Ex: [GEOMETRIC & ALGEBRAIC MULTIPLICITY]

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \rightarrow \det(\lambda I - A) = 0$$

$$\Rightarrow (\lambda-2)[\lambda(\lambda-3)+2] = 0$$

$\lambda_1 = 2$

$\Rightarrow \lambda_2 = 2$

$\Rightarrow \lambda_3 = 1$

$$\Rightarrow (\lambda-2)^{2}(\lambda-1)^{1} = 0$$

$\Rightarrow$ Algebraic multipl. of $\lambda = 2$ is 2 //

A.M. of $\lambda = 1$ is 1 //

· Eigenvector for $\lambda = 2$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = s \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

⟶ Base has 2 vectors
⟹ geom. mult. is $\underline{\underline{2}}$

· Eigenvectors for $\lambda = 1$:

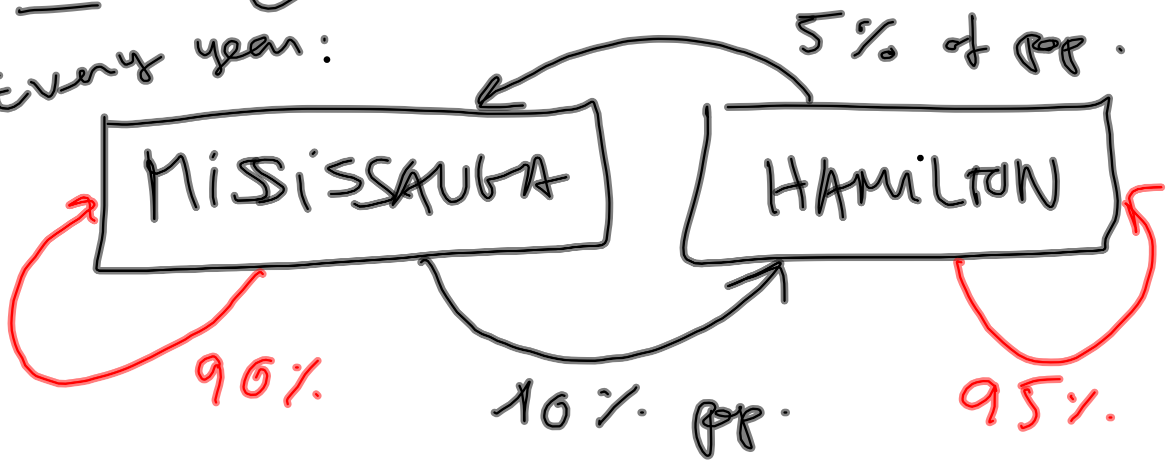
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = s \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

⟶ Base has 1 vector
⇓
g. m. is 1

⟹ A is Diagonalizable

• **Ex**: [DYNAMICAL SYSTEM]

Every year:



Variables

m : population in Mississauga in thousands

h : " " Hamilton.

Ex:
$$\left.\begin{aligned} m(t=0) &= 300 \\ h(t=0) &= 100 \end{aligned}\right\} \to \text{State of the variables at } t=0$$

State vector at $t=0$:

$$\begin{pmatrix} m(t=0) \\ h(t=0) \end{pmatrix} = \begin{pmatrix} 300 \\ 100 \end{pmatrix}$$

Translating migration scheme into equations:

$$m(t=1) = 0.9\, m(t=0) + 0.05\, h(t=0)$$
$$h(t=1) = 0.1\, m(t=0) + 0.95\, h(t=0)$$

$\hookrightarrow$ any year k and $k+1$:

$$m(k+1) = 0.9\, m(k) + 0.05\, h(k)$$
$$h(k+1) = 0.1\, m(k) + 0.95\, h(k)$$

In vector/matrix form :

$$\begin{pmatrix} m(k+1) \\ h(k+1) \end{pmatrix} = \underbrace{\begin{pmatrix} 0.9 & 0.05 \\ 0.1 & 0.95 \end{pmatrix}}_{\text{(Transition matrix)}} \begin{pmatrix} m(k) \\ h(k) \end{pmatrix}$$

Ex : [Stochastic processes, Markov Chain]

↳ Rewrite previous examples from the probability point of view

$$\left.\begin{array}{l} M(0) = 300 \\ H(0) = 100 \end{array}\right\} \to \text{TOTAL POP.} \approx 400 \text{ (in thousands)}$$

$$\to x_1(0) = \frac{300}{400} = \underline{0.75} \quad \text{(fraction of pop. in M.)}$$

$$\to x_2(0) = \frac{100}{400} = \underline{0.25} \quad \text{(fraction of pop. in H.)}$$

Note, $x_1(0)$ and $x_2(0)$ can also be interpreted as probabilities:

$x_1(0)$: prob. that a person is living in M. at $t=0$.

$x_2(0)$: " " " " " " H at $t=0$.

Then, $\vec{X} = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$

is a probability vector !

$X_1(0) + X_2(0) = 1$, and

$\vec{X}(1), \vec{X}(2), \ldots$ also prob. vectors.

$\Rightarrow$ we can write the DS as an Markov Chain:

$$\begin{pmatrix} X_1(k+1) \\ X_2(k+1) \end{pmatrix} = \begin{pmatrix} 0.9 & 0.05 \\ 0.1 & 0.95 \end{pmatrix} \begin{pmatrix} X_1(k) \\ X_2(k) \end{pmatrix}$$

TRANS. MATRIX
IS A STOCHASTIC MATRIX.

Interpretation of matrix entries:

$p_{11} = 0.9 =$ prob. that a person living in M. at $t=k$ remains livin in M at $t=k+1$.

$p_{21} = 0.1 =$ prob. that a person living in M. at $t=k$ will live in H. at time $t=k+1$.

p_{12}, p_{22} similarly.

$\underline{\underline{t=1}}$:
$$\begin{pmatrix} x_1(1) \\ x_2(1) \end{pmatrix} = \begin{pmatrix} 0.9 & 0.05 \\ 0.1 & 0.95 \end{pmatrix} \begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix} = \begin{pmatrix} 0.6875 \\ 0.3125 \end{pmatrix}$$

$x_1(0)$

$x_2(0)$

$\underline{\underline{t=2:}}$

$$\begin{pmatrix} x_1(2) \\ x_2(2) \end{pmatrix} = \begin{pmatrix} 0.9 & 0.05 \\ 0.1 & 0.95 \end{pmatrix} \begin{pmatrix} x_1(1) \\ x_2(1) \end{pmatrix} =$$

$$= \begin{pmatrix} 0.6344 \\ 0.3656 \end{pmatrix}$$

$t=3, \quad t=4, \ldots$

$$\begin{pmatrix} x_1(2) \\ x_2(2) \end{pmatrix} = \begin{pmatrix} 0.9 & 0.05 \\ 0.1 & 0.95 \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 0.9 & 0.05 \\ 0.1 & 0.95 \end{pmatrix} \begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}}_{= \begin{pmatrix} x_1(1) \\ x_2(1) \end{pmatrix}}$$

• **Ex** : [Convergence or Long term behaviour of a regular Markov chain]

$$(I - P)\vec{q} = 0$$

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 9/10 & 1/20 \\ 1/10 & 19/20 \end{pmatrix}\right]\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1/10 & -1/20 \\ -1/10 & 1/20 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\hookrightarrow \begin{pmatrix} \frac{1}{10} & -1/20 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \frac{1}{10} q_1 - \frac{1}{20} q_2 = 0 \quad \Rightarrow$$

$$\Rightarrow \quad q_1 = \frac{10}{20} q_2 = \frac{1}{2} q_2$$

Let, $q_2 = s \Rightarrow q_1 = \frac{1}{2} s$

$$\Rightarrow \quad \vec{q} = \begin{pmatrix} \frac{1}{2} s \\ s \end{pmatrix}$$

Because $\vec{q}$ is a prob. vector:

$$q_1 + q_2 = 1 \quad \Rightarrow \quad \frac{1}{2} s + s = 1$$

$$\Rightarrow s \underbrace{\left(\frac{1}{2} + 1\right)}_{3/2} = 1 \Rightarrow \boxed{s = 2/3}$$

Say $\vec{q} = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$

$\Rightarrow$ At the end of time the prob. of a person living in M is $1/3$ and in H is $2/3$

(at $t=0$ we had $3/4$ and $1/4$, respectively)

and

$$P^k \longrightarrow Q = \begin{pmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{pmatrix}$$

as $k \to \infty$

You can check (with Matlab):

$$P^{32} = \begin{pmatrix} 0.3370 & 0.3345 \\ 0.6630 & 0.6685 \end{pmatrix}$$

<u>Ex</u>: [Basic operations with complex]

$$z_1 = 2+3i \quad , \quad z_2 = 1 + 5i$$

<u>Addition</u>:

$$z_1 + z_2 = 3 + 8i$$

<u>Substraction</u>: $z_1 - z_2 =$

$$= 1 - 2i$$

Multip. by scalar:

$$5z_1 = 10 + 15i$$

Multiplication:

$$z_1 \cdot z_2 = (2+3i)(1+5i)$$

$$= 2 + 10i + 3i + 3 \cdot 5 \cdot i^2 =$$

$$= -13 + 13i$$

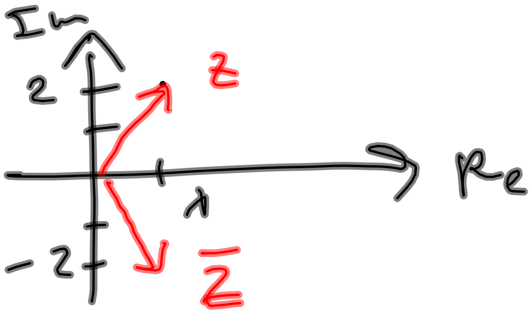
$$-1 = \left(\sqrt{-1}\right)^2 = i^2$$

NOTE: If $Re(z)=0$, the complex number z is called "pure complex number"

Ex: $\pi i, -2i, \sqrt{13}\, i, \ldots$

• **Ex**: [conjugate]

$z = 1 + 2i \longrightarrow \bar{z} = 1 - 2i$



• **Ex**: [modulus]

$z = 1 + 2i \longrightarrow |z| = \sqrt{1^2 + 2^2} =$

$= \sqrt{5}$

• **Ex**: [verify $z\bar{z} = |z|^2$]

Let $z = a + bi$

$$z\bar{z} = (a+bi)(a-bi)$$

$$= a^2 - \cancel{abi} + \cancel{bia} - b^2 \overset{-1}{\overset{\shortparallel}{i^2}}$$

$$= a^2 + b^2 = \left(\sqrt{a^2+b^2}\right)^2 = |z|^2$$

<u>Ex</u>: [verify $z = \frac{z_1}{z_2} = \frac{1}{|z_2|^2} \bar{z}_2 z_1$]

$z = \frac{z_1}{z_2} \longrightarrow z_1 = z_2 \cdot z$

we know $z_2 \bar{z}_2 = |z_2|^2$, so

$z_1 = z_2 z = \frac{|z_2|^2}{\bar{z}_2} \cdot z \Rightarrow$

$\Rightarrow \boxed{z = \frac{1}{|z_2|^2} \bar{z}_2 \cdot z_1}$

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• **Ex**: [Division]

Ef $z_1 = 1+2i$, $z_2 = 1+3i$

$$\frac{z_1}{z_2} = \frac{1}{\left(\sqrt{1^2+3^2}\right)^2} \cdot \overset{= \overline{z_2}}{(1-3i)} \cdot \overset{= z_1}{(1+2i)}$$

$$\left(\sqrt{1^2+3^2}\right)^2 = |z_2|^2$$

$$= \frac{1}{10}\left[1 + 2i - 3i - 6\overset{= -1}{i^2}\right]$$

$$= \frac{1}{10}\left[7 - 1i\right] =$$

$$= \frac{7}{10} - \frac{1}{10} i$$

· **Ex**: $\left[\ \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2\ \right]$

$z_1 = a + bi\ ,\quad z_2 = c + di$

$z_1 + z_2 = (a+c) + (b+d)i$

$\overline{z_1 + z_2} = (a+c) - (b+d)i$

$\bar{z}_1 + \bar{z}_2 = a - bi + c - di$

$= (a+c) - (b+d)i$