lambda - c \end{vmatrix} =$$

$$= (\lambda - a)(\lambda - b)(\lambda - c) = 0$$

$\hookrightarrow \lambda_1 = a, \lambda_2 = b, \lambda_3 = c$

If $\lambda_1, \lambda_2$ or $\lambda_3$ are zero $\Rightarrow$ A has a row of 0's $\Rightarrow$ A is not invertible

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Show that $\det(A) = \det(P^{-1}AP)$

$\det(MD) = \det(M) \cdot \det(D)$

$B = P^{-1}AP$

$\Downarrow$

$\det(B) = \det(P^{-1}AP) =$

$= \det(P^{-1})\det(A)\det(P) =$

$= \dfrac{1}{\cancel{\det(P)}} \cdot \det(A) \cdot \cancel{\det(P)} =$

$= \det(A)$

$\det(C^{-1}) = \dfrac{1}{\det(C)}$

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Ex : [A diagonalizable $\Longleftrightarrow$ n L.I. vectors]

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \rightarrow \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \\ \lambda_4 = 4 \end{array}$$

$$\lambda_1 = 1 \rightarrow \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} t \\ 0 \\ 0 \\ 0 \end{pmatrix} = \vec{v}_1$$

$$\left[ 1 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \right] \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\hookrightarrow \left. \begin{array}{l} y = z = w = 0 \\ x = t \end{array} \right\}$$

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$$\lambda_2 = 2 \longrightarrow \vec{v}_2 = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda_3 = 3 \longrightarrow \vec{v}_3 = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ r \\ 0 \end{pmatrix}$$

$$\lambda_4 = 4 \longrightarrow \vec{v}_4 = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ q \end{pmatrix}$$

$\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$
are 4 linearly independent eigenvectors

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A is a $4\times4$ matrix $\Rightarrow$ A has 4 linearly independent vecters

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}\}$

$\hookrightarrow$ is called a <u>linearly independent set</u>

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Ex : [DIAGONALIZE A 2x2 MATRIX]

STEP 1 :

$$A = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \to \lambda_1 = 2 \to \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\to \lambda_2 = -2 \to \vec{v}_2 = \begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix}$$

$\to$ Matrix is 2x2 and we have 2 linearly independent vectors ($\vec{v}_1$ & $\vec{v}_2$) $\Rightarrow$

$\Rightarrow$ A is diagonalizable!

STEP 2 :

$$P = \begin{pmatrix} 1 & -1/3 \\ 1 & 1 \end{pmatrix}$$

STEP3 : Find diagonal matrix $P^{-1}AP$

$$P^{-1} = \frac{1}{1+1/3}\begin{pmatrix} 1 & 1/3 \\ -1 & 1 \end{pmatrix} =$$

$$= \frac{3}{4}\begin{pmatrix} 1 & 1/3 \\ -1 & 1 \end{pmatrix}$$

$$P^{-1}AP = \frac{3}{4}\begin{pmatrix} 1 & 1/3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1/3 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{3}{4}\begin{pmatrix} 1 & 1/3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2/3 \\ 2 & -2 \end{pmatrix}$$

$$= \frac{3}{4}\begin{pmatrix} 8/3 & 0 \\ 0 & -8/3 \end{pmatrix} = \frac{8}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= 2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

Ex: [Matrix non-diagonalizable]

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow \lambda_1 = -1, \quad \lambda_2 = 2$$

$$\Rightarrow \vec{v}_1 = \begin{pmatrix} -t \\ t \\ 0 \end{pmatrix}$$

$$\Rightarrow \vec{v}_2 = \begin{pmatrix} 2t \\ t \\ 0 \end{pmatrix}$$

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Ex: [1 eigenvalue of A $\Rightarrow$ $\lambda^k$ is eigenvalue of $A^k$]

$$A = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \longrightarrow \lambda_1 = 2, \quad \lambda_2 = -2$$

$$A^2 = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} =$$

$$= \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \quad \begin{matrix} \to (2)^2 \\ \to (-2)^2 \end{matrix}$$