• DERIVATION OF:

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

$$\frac{z_1}{z_2} = \frac{1}{|z_2|^2} z_1 \bar{z}_2 = \frac{1}{\left(\sqrt{\underbrace{r_2^2\cos^2\theta_2 + r_2^2\sin^2\theta_2}_{\to\, r_2^2(\underbrace{\cos^2\theta_2 + \sin^2\theta_2}_{=1})}}\right)^2} z_1 \bar{z}_2 =$$

$$= \frac{1}{r_2^{\cancel{2}}} \cdot r_1(\cos\theta_1 + i\sin\theta_1) \cdot \cancel{r_2}(\cos\theta_2 - i\sin\theta_2)$$

$$= \frac{r_1}{r_2}\big(\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2$$
$$1 = -\left(\sqrt{-1}\right)^2 = - i^2 \sin\theta_1\sin\theta_2\big)$$

$$= \frac{r_1}{r_2}\left[\underbrace{\cos\theta_1\cos\theta_2}_{} + \underbrace{\sin\theta_1\sin\theta_2}_{} + i\left(\underbrace{\sin\theta_1\cos\theta_2}_{} - \underbrace{\cos\theta_1\sin\theta_2}_{}\right)\right.$$

$$\triangleright = \frac{1}{2}\left[\cancel{\cos(\theta_1-\theta_2)} + \cos(\theta_1+\theta_2)\right]$$

$$\triangleright = \frac{1}{2}\left[\cos(\theta_1-\theta_2) - \cancel{\cos(\theta_1+\theta_2)}\right]$$

$$\triangleright = \frac{1}{2}\left[\cancel{\sin(\theta_1+\theta_2)} + \sin(\theta_1-\theta_2)\right]$$

$$\triangleright = \frac{1}{2}\left[\cancel{\sin(\theta_1+\theta_2)} - \sin(\theta_1-\theta_2)\right]$$

$$= \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right] \;//$$

Derivation n-th root formula:

We have $z = r(\cos\theta + i\sin\theta)$

the n-th root

$$z^{1/n} = w \longrightarrow \underline{z = w^n}$$

Let,

$$w = \rho(\cos\alpha + i\sin\alpha)$$

$$\rho^n(\cos n\alpha + i\sin n\alpha) = \underline{r(\cos\theta + i\sin\theta)}$$

$$\rho^n = r \longrightarrow \boxed{\rho = \sqrt[n]{r}}$$

$$\left.\begin{array}{l}\cos(n\alpha) = \cos\theta \\ \sin(n\alpha) = \sin\theta\end{array}\right\} \rightarrow n\alpha = \theta + 2\pi k$$

$$k = 0, \pm 1, \pm 2, \ldots$$

So, $\boxed{\alpha = \frac{\theta}{n} + \frac{2\pi k}{n}}$

$$\boxed{w = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i \sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right]}$$

$k = 0, \pm 1, \pm 2, \ldots$

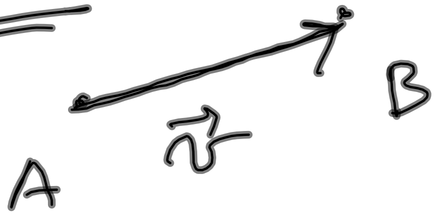
But we realize after $k = n-1$ values repeat. So,

$$\boxed{k = 0, 1, \ldots, n-1}$$

(We have $\underline{\underline{n}}$ different roots)

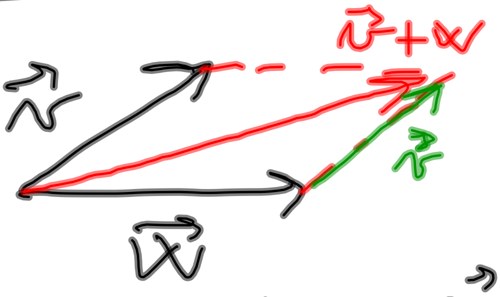
VECTORS IN 2-SPACE, 3-SPACE, n-SPACE

- Vectors

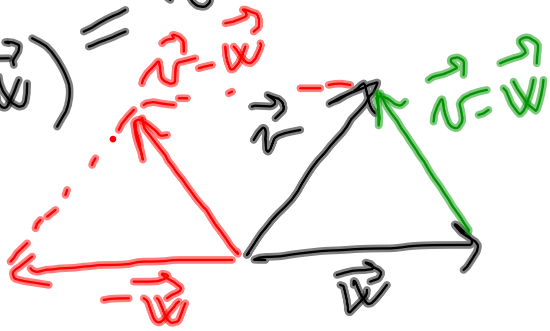


$\vec{v} = \overrightarrow{AB}$

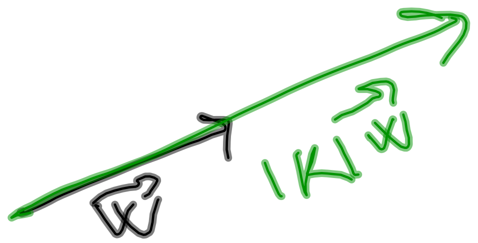
- Addition of vectors: $(\vec{v} + \vec{w})$



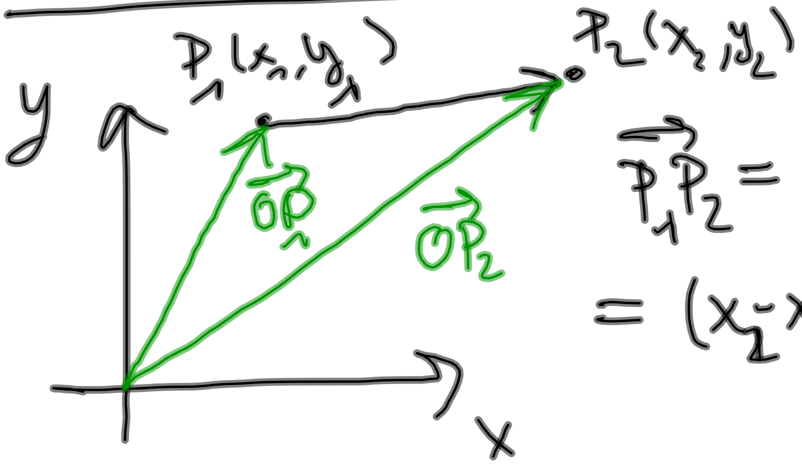
- Subtraction: $(\vec{v} - \vec{w}) = \vec{v} + (-\vec{w})$



- Multiplic. by scalar : $|k|\vec{w}$



- Vectors whose initial point is not at the origin :



$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$$

$$= (x_2 - x_1, y_2 - y_1)$$

• **Ex**: [linear combination]

$$\vec{u} = 3\vec{v}_1 + 6\vec{v}_2 - \pi\vec{v}_3 + \sqrt{2}\,\vec{v}_4$$

We say that $\vec{u}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ & $\vec{v}_4$ with coef. $3, 6, -\pi$ and $\sqrt{2}$.

We can write vectors in different $\underbrace{\text{matlab}}$ ways:

$\vec{v} = (v_1, v_2, \dots, v_n)$ or $\vec{v} = [v_1\ v_2\ \cdots\ v_n]$

or $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$

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• **Ex**: [norm]

If $\vec{v} = (-1, 2, 0, 3)$ in $\mathbb{R}^4$, then

$$\|\vec{v}\| = \sqrt{(-1)^2 + 2^2 + 0^2 + 3^2} = \sqrt{14}$$

• DERIVATION of $\|k\vec{v}\| = |k|\,\|\vec{v}\|$:

$$k\vec{v} = (kv_1, kv_2, \dots, kv_n)$$

$$\|k\vec{v}\| = \sqrt{k^2v_1^2 + k^2v_2^2 + \dots + k^2v_n^2} =$$

$$= \sqrt{k^2(v_1^2 + v_2^2 + \dots + v_n^2)} = |k|\,\|\vec{v}\|$$

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• <u>Ex</u>: [normalization]

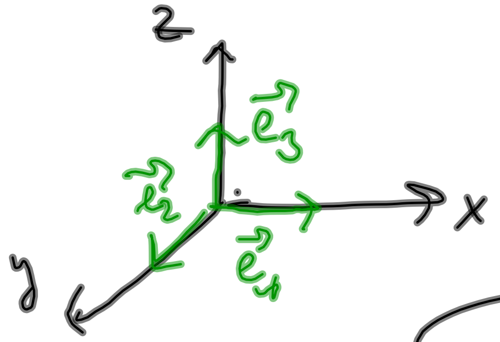
If $\vec{v} = (-1, 2, 0, 3)$ in $\mathbb{R}^4$

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\sqrt{14}}(-1, 2, 0, 3)$$

$$= \left(-\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, 0, \frac{3}{\sqrt{14}}\right)$$

- **Ex**: [standard unit vectors]

In 3D:



$$\left.\begin{aligned} \vec{e}_1 &= (1,0,0) \\ \vec{e}_2 &= (0,1,0) \\ \vec{e}_3 &= (0,0,1) \end{aligned}\right\}$$

$\Rightarrow$ Sometimes called $\vec{i}, \vec{j}, \vec{k}$ (physics)

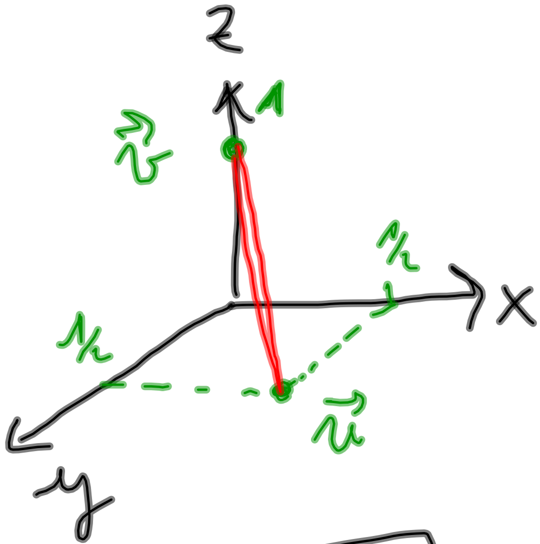
Any vector in $\mathbb{R}^3$ can be expressed as a linear combination of $\vec{i}, \vec{j}, \vec{k}$ $\Rightarrow$ $\vec{v} = (\pi, -1, 30) =$

$$= \pi\vec{i} - 1\vec{j} + 30\vec{k}$$

• <u>Ex</u> : [distance]

$$\vec{u} = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$$

$$\vec{v} = (0, 0, 1)$$



$$\left\langle\!\left\langle\, d(\vec{u}, \vec{v}) = \sqrt{\left(\frac{1}{2}-0\right)^2 + \left(\frac{1}{2}-0\right)^2 + (0-1)^2} = \sqrt{\frac{1}{4}+\frac{1}{4}+1} = \sqrt{\frac{3}{2}} \,\right\rangle\!\right\rangle$$

• **Ex**: [dot product by components]

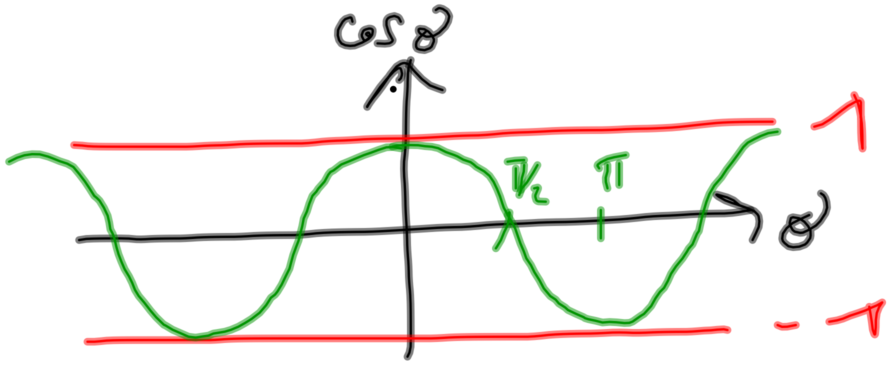
$$\vec{u} = (1, 0, 2, 1) \qquad \vec{v} = \left(0, 1, \tfrac{1}{2}, 3\right)$$

$$\vec{u} \cdot \vec{v} = 1 \cdot 0 + 0 \cdot 1 + 2 \cdot \tfrac{1}{2} + 1 \cdot 3 =$$
$$= 4$$

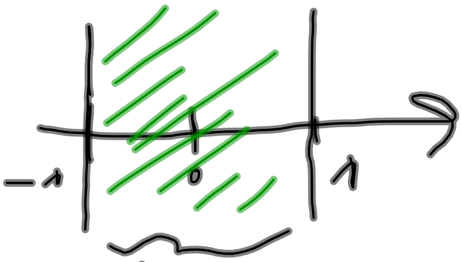
COMMENT: [Cauchy-Schwars inequality]

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos \theta$$

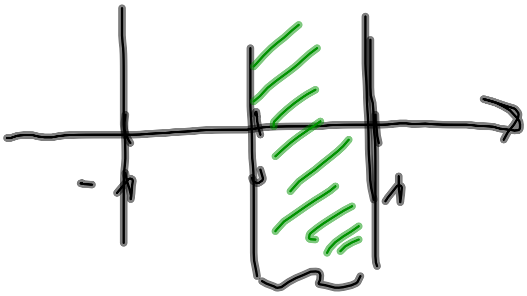
$$\longrightarrow \cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$$



$$\Rightarrow \quad -1 \leq \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} \leq 1$$



$$\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$$



$$\frac{|\vec{u}\cdot\vec{v}|}{\|\vec{u}\|\,\|\vec{v}\|} = \left|\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right|$$

$$\frac{|\vec{u}\cdot\vec{v}|}{\|\vec{u}\|\,\|\vec{v}\|} \leq 1$$

$$\boxed{|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\,\|\vec{v}\|}$$

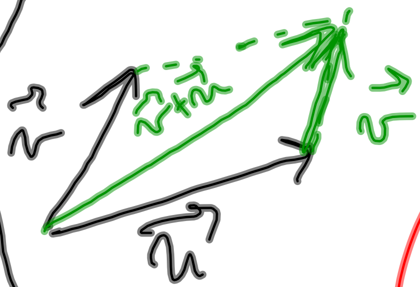
Cauchy-Schwarz Inequality.

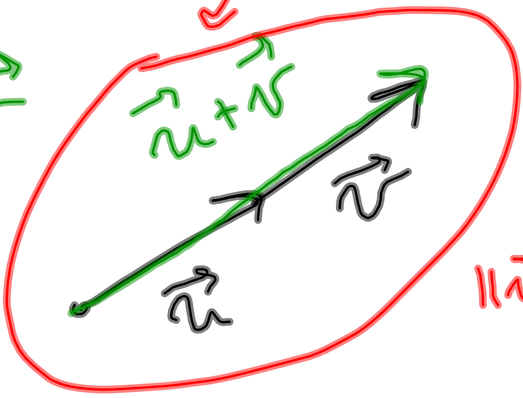
COMMENT:

(vectors)

TRIANGLE INEQUALITY:

$$\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$$





$\|\vec{u}+\vec{v}\| =$
$= \|\vec{u}\| + \|\vec{v}\|$

The sum of the lengths of two sides of a triangle is at least as large as the third.

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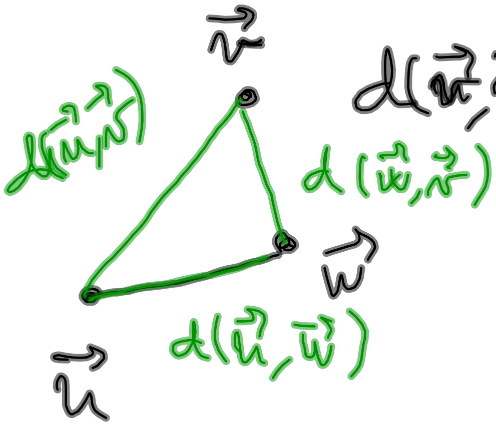
- **TRIANGLE INEQUALITY**: (DISTANCES)



$$d(\vec{u},\vec{v}) \leq d(\vec{u},\vec{w}) + d(\vec{v},\vec{w})$$

"shortest distance between two points is a straight line".

- <u>Parallelogram equation for vectors:</u>

$$\|\vec{u}+\vec{v}\|^2+\|\vec{u}-\vec{v}\|^2 =$$

$\|w\| = \sqrt{\vec{w}\cdot\vec{w}}$

$$= \left(\cancel{\sqrt{(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})}}\right)^{\cancel{2}} + \left(\cancel{\sqrt{(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})}}\right)^{\cancel{2}} =$$

$$= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) + (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) =$$

$$= \vec{u}\cdot\vec{u} + \cancel{\vec{u}\cdot\vec{v}} + \cancel{\vec{v}\cdot\vec{u}} + \vec{v}\cdot\vec{v} +$$

$$+ \vec{u}\cdot\vec{u} - \cancel{\vec{u}\cdot\vec{v}} - \cancel{\vec{v}\cdot\vec{u}} + \vec{v}\cdot\vec{v}$$

$$= 2(\vec{u}\cdot\vec{u}) + 2(\vec{v}\cdot\vec{v}) =$$

$\|\vec{v}\|^2 = \vec{v}\cdot\vec{v}$

$$= 2\left(\|\vec{u}\|^2 + \|\vec{v}\|^2\right)$$

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· Relation between the norm and the dot product:

$$\vec{u}\cdot\vec{v} = \frac{1}{4}\|\vec{u}+\vec{v}\|^2 - \frac{1}{4}\|\vec{u}-\vec{v}\|^2 = ⊛$$

$$\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})(\vec{u}+\vec{v}) =$$

$$= \vec{u}\cdot\vec{u} + 2(\vec{u}\cdot\vec{v}) + \vec{v}\cdot\vec{v}$$

$$\|\vec{u}-\vec{v}\|^2 = \vec{u}\cdot\vec{u} - 2(\vec{u}\cdot\vec{v}) + \vec{v}\cdot\vec{v}$$

$$= \frac{1}{4}\left[\cancel{\vec{u}\cdot\vec{u}} + 2(\vec{u}\cdot\vec{v}) + \cancel{\vec{v}\cdot\vec{v}}\right]$$

$$- \frac{1}{4}\left[\cancel{\vec{u}\cdot\vec{u}} - 2(\vec{u}\cdot\vec{v}) + \cancel{\vec{v}\cdot\vec{v}}\right] =$$

$$= \frac{1}{2}(\vec{u}\cdot\vec{v}) + \frac{1}{2}(\vec{u}\cdot\vec{v}) =$$

$$= \vec{u}\cdot\vec{v}$$